(p, q, r)-generations and Conjugacy Class ranks of Certain Simple Groups of the form, $Sp(6, 2)$, M_{23} and A_{11}

by

MALEBOGO JOHN MOTALANE

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SUPERVISOR: Prof. TT Seretlo (University of Limpopo)

CO-SUPERVISOR: Dr. ABM Basheer (University of Al-Jouf, Saudi Arabia)

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DECLARATION

I declare that thesis hereby submitted to the University of Limpopo, for the degree **Doctor of Philosophy (Mathematics)** has not previously been submitted by me for a degree at this or any other university; that it is my work in design and in execution, and that all material contained herein has been duly acknowledeged.

$\frac{Motalane,\quad MJ\quad (Mr)}{Surname, Initials(title)}$	$\frac{September\quad 2021}{Date}$

Abstract

A finite group G is called (l, m, n)-*generated*, if it is a quotient group of the triangle group $T(l, m, n) = \langle x, y, z | x^l = y^m = z^n = xyz = 1 \rangle$. In [43], Moori posed the question of finding all the (p, q, r) triples, where p, q and r are prime numbers, such that a non-abelian finite simple group G is a (p, q, r)-generated. In this thesis, we will establish all the (p, q, r)-generations of the following groups, the Mathieu sporadic simple group M_{23}, the alternating group A_{11} and the symplectic group $Sp(6, 2)$.

Let X be a conjugacy class of a finite group G. The *rank* of X in G, denoted by $rank(G : X)$, is defined to be the minimum number of elements of X generating G. We investigate the ranks of the non-identity conjugacy classes of the above three mentioned finite simple groups. The Groups, Algorithms and Programming (GAP) [26] and the Atlas of finite group representatives [55] are used in our computations.

Dedication

I would like to thank my wife Hellen and my children Katlego, Mohlapa, Ngokwana, Maila and Kgaogelo for their support and understanding during my studies.

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List of Symbols and Notations

Throughout this thesis all groups will be assumed to be finite, unless otherwise stated. We will use the notation and terminology from these two Atlases [20] and [38].

G	groups
1_G	the identity element of G
H, K, M	maximal subgroups of G
$H \cong G$	H is isomorphic to G
$\mathbb{F}$	a field
nX	a general conjugacy class of G with representatives of order n
$\lvert g \rvert$	order of $g \in G$
$[g]_G$	a conjugacy class of G with representative g
$\lvert \Omega \rvert$	the cardinality of the set Ω
1_G	the identity character of G
D_n	dihedral group of order $2n$
S_n	the symmetric group on n symbols

$GF(q)$	the Galois field of q elements
$V(n,q)$	a vector space of dimension n over $GF(q)$
$Sp(2n,q)$	symplectic group of dimension $2n$ over $GF(q)$
M_{23}	Sporadic Mathieu group acting on 23 points
A_n	alternating group of order $\frac{n!}{2}$

CHAPTER 1

Introduction

According to [54], the classification theorem for finite simple groups states that every finite simple group is isomorphic to one of the following:

(a) a cyclic group C_p of prime order p;

(b) an alternating group A_n, for $n \geq 5$;

(c) a classical group:

linear: $PSL_n(q), n \geq 2$, except $PSL_2(2)$ and $PSL_2(3)$;

unitary: $PSU_n(q), n \geq 3$, except $PSU_3(2)$;

symplectic: $PSp_{2n}(q), n \geq 2$, except $PSp_4(2)$;

orthogonal:

$P\Omega_{2n+1}(q), n \geq 3$, q is odd;

$P\Omega^+_{2n}(q), n \geq 4$;

$P\Omega^-_{2n}(q), n \geq 4$;

where q is a power p^a of a prime p;

(d) an exceptional group of Lie type:

$G_2(q),\ q > 2;\ F_4(q);\ E_6(q);\ {}^2E_6(q);\ {}^3D_4(q);\ E_7(q);\ E_8(q)$

where q is a prime power, or

${}^2B_2(2^{2n+1}),\ n \geq 1;\ {}^2G_2(3^{2n+1}),\ n \geq 1;\ {}^2F_4(2^{2n+1}),\ n \geq 1$

or the Tits group ${}^2F_4(2)'$;

(e) One of the 26 sporadic simple groups

the five Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24};

the seven Leech lattice groups Co_1, Co_2, Co_3, McL, HS, Suz, J_2;

the three Fischer groups Fi_{22}, Fi_{23}, Fi_{24}';

the five Monstrous groups $\mathbb{M}$, $\mathbb{B}$, Th, HN, He;

the six pariahs J_1, J_3, J_4, $O'N$, Ly, Ru.

1.1. Motivation

In this thesis, we investigate the generation of groups by their triples using structure constant method. We are also interested in generation of finite simple groups by minimal number of elements from a given non-identity conjugacy class of a group. This minimal number is called the rank of that conjugacy class for that group. Our targeted groups are the Mathieu simple group M_{23}, the alternating group A_{11} and the symplectic group $Sp(6, 2)$.

Generation of finite groups by suitable subsets is of great interest and has many applications to groups and their representations. For example, the computations of the genus of simple groups can be reduced to the generation of the relevant simple groups (see Woldar [57] for details). Also Di Martino et al. [41] established a useful connection between generation of groups by conjugate elements and the existence of elements represented by almost cyclic matrices. Their motivation was to study irreducible projective representations of sporadic simple groups.

Following Basheer and Moori [13], the study of generating sets in finite groups has a rich history, with numerous applications. If G is a finite non-abelian simple group then the group G is said to be 2-*generated* if it can be generated by two elements. This has been known for a long time in the case of the alternating groups in 1901 by Miller [42]. In 1962 the result was extended by Steinberg [51] to the groups of Lie type, where he gave a unified treatment for the 2-generation of the Chevalley and the Twisted groups. Before this the 2-generation of certain families of groups of Lie type were known (e.g., $PSL(n, \mathbb{F})$ and $Sp(2n, \mathbb{F})$). In 1984, Aschbacher and Guralnick [10] completed the problem of determining which of the finite simple groups are 2-generated by analyzing the sporadic groups that had not already been settled by other authors. They showed that any sporadic simple group can be generated by an involution (an element of order 2) and another suitable element. In 2017, King [39] wrote a paper giving a refinement where it was shown that every finite non-abelian simple group is generated by an involution and an element of a prime order.

The topic of generation of finite simple groups is fairly rich. In this thesis we cover generation of some finite simple groups by methods of (p, q, r)-generation. The following are a few examples of the problems concerning the generation of finite simple groups that may be found in the literature:

- In his PhD Thesis [53], Ward considered the problem of generating a non-abelian finite simple group by a set of conjugate involutions whose product is the identity. More specifically, he considers the problem of which groups have the property of being generated by the product of 5 conjugate involutions whose product is the identity and which simple groups have the property that they can be generated by 3 conjugate involutions, a, b

and c such that ab is also conjugate to a. The second of these properties can easily be shown to imply the first.

- A group is said to be $\frac{3}{2}$-*generated* if every non-trivial element is contained in a generating pair. Guralnick and Kantor [34] showed that every finite simple group is $\frac{3}{2}$-generated. In [18], Breuer et. al., conjectured that any finite group is $\frac{3}{2}$-generated if and only if every proper quotient is cyclic and the recent work of Guralnick [33] reduces this conjecture to almost simple groups. In 2017, a paper by S. Harper [37] extended the results to almost simple symplectic and odd-dimensional orthogonal groups.

- Guralnick et. al. [34, 35] were interested in probabilistic random generation of a finite simple group using elements of a fixed conjugacy class of the group. In fact Burness, Guralnick, Kantor, Liebeck, Saxl and Shalev have a pioneering role in the problem of the probabilistic random generation.

In addition to the above methods of generating a group G, there are many other problems concerning generating sets of groups. In this thesis we restrict ourselves to the areas of generation of finite non-abelian simple groups through

- the **ranks** of non-trivial conjugacy classes of elements,

- the (p, q, r)-**generations**.

1.2. Literature review

A finite group G is said to be (l, m, n)-*generated*, if $G = \langle x, y \rangle$, with $o(x) = l$, $o(y) = m$ and $o(xy) = o(z) = n$. Here $[x] = lX$, $[y] = mY$ and $[z] = nZ$, where $[x]$ is the conjugacy class of X in G containing elements of order l. The same applies to $[y]$ and $[z]$. When we refer to (lX, mY, nZ)-generations, we mean generations of all possible triples of conjugacy classes of elements of orders l, m and n respectively. In this case G is also a quotient group of the triangular group $T(l, m, n)$ and, by definition of the triangular group, G is also $(\sigma(l), \sigma(m), \sigma(n))$-generated group for any $\sigma \in S_3$. Therefore we may assume that $l \leq m \leq n$. Also a finite simple group G is said to be (p, q, r)-*generated* if a group G can be generated by two elements x and y of respective prime orders p and q such that xy has a prime order r.

For more information on (p, q, r)-generations, the reader is referred this series of papers [27, 28, 29, 30, 31, 32, 43, 44, 45] and [46]. Moori and Ganief established all possible (p, q, r)-generations of the sporadic groups J_1, J_2, J_3, HS, McL, Co_3, Co_2 and F_{22}, where p, q and r are distinct prime divisors of the order of a group. Ashrafi in [11, 12] established all possible (p, q, r)-generations for the sporadic simple groups He and HN. The (p, q, r)-generations of the sporadic simple groups Co_1, Ru, $O^{'}N$ and Ly were calculated by Darafsheh and his co-authors in these papers [21, 22, 23] and [24]. The motivation for this study is outlined in these papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques. In establishing all the (p, q, r)-generations of a group, we follow the methods used in [16] and [17] and also methods used in the recent papers [7] and [8] by Ali, Ibrahim and Woldar.

If nX be a non-identity conjugacy class of element of a finite simple group G, then $G = \langle nX \rangle$. Let G be a finite group and nX a conjugacy class of non-identity elements of G. We define the rank of G with respect to the conjugacy class nX to be the minimum number of elements of G in nX generating the entire group G and it is denoted by $rank(G : nX)$. One of the applications of ranks of conjugacy classes of a finite group is that they are used in the computations of the covering number of the finite simple group (see Zisser [59]).

Moori in various articles [45, 46] and [47]), computed the ranks of involution classes of the Fischer sporadic simple group Fi_{22}. He proved that $rank(Fi_{22} : 2A) \in \{5, 6\}$ and $rank(Fi_{22} : 2B) = 3 = rank(Fi_{22} : 2C)$. Hall and Soicher [36] found that $rank(Fi_{22} : 2A) = 6$. Ali in [1, 2] computed the ranks of the Fischer group Fi_{22} and both simple sporadic groups $O'N$ and Ly. Ali and Ibrahim in [3, 4, 5] determined the ranks of Conway group Co_1, the Higman-Sims group HS, McLaughlin group McL, Conway's sporadic simple groups Co_2 and Co_3. More recently Ibrahim and his co-authors in [6], computed the ranks of the Fischer group $Fi_{24}^{'}$ and the Baby Monster group $\mathbb{B}$. In determining the rank for each non-identity conjugacy class of a group G, we follow the methods in in [14] and [48] and the notation used in [20].

Note that, in general, if G is a $(2, 2, n)$-generated group, then G is a dihedral group and therefore G is not simple. Also by [19], if G is a non-abelian (lX, mY, nZ)-generated group, then either $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \geq 1$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. If a simple group G is (lX, mY, nZ)-generated with $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \geq 1$ then $G \cong A_5$. Thus for our purpose of establishing the (p, q, r)-generations of G, the only cases we need to consider are when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. None of the groups we are dealing is isomorphic to A_5. None of these groups we are dealing with is $(2, 2, n)$-generated, thus the rank of an involution class of any of these groups cannot be 2. In most cases two involutions generate a dihedral group. Thus the lower bound of the rank of an involution class in a finite group $G \neq D_{2n}$ (the dihedral group of order $2n$) is 3.

1.3. Thesis outlines

In this thesis we are mainly concerned with the (p, q, r)-generations of a group G, where p, q and r are prime numbers (not necessarily distinct) dividing the order G. We find the conjugacy classes ranks for each targeted group G. The thesis is structured into five chapters.

Chapter 1, which is the current chapter is an introduction to the thesis, which is itself divided into three sections. In Sections 1.1 and 1.2 we introduce the purposes of the thesis, the ideas and background behind the (p, q, r)-generations and the conjugacy classes ranks. Section 1.3 which is the current section, describes the structure of this thesis.

In Chapter 2 we discuss the structure constant method and use some of the results to find ranks. This chapter is divided into two sections. We first give important definitions and basic concepts in Section 2.1. Section 2.2 is mainly concerned with using some of the basic results in computing the ranks.

In Chapter 3, we discuss the symplectic simple group $Sp(6, 2)$. This chapter is divided into three sections. In Section 3.1, we give some information about the group $Sp(6, 2)$ that may be required in some of the computations. We discuss triple generation of $Sp(6, 2)$ in Section 3.2 and this section is divided into three subsections. The discussions on the $(2, q, r)$- and $(3, q, r)$-generations of group $Sp(6, 2)$ are discussed in the respective Sections 3.2.1 and 3.2.2. Other results in Section 3.2.3 deals with the $(5, 5, r)$-, $(5, 7, r)$- and $(7, 7, 7)$-generations for the group $Sp(6, 2)$. In Section 3.3, we find the ranks of all the conjugacy classes except the identity element in $Sp(6, 2)$. The proofs of the results for the ranks of each non-identity conjugacy class of $Sp(6, 2)$ in this section are done in Propositions 3.3.1 to 3.3.14.

In Chapter 4, we discuss one of the 26 sporadic simple group called the Mathieu group M_{23}. This chapter is divided into three sections. Section 4.1 gives information about the sporadic simple group called the Mathieu group M_{23}. We discuss the (p, q, r)-generations of the Mathieu group M_{23} in Section4.2. This section is divided into three subsection, namely, 4.2.1, 4.2.2 and 4.2.3. In Sections 4.2.1 and 4.2.2, we discuss the $(2, q, r)$- and $(3, q, r)$-generations of the group M_{23}, respectively. Section 4.2.3 named "Other results" investigates the $(5, q, r)$-, $(7, q, r)$-, $(11, q, r)$- and $(23, q, r)$-generations of the group M_{23}. In Section 4.3, we find the ranks of all the conjugacy classes except the identity element of M_{23}. The values of the ranks in this section are proved in the Propositions 4.3.1 to 4.3.3.

In Chapter 5, we discuss the alternating group A_{11}. This chapter is divided into three sections, namely, 5.1, 5.2 and 5.3. The basic information about the alternating group A_{11} is given in Section 5.1. The (p, q, r)-generations of the alternating group A_{11} is discussed in Section 5.2 divided into three subsections, namely, 5.2.1, 5.2.2 and 5.2.3. The $(2, q, r)$- and $(3, q, r)$-generations of the alternating group A_{11} are discussed in the respective Sections 5.2.1 and 5.2.2. Other results in Section 5.2.3 refers to the investigations on triple generations of $(5, q, r)$, $(7, q, r)$ and $(11, q, r)$.

The Appendix A is divided into three sections. In Section A.1, we put tables for relevant structure constants for the symplectic simple group $Sp(6, 2)$ are found in Tables A.1 to A.8. In Section A.2, we listed the tables for relevant structure constants for the Mathieu group M_{23} in Tables A.9 to A.12. Finally, Section A.3 provides tables for the relevant structure constants of the alternating group A_{11} in Tables A.13 to A.16.

It worth mentioned that an article titled "The (p, q, r)-generations of the alternating group A_{11}" has been published online (article in press) by Khayyam journal of mathematics. The aticle titled "The (p, q, r)-generations of the symplectic group $Sp(6, 2)$" has been accepted by Algebraic structures and their applications. We would like to mention that the (p, q, r)-generations of the Mathieu group M_{23} have been submitted for publication and its status is under review. Ones more the conjugacy classes ranks of the following groups, namely, symplectic group $Sp(6, 2)$, Mathieu group M_{23} and the alternating group A_{11} have both been submitted for publications and all their status are also under review.

CHAPTER 2

Preliminaries

2.1. The structure constant method

Let G be a finite group and $C_1, C_2, \cdots, C_k$ for $k \geq 3$ (not necessarily distinct) be conjugacy classes of G with $g_1, g_2, \cdots, g_k$ being representatives for these classes respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G through the formula

$$\Delta_G(C_1, C_2, \cdots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \tag{2.1}$$

Also for a fixed $g_k \in C_k$ we denote by $\Delta^*_G(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ satisfying

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \cdots, g_{k-1} \rangle. \tag{2.2}$$

Definition 2.1.1. *If $\Delta^*_G(C_1, C_2, \cdots, C_k) > 0$, the group G is said to be $(C_1, C_2, \cdots, C_k)$*-

generated.

Furthermore, if H is any subgroup of G containing a fixed element $h_k \in C_k$, we let $\Sigma_H(C_1, C_2, \cdots, C_k)$ be the total number of distinct tuples $(h_1, h_2, \cdots, h_{k-1})$ such that

$$h_1 h_2 \cdots h_{k-1} = h_k \quad \text{and} \quad \langle h_1, h_2, \cdots, h_{k-1}\rangle \leq H. \tag{2.3}$$

The value of $\Sigma_H(C_1, C_2, \cdots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of H-conjugacy classes $c_1, c_2, \cdots, c_k$ such that $c_i \subseteq H \cap C_i$.

Theorem 2.1.1. *Let G be a finite group and H be a subgroup of G containing a fixed element g such that $\gcd(o(g), [N_G(H){:}H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \cdots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See Ganief and Moori [27, 29, 32]. □

By [15], the above number $h(g, H)$ is useful in giving a lower bound for $\Delta^*_G(C_1, C_2, \cdots, C_k)$, namely $\Delta^*_G(C_1, C_2, \cdots, C_k) \geq \Theta_G(C_1, C_2, \cdots, C_k)$, where

$$\Theta_G(C_1, C_2, \cdots, C_k) = \Delta_G(C_1, C_2, \cdots, C_k) - \sum h(g_k, H)\Sigma_H(C_1, C_2, \cdots, C_k), \tag{2.4}$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1, C_2, \cdots, C_k$. Since we have all the maximal subgroups of the sporadic simple groups except for $G = \mathbb{M}$ the Monster group, it is possible to build a small subroutine in GAP [26] to compute the values of $\Theta_G = \Theta_G(C_1, C_2, \cdots, C_k)$ for any collection of conjugacy classes and any finite simple group.

Remark 2.1.1. It can be easily noted that the upper bound of $\Delta_G^*(C_1, C_2, \cdots, C_k)$ is $\Delta_G(C_1, C_2, \cdots, C_k)$. Preciously, the value of $\Delta_G^*(C_1, C_2, \cdots, C_k)$ will range between 0 and the value of $\Delta_G(C_1, C_2, \cdots, C_k)$.

Theorem 2.1.2 ([32])**.** *Let G be a finite group and let l, m and n be integers that are pairwise co-prime. Then for any integer t co-prime to n, we have*

$$\Delta(lX, mY, nZ) = \Delta(lX, mY, (nZ)^t).$$

Remark 2.1.2. The above Theorem 2.1.2 is saying that a group G is (lX, mY, nZ)-generated if and only if G is $(lX, mY, (nZ)^t)$-generated.

We see that $(7A)^{-1} = 7B$, $(11A)^{-1} = 11B$ and $(23A)^{-1} = 23B$ in M_{23}. As an application of the above theorem, the group M_{23} is $(p, q, 7A)$-generated if and only if it is $(p, q, 7B)$-generated, is $(p, q, 11A)$-generated if and only if it is $(p, q, 11B)$-generated and it is also $(p, q, 23A)$-generated if and only if it is $(p, q, 23B)$-generated. Therefore, it is sufficient to check the $(p, q, 7A)$-, $(p, q, 11A)$- and $(p, q, 23A)$-generations of M_{23}.

We see that $(11A)^{-1} = 11B$ in A_{11}. As an application of the above theorem, the group A_{11} is $(p, q, 11A)$-generated if and only if it is $(p, q, 11B)$-generated. Therefore, it is sufficient to consider only the $(p, q, 11A)$-generations of A_{11}.

Lemma 2.1.3, Theorems 2.1.4 and 2.1.5 are in some cases useful in establishing non-generation of finite groups.

Lemma 2.1.3. *Let G be a finite centerless group. If $\Delta_G^*(C_1, C_2, \cdots, C_k) < |C_G(g_k)|$, $g_k \in C_k$, then $\Delta_G^*(C_1, C_2, \cdots, C_k) = 0$ and therefore G is not $(C_1, C_2, \cdots, C_k)$-generated.*

Proof. See [14]. $\square$

Theorem 2.1.4 (Ree [49]). *Let G be a transitive permutation group generated by permutations $g_1, g_2, \cdots, g_s$ acting on a set of n elements such that $g_1 g_2 \cdots g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \leq i \leq s$, then $\sum_{i=1}^{s} c_i \leq (s-2)n + 2$.*

Theorem 2.1.5 (Scott [50]). *Let $g_1, g_2, \cdots, g_s$ be elements generating a group G with $g_1 g_2 \cdots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with $\dim \mathbb{V} = n \geq 2$. Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i \rangle$ on $\mathbb{V}$ and let d_i be the co-dimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i \rangle}$ being the trivial character of the cyclic group $\langle g_i \rangle$, the co-dimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([25]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi\downarrow^{G}_{\langle g_i \rangle}, \mathbf{1}_{\langle g_i \rangle} \right\rangle \\ &= \chi(1_G) - \frac{1}{|\langle g_i \rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \end{aligned} \tag{2.5}$$

2.2. Ranks

Lemma 2.2.1, Theorem 2.2.3, Corollaries 2.2.2 and 2.2.4 can be used to determine the ranks (G, nX) of the finite group G.

Lemma 2.2.1 ([9]). *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $((\underbrace{lX, lX, \ldots, lX}_{m-times}), (nZ)^m)$-generated.*

Corollary 2.2.2 ([9]). *Let G be a finite simple group such that G is $(lX, mYnZ)$-generated. Then $rank(G : lX) \leq m$.*

Proof. The result follows immediately from Lemma 2.2.1. $\square$

Theorem 2.2.3 ([32])**.** *Let G be a $(2X, sY, tZ)$-generated simple group, then G is $(sY, sY, (tZ)^2)$-generated.*

Corollary 2.2.4. *Let G be a finite simple group such that G is $(2X, mY, nZ)$-generated. Then $rank(G : mY) = 2$.*

Proof. If G is (lX, mY, nZ)-generated so by Lemma 2.2.1 we obtained that G is $(mY, mY, (nZ)^m)$-generated. Hence the result follows. □

CHAPTER 3

The Symplectic group $Sp(6, 2)$

In this chapter, we are interested in two kinds of generations of the group $Sp(6, 2)$, namely, the (p, q, r)-generations and the *ranks* of the conjugacy classes of $Sp(6, 2)$. For (p, q, r)-generations, we prove Theorem 3.0.1 in Section 3.2 and its subsections. We also prove Theorem 3.0.2 in Section 3.3 and its subsections.

Theorem 3.0.1. *With the notation being as in the Atlas [20], the symplectic group* $Sp(6, 2) \cong S_6(2)$ *is generated by the triples* (lX, mY, nZ), l, m *and* n *primes dividing* $|Sp(6, 2)|$, *except for the cases* $(lX, mY, nZ) \in \{(2M, 3N, 7A), (2M, 5A, 5A), (2N, 5A, 7A), (3N, 3N, 5A), (3M, 3N, 7A), (3O, 5A, 5A), (3A, 5A, 7A)\}$, *for all* $M \in \{A, B, C, D\}$, $O \in \{A, B\}$ *and* $N \in \{A, B, C\}$.

Theorem 3.0.2. *Let* G *be the symplectic group* $Sp(6, 2)$. *Then*

1. $rank(G : 2A) = 7$,
2. $rank(G : 2X) = rank(G : 3A) = 3$ *for* $X \in \{B, C\}$,
3. $rank(G : 2D) = rank(G : 3B) = rank(G : 4A) = rank(G : 4B) = rank(G : 6A) = 4$,
4. $rank(G : nX) = 2$ *for all* $nX \notin \{1A, 2A, 2B, 2C, 2D, 3A, 3B, 4A, 4B, 4C, 6A\}$.

3.1. Introduction

There are six classical simple groups, namely, linear, unitary and symplectic groups, and three families of orthogonal groups. The classical groups are defined in terms of groups of matrices over fields. Let $\mathbb{F}$ be a field. Then the general linear group $GL(n,\mathbb{F})$ is the group of invertible $n\times n$ matrices with entries in $\mathbb{F}$ under matrix multiplication. A linear group $L(n,\mathbb{F})$ is a closed subgroup of $GL(n,\mathbb{F})$.

The group $Sp(6,2)$ is a group of 6×6 symplectic matrices with entries 0 and 1, and with the matrix multiplication as the operation. Since all symplectic matrices have determinant 1, the symplectic group $Sp(6,2)$ is a subgroup of the special linear group $SL(6,2)$. The symplectic group $Sp(6,2)$ has order $1451520 = 2^9\times 3^4\times 5\times 7$. By the Atlas [20] the group $Sp(6,2)$ has exactly 30 conjugacy classes of its elements and 8 conjugacy classes of its maximal subgroups. Representatives of conjugacy classes of the maximal subgroups can be taken as follows:

$$\begin{array}{lll} H_1 = U_4(2){:}2 & H_2 = S_8 & H_3 = 2^5{:}S_6 \\ H_4 = U_3(3){:}2 & H_5 = 2^6{:}L_3(2) & H_6 = (2^2\times 2^{1+4}){:}(S_3\times S_3) \\ H_7 = S_3\times S_6 & H_8 = L_2(8){:}3. & \end{array}$$

In this chapter, we will use G instead of $Sp(6,2)$, unless stated otherwise. For the sake of computations with Gap [26], we use a permutation presentation for G. By the electronic Atlas of Wilson [55], G can be generated in terms of permutations on 28 points. Generators g_1 and g_2 can be taken as follows:

$$\begin{aligned} g_1 &= (2,3)(6,7)(9,10)(12,14)(17,19)(20,22), \\ g_2 &= (1,2,3,4,5,6,8)(7,9,11,13,16,18,14)(10,12,15,17,20,19,21)(22,23,24,25, \\ &\quad 26,27,28), \end{aligned}$$

with $o(g_1)=2,\ o(g_2)=7$ and $o(g_1g_2)=9$.

Table 3.1 gives all the values of $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$ for nX classes of prime order for the G with $\dim(\mathbb{V}) = 7$. This table will be referred to when we are proving non-generation of a triple for the group G. In Table 3.3, we list the values of the cyclic structure for each conjugacy of G which containing elements of prime order together with the values of both c_i and d_i obtained from Ree and Scotts theorems, respectively.

In Table 3.4 we list the representatives of classes of the maximal subgroups together with the orbits lengths of $Sp(6, 2)$ on these groups and the permutation characters.

Table 3.5 gives us the partial fusion maps of classes of maximal subgroups into the classes of $Sp(6, 2)$. These will be used in our computations.

Table 3.1: $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, nX is a non-trivial class of G and $\dim(\mathbb{V}) = 7$

nX	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$4E$	$5A$	$6A$	$6B$
d_{nX}	6	4	2	4	2	6	4	4	4	6	6	4	4	6	4
nX	$6C$	$6D$	$6E$	$6F$	$6G$	$7A$	$8A$	$8B$	$9A$	$10A$	$12A$	$12B$	$12C$	$15A$	
d_{nX}	6	4	6	6	6	6	6	6	6	6	6	6	6	6	

Table 3.2: $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, nX is a non-trivial class of $Sp(6, 2)$ and $\dim(\mathbb{V}) = 15$.

nX	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$4E$	$5A$	$6A$	$6B$	$6C$	$6D$
d_{nX}	10	4	6	8	10	12	8	10	12	10	8	10	12	14	12	12	12
nX	$6E$	$6F$	$6G$	$7A$	$8A$	$8B$	$9A$	$10A$	$12A$	$12B$	$12C$	$15A$					
d_{nX}	12	10	12	12	12	12	14	14	14	14	14	14					

Table 3.3: Cycle structures of conjugacy classes of G

nX	Cycle Structure	c_i	d_i
$1A$	1^{28}	28	0
$2A$	$1^{16}2^{6}$	22	6
$2B$	$1^{4}2^{12}$	16	12
$2C$	$1^{8}2^{10}$	18	10
$2D$	$1^{4}2^{12}$	16	12
$3A$	$1^{10}3^{6}$	16	12
$3B$	$1\ 3^{9}$	10	18
$3C$	$1\ 3^{9}$	10	18
$4A$	$1^{4}4^{6}$	10	18
$4B$	$1^{2}2^{3}4^{5}$	10	18
$4C$	$1^{6}2\ 4^{5}$	12	16
$4D$	$2^{4}4^{5}$	9	19
$4E$	$1^{2}2^{3}4^{5}$	10	18
$5A$	$1^{3}5^{5}$	8	20
$6A$	$1^{4}2^{3}3^{4}6$	12	16
$6B$	$1^{4}2^{3}6^{3}$	10	18
$6C$	$1\ 3\ 6^{4}$	6	22
$6D$	$1^{2}2^{4}3^{2}6^{3}$	11	17
$6E$	$1\ 3^{5}6^{2}$	8	20
$6F$	$1\ 3\ 6^{4}$	6	22
$6G$	$1\ 3\ 6^{4}$	6	22
$7A$	7^{4}	4	24
$8A$	$1^{2}2\ 8^{3}$	6	22
$8B$	$4\ 8^{3}$	4	24
$9A$	$1\ 9^{3}$	4	24
$10A$	$1\ 2\ 5^{5}$	7	21
$12A$	$1^{2}4^{2}6\ 12$	6	22
$12B$	$2\ 3^{2}4^{2}12$	6	22
$12C$	$1\ 3\ 12^{2}$	4	24
$15A$	$3\ 5^{3}$	4	24

Table 3.4: Maximal subgroups of $Sp(6,2)$

Maximal Subgroup	Order	Orbit Lengths	Character
$U_4(2){:}2$	$2^7 \cdot 3^4 \cdot 5$	[1,27]	$1a + 27a$
S_8	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	[28]	$1a + 35b$
$2^5{:}S_6$	$2^9 \cdot 3^2 \cdot 5$	[12,16]	$1a + 27a + 35b$
$U_3(3){:}2$	$2^6 \cdot 3^3 \cdot 7$	[28]	$1a + 35a + 84a$
$2^6{:}L_3(2)$	$2^9 \cdot 3 \cdot 7$	[28]	$1a + 15a + 35b + 84a$

Table 3.4 continued

Maximal Subgroup	Order	Orbit Lengths	Character
$(2\cdot 2^6){:}(S_3\times S_3)$	$2^9\cdot 3^2$	[4,24]	$1a+27a+35b+84a+168a$
$S_3\times S_6$	$2^5\cdot 3^3\cdot 5$	[10,18]	$1a+27a+35b+105b+168a$
$L_2(8){:}3$	$2^3\cdot 3^3\cdot 7$	[28]	$1a+70a+84a+105b+280a+420a$

Table 3.5: The partial fusion maps into $Sp(6,2)$

$U_4(2){:}2$-class	2a	2b	2c	2d	3a	3b	3c	5a								
$\rightarrow$ G	2A	2B	2C	2D	3B	3A	3C	5A								
h								3								
S_8-class	2a	2b	2c	2d	3a	3b	5a	7a								
$\rightarrow$ $Sp(6,2)$	2A	2B	2C	2D	3A	3C	5A	7A								
h							1	1								
$2^5{:}S_6$-class	2a	2b	2c	2d	2e	2f	2g	2h	2i	2j	3a	3b	5a			
$\rightarrow$ $Sp(6,2)$	2A	2C	2B	2C	2A	2B	2D	2D	2C	2D	3A	3C	5A			
h													3			
$U_3(3){:}2$-class	2a	2b	3a	3b	7a											
$\rightarrow$ $Sp(6,2)$	2B	2D	3B	3C	7A											
h					1											
$2^6{:}L_3(2)$-class	2a	2b	2c	2d	2e	2f	2g	3a	7a	7b						
$\rightarrow$ $Sp(6,2)$	2A	2B	2C	2D	2C	2B	2D	3C	7A	7A						
h									1	1						
$2\cdot 2^6{:}(S_3\times S_3)$-class	2a	2b	2c	2d	2e	2f	2g	2h	2i	2j	2k	2l	2m	3a	3b	3c
$\rightarrow$ $Sp(6,2)$	2B	2C	2A	2C	2B	2A	2D	2D	2C	2B	2D	2C	2D	3A	3B	3C
h														15	9	3
$S_3\times S_6$-class	2a	2b	2c	2d	2e	2f	2g	3a	3b	3c	3d	3e	5a			
$\rightarrow$ $Sp(6,2)$	2A	2A	2B	2C	2C	2D	2D	3A	3A	3C	3C	3B	5A			
h													1			
$L_2(8){:}3$-class	2a	3a	3b	3c	7a											
$\rightarrow$ $Sp(6,2)$	2D	3B	3C	3C	7A											
h					1											

3.2. The (p,q,r)-generations of $Sp(6,2)$

Let $tX,\ t\in\{2,3,5,7\}$ be a conjugacy class of G and c_i be the number of disjoint cycles in a representative of pX. The group G is not $(2Y,2Z,pX)$-generated, for if G is $(2Y,2Z,pX)$-generated, then G is a dihedral group and thus is not simple for all $Y,Z\in\{A,B\}$. Also we know that if G is (lX,mY,nZ)-generated with $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}\geq 1$ and G is simple, then $G\cong A_5$, but $G \cong Sp(6,2)$ and $Sp(6,2) \ncong A_5$. Hence if G is (lX, mY, nZ)-generated, then we must have $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.

From the Atlas of finite group representations [55], we see that G is acting on 28 points, implies that $n = 28$ and since our generation is triangular, we have $s = 3$. Hence by Ree's Theorem [49] if G is (l, m, n)-generated, then $\sum c_i \leq 30$.

3.2.1 $(2, q, r)$-generations

Now the $(2, q, r)$-generations of G comprises the cases $(2, 3, r)$-, $(2, 5, r)$- and $(2, 7, r)$-generations.

$(2, 3, r)$-generations

The condition $\frac{1}{2} + \frac{1}{3} + \frac{1}{r} < 1$ shows that $r \geq 7$. Thus we have to consider the cases $(2X, 3Y, 7A)$ for $X \in \{A, B, C, D\}$ and $Y \in \{A, B, C\}$.

Proposition 3.2.1. *The group G is*

(i) not $(2X, 3Y, 7A)$, $(2Z, 3C, 7A)$-generated for all $X \in \{A, B, C, D\}$, $Y \in \{A, B\}$ and $Z \in \{A, B, C\}$,

(ii) $(2D, 3C, 7A)$-generated.

Proof. (i) By [40, Theorem 2], G is a Hurwitz group, we have to consider the triples $(2X, 3Y, 7A)$ for $X \in \{A, B, C, D\}$ and $Y \in \{A, B, C\}$. If G is a $(2A, 3A, 7A)$-generated group, then we must have $c_{2A} + c_{3A} + c_{7A} \leq 30$. From Table 3.3 we see that $c_{2A} + c_{3A} + c_{7A} = 22 + 16 + 4 = 42 > 30$ and by Ree's Theorem [49], it follows that G is not $(2A, 3A, 7A)$-generated. Similarly, by applying Ree's Theorem, the group G is not generated by these triples $(2A, 3B, 7A)$, $(2A, 3C, 7A)$, $(2B, 3A, 7A)$, $(2C, 3A, 7A)$, $(2C, 3B, 7A)$, $(2C, 3C, 7A)$ and $(2D, 3A, 7A)$. By Table A.2 we have

$\Delta_G(2B,3B,7A) = 0$ and when we apply Lemma 2.1.3, it follows that the group G is not $(2B,3B,7A)$-generated.

By Table 3.4 we see that only four maximal subgroups of G have each an element of order 7, namely, H_2, H_4, H_5 and H_8. The non-empty intersection with all the conjugacy classes for these four maximal subgroups does not contain elements of order 7.

We have $PSL_3(2)$ and 7:6 are the only groups having elements of order 7 after taking non-empty intersection with all the conjugacy classes for any three maximal subgroups of G.

We have $2^3{:}PSL_3(2)$, $PSL_3(2){:}2$, $PSL_3(2)$ (2-copies) and 7:6 (2-copies) are the only groups having elements of order 7 after taking non-empty intersection with all the conjugacy classes for any two maximal subgroups of G.

The groups $2^3{:}PSL_3(2)$ and $PSL_3(2)$ will not have any contribution because their elements of order 2 does not fuse to the class $2B$ of the group G. The groups H_2, H_4, H_8, $PSL_3(2){:}2$ and 7:6 will also not have any contributions here because their relevant structure constants are all zeros. We obtained that $\Sigma_{H_5}(2x,3a,7y) = \Delta_{H_5}(2d,3a,7a) + \Delta_{H_5}(2d,3a,7b) = \Delta_{H_5}(2g,3a,7a) + \Delta_{H_5}(2g,3a,7b) = 0+0+14+0 = 14$ and the value of h for each contributing group is 1. We then have $\Delta^*_G(2D,3B,7A) = \Delta_G(2D,3B,7A) - \Sigma_{H_5}(2x,3c,7y) = 14-14 = 0$, proving that G is not $(2D,3B,7A)$-generated. By Table 3.2, the group G acts on a 15-dimensional irreducible complex module $\mathbb{V}$ and we have $d_{2B} + d_{3C} + d_{nX} = 4+8+d_{nX} < 2\times 15$ for all nX where nX is any conjugacy class dividing $|Sp(6,2)|$. By applying Scott's Theorem [50], we conclude that G is not $(2B,3C,nX)$-generated.

(ii) By Table A.4 we have $\Delta_G(2D,3C,7A) = 28$. We then have $\Delta^*_G(2D,3C,7A) \geq \Delta_G(2D,3C,7A) - \Sigma_{H_5}(2x,3y,7z) = 28-14 = 14$, proving that G is $(2D,3C,7A)$-generated. □

$(2,5,r)$**-generations**

The condition $\frac{1}{2}+\frac{1}{5}+\frac{1}{r}<1$ shows that $r>\frac{10}{3}$. Thus we have to consider the cases $(2X,5A,5A)$ and $(2X,5A,7A)$, for $X\in\{A,B,C,D\}$.

Proposition 3.2.2. *The group G is not a $(2X,5A,5A)$-generated group for all $X\in\{A,B,C,D\}$.*

Proof. If G is a $(2X,5A,5A)$-generated group, then we must have $c_{2X}+c_{5A}+c_{5A}\leq 30$. Since by Table 3.3 we have $c_{2X}\in\{16,18,22\}$, it follows that $c_{2X}+c_{5A}+c_{5A}=c_{2X}+8+8>30$ for any $X\in\{A,B,C,D\}$ and by Ree's Theorem [49] we conclude that G is not $(2X,5A,5A)$-generated group, for all $X\in\{A,B,C,D\}$. □

Proposition 3.2.3. *The group G is*

(i) not $(2X,5A,7A)$-generated for $X\in\{A,B\}$,

(ii) $(2Y,5A,7A)$-generated for $Y\in\{C,D\}$.

Proof. (i) By Table A.1 we see that $\Delta_G(2A,5A,7A)=0$, it follows that G is not $(2A,5A,7A)$-generated.

We prove that G is not $(2B,5A,7A)$-generated. Look at Proposition 3.2.1, we see that the groups $2^3{:}PSL_3(2)$, $PSL_3(2){:}2$, $PSL_3(2)$ and 7:6 have elements of order 7 and none will be considered here since none of these groups have elements of order 5. By Table 3.4, H_2 is the only maximal subgroup containing elements of orders 2, 5 and 7. We have $\Sigma_{H_2}(2b,5a,7a)=7$ and $h(7A,H_2)=1$. Since by Table A.2 we have $\Delta_G(2B,5A,7A)=7$, it follows that $\Delta_G^*(2B,5A,7A)=\Delta_G(2B,5A,7A)-\sum_{H_2}(2b,5a,7a)=7-7=0$, proving that G is not $(2B,5A,7A)$-generated.

(ii) As stated ealier, only H_2 will have a contribution because it contains elements of orders 2, 5 and 7. By GAP, we have $\Sigma_{H_2}(2c,5a,7a)=7$ and by Table A.3, we have $\Delta_G(2C,5A,7A)=14$ so that $\Delta^*_G(2C,5A,7A)\geq\Delta_G(2C,5A,7A)-\Sigma_{H_2}(2c,5a,7a)=14-7=7>0$, proving that G is $(2C,5A,7A)$-generated.

By Table A.4 we have $\Delta_G(2D,5A,7A)=98$. Although H_2 is the only maximal subgroup meeting the $2D$, $5A$, $7A$ classes of G, it will not have any contribution since its relevant structure constant is zero. We then obtained that $\Delta^*_G(2D,5A,7A)=\Delta_G(2D,5A,7A)=98>0$, proving that G is $(2D,5A,7A)$-generated. □

$(2,7,r)$-generations

We have to check the generation of G through the triples $(2A,7A,7A)$, $(2B,7A,7A)$, $(2C,7A,7A)$ and $(2D,7A,7A)$.

Proposition 3.2.4. *The group G is $(2X,7A,7A)$-generated for $X\in\{A,B,C,D\}$.*

Proof. As in Proposition 3.2.1, subgroups H_2, H_4, H_5, H_8, $2^3{:}PSL_3(2)$, $PSL_3(2){:}2$, $PSL_3(2)$ and 7:6 are the only ones having elements of order 7.

By Table A.1 we have $\Delta_G(2A,7A,7A)=14$. Out of all the subgroups having elements of order 7, only $2^3{:}PSL_3(2)$, H_2 and H_5 meet the $2A$, $7A$ classes of G. The maximal subgroup H_2 will not have any contribution here since its relevant structure constant is zero. We obtained that $\Sigma_{2^3:PSL_3(2)}(2x,7y,7z)=\Delta_{2^3:PSL_3(2)}(2a,7a,7a)+\Delta_{2^3:PSL_3(2)}(2a,7a,7b)+\Delta_{2^3:PSL_3(2)}(2a,7b,7b)+\Delta_{2^3:PSL_3(2)}(2c,7a,7a)+\Delta_{2^3:PSL_3(2)}(2c,7a,7b)+\Delta_{2^3:PSL_3(2)}(2c,7b,7b)=7+0+7+0+14+0=28$ and $\Sigma_{H_5}(2a,7x,7y)=\Delta_{H_5}(2a,7a,7a)+\Delta_{H_5}(2a,7a,7b)+\Delta_{H_5}(2a,7b,7b)=7+0+7=14$. Since the value of h for each of these contributing subgroups is 1, we then obtain that $\Delta^*_G(2A,7A,7A)\geq\Delta_G(2A,7A,7A)-\Sigma_{H_5}(2a,7x,7x)+$

$\Sigma_{2^3:PSL_3(2)}(2x,7y,7z) = 14-14+28 = 28 > 0$, proving that G is $(2A,7A,7A)$-generated.

By Table A.2 we have $\Delta_G(2B,7A,7A) = 70$. Out of all the subgroups having elements of order 7, only 7:6 does not meet the $2B$, $7A$ classes of G. We obtained that $\Sigma_{2^3PSL_3(2)}(2x,7y,7z) = \Delta_{2^3PSL_3(2)}(2a,7a,7a)+\Delta_{2^3PSL_3(2)}(2a,7a,7b)+\Delta_{2^3PSL_3(2)}(2a,7b,7b)+\Delta_{2^3PSL_3(2)}(2b,7a,7a)+\Delta_{2^3PSL_3(2)}(2b,7a,7b)+\Delta_{2^3PSL_3(2)}(2b,7b,7b) = 7+0+7+0+14+0 = 28$, $\Sigma_{PSL_3(2):2}(2b,7a,7a) = 7$, $\Sigma_{PSL_3(2)}(2a,7x,7y) = \Delta_{PSL_3(2)}(2a,7a,7a)+\Delta_{PSL_3(2)}(2a,7a,7b)+\Delta_{PSL_3(2)}(2a,7b,7b) = 0+7+0 = 7$, $\Sigma_{H_2}(2b,7a,7a) = 35$, $\Sigma_{H_4}(2a,7a,7a) = 14$ and $\Sigma_{H_5}(2x,7y,7z) = \Delta_{H_5}(2b,7a,7a)+\Delta_{H_5}(2b,7a,7b)+\Delta_{H_5}(2b,7b,7b)+\Delta_{H_5}(2f,7a,7a)+\Delta_{H_5}(2f,7a,7b)+\Delta_{H_5}(2f,7b,7b) = 7+0+7+0+28+0 = 42$. Since the value of h for each of these contributing subgroups is 1, we then obtain that $\Delta_G^*(2B,7A,7A) \geq \Delta_G(2B,7A,7A)-\Sigma_{H_2}(2b,7a,7a)-\Sigma_{H_4}(2a,7a,7a)-\Sigma_{H_5}(2x,7y,7z)+\Sigma_{2^3PSL_3(2)}(2x,7y,7z)+\Sigma_{PSL_3(2):2}(2b,7a,7a)+2\times\Sigma_{PSL_3(2)}(2a,7x,7y)-\Sigma_{PSL_3(2)}(2a,7x,7y) = 70-35-14-42+28+7+2(7)-7 = 21 > 0$, proving that G is $(2B,7A,7A)$-generated.

By Table A.3 we have $\Delta_G(2C,7A,7A) = 210$. Of all the subgroups of G having elements of order 7, only H_2, H_5 and $2^3{:}PSL_3(2)$ meet the $2C$, $7A$ classes of G. We obtained that $\Sigma_{H_2}(2c,7a,7a) = 70$, $\sum_{H_5}(2x,7y,7z) = \Delta_{H_5}(2c,7a,7a)+\Delta_{H_5}(2c,7a,7b)+\Delta_{H_5}(2c,7b,7b)+\Delta_{H_5}(2e,7a,7a)+\Delta_{H_5}(2e,7a,7b)+\Delta_{H_5}(2e,7b,7b) = 21+0+21+0+28+0 = 70$ and $\sum_{2^3:PSL_3(2)}(2b,7x,7y) = \Delta_{2^3:PSL_3(2)}(2b,7a,7a)+\Delta_{2^3:PSL_3(2)}(2b,7a,7b)+\Delta_{2^3:PSL_3(2)}(2b,7b,7b) = 0+14+0 = 14$. The value of h for all contributing subgroups is 1. We then get that $\Delta_G^*(2C,7A,7A) \geq \Delta_G(2C,7A,7A)-\sum_{H_2}(2c,7a,7a)-\sum_{H_5}(2x,7y,7z)+\sum_{2^3:PSL_3(2)}(2b,7x,7y) - \sum_{2^3:PSL_3(2)}(2b,7x,7y) = 210-70-70+14-14 = 70 > 0$, proving that G is $(2C,7A,7A)$-generated.

By Table A.4 we have $\Delta_G(2D,7A,7A) = 560$. The groups H_2, H_4, $PSL_3(2){:}2$ and 7:6 have their relevant structure constant all zero. The elements of order 2 of these groups $2^3{:}PSL_3(2)$ and $PSL_3(2)$ do not fuse to the class $2D$ of G. Thus, only the maximal subgroups H_5 and H_8 have contributions here. We obtained that $\sum_{H_5}(2x,7y,7z) = \Delta_{H_5}(2d,7a,7a) + \Delta_{H_5}(2d,7a,7b) + \Delta_{H_5}(2d,7b,7b) + \Delta_{H_5}(2g,7a,7a) + \Delta_{H_5}(2g,7a,7b) + \Delta_{H_5}(2g,7b,7b) = 28+0+28+0+56+0 = 112$ and $\sum_{H_8}(2a,7a,7a) = 28$. The value of h for all contributing subgroups is 1. We then get $\Delta^*_G(2D,7A,7A) \geq \Delta_G(2D,7A,7A) - \sum_{H_5}(2x,7y,7z) - \sum_{H_8}(2a,7a,7a) = 560 - 112 - 28 = 420 > 0$, proving that G is $(2D,7A,7A)$-generated. □

3.2.2 $(3,q,r)$-generations

The condition $\frac{1}{3}+\frac{1}{3}+\frac{1}{r} < 1$ shows that $r > 3$. For the $(3,q,r)$-generations, we end up having the following cases: $(3X,3Y,5A)$-, $(3X,3Y,7A)$-, $(3X,5A,5A)$-, $(3X,5A,7A)$- and $(3X,7A,7A)$-generations.

$(3,3,r)$-generations

Proposition 3.2.5. *The group G is not $(3X,3Y,5A)$-generated group for all $X,Y \in \{A,B,C\}$.*

Proof. The group G acts on a 7-dimensional irreducible complex module $\mathbb{V}$. By Scott's Theorem [50] applied to the module $\mathbb{V}$ and using the Atlas of finite groups [20], we see that $d_{3A} = \dim(\mathbb{V}/C_{\mathbb{V}}(3A)) = \frac{2(7-4)}{3} = 2$, $d_{3B} = \dim(\mathbb{V}/C_{\mathbb{V}}(3B)) = \frac{2(7+2)}{3} = 6$, $d_{3C} = \dim(\mathbb{V}/C_{\mathbb{V}}(3C)) = \frac{2(7-1)}{3} = 4$ and $d_{5A} = \dim(\mathbb{V}/C_{\mathbb{V}}(5A)) = \frac{4(7-2)}{5} = 4$. Since $d_{3X} \in \{2,4,6\}$ above, it follows that $d_{3A} + d_{3X} + d_{5A} < 14$ and by Scott's Theorem G is not $(3A,3X,5A)$-generated for all $X \in \{A,B,C\}$. Again by Scott's Theorem, G is not $(3C,3C,5A)$-generated because $d_{3C}+d_{3C}+d_{5A} = 12 < 14$. By Table A.6 we see that $\Delta_G(3B,3B,5A) = \Delta_G(3B,3C,5A) = 10 < 30 = |C_G(5A)|$. Using Lemma 2.1.3, the group G is not $(3B,3X,5A)$-generated for $X \in \{B,C\}$. □

Proposition 3.2.6. *The group G is*

(i) not $(3X,3Y,7A)$-generated for $X \in \{A,B\}$ and $Y \in \{A,B,C\}$,

(ii) $(3C,3C,7A)$-generated.

Proof. (i) By Table A.5 we have $\Delta_G(3A,3A,7A) = \Delta_G(3A,3B,7A) = 0$, it follows that G is not $(3A,3X,7A)$-generated for all $X \in \{A,B\}$. As in Proposition 3.2.1, subgroups H_2, H_4, H_5, H_8, $2^3{:}PSL_3(2)$, $PSL_3(2){:}2$, $PSL_3(2)$ and 7:6 are the only ones having elements of order 7.

By the same Table A.5 we have $\Delta_G(3A,3C,7A) = 7$. The maximal subgroup H_2 is the only one meeting the classes $3A$, $3C$ and $7A$ of G. We obtained that $\sum_{H_2}(3a,3b,7a) = 7$ and we have $h(7A,H_2) = 1$. We obtain $\Delta^*_G(3A,3C,7A) = \Delta_G(3A,3C,7A) - \sum_{H_2}(3a,3b,7a) = 7-7 = 0$, proving that the group G is not $(3A,3C,7A)$-generated.

By Table A.6 we have $\Delta_G(3B,3B,7A) = 7$. Although the maximal subgroups H_4 and H_8 are the only ones meeting the $3B$, $7A$ classes of G, the maximal subgroup H_4 will not contribute because its relevant structure constant is zero. We obtained that $\sum_{H_8}(3a,3a,7a) = 7$ and we have $h(7A,H_8) = 1$. Thus we obtain $\Delta^*_G(3B,3B,7A) = \Delta_G(3B,3B,7A) - \sum_{H_8}(3a,3a,7a) = 7-7 = 0$, proving that G is not $(3B,3B,7A)$-generated.

By Table A.6 we have $\Delta_G(3B,3C,7A) = 7$. Although H_4 and H_8 meet the $3B$, $3C$, $7A$ classes of G, only H_4 has a contribution because the relevant structure constant of H_8 is zero. We obtained that $\sum_{H_4}(3a,3b,7a) = 7$. Thus we obtain $\Delta^*_G(3B,3C,7A) = \Delta_G(3B,3C,7A) - \sum_{H_4}(3a,3b,7a) = 7-7 = 0$, proving that G is not $(3B,3C,7A)$-generated.

(ii) By Proposition 3.2.1, G is $(2B,3C,7A)$-generated. It follows by Theorem 2.2.3 that G is $(3C,3C,(7A)^2)$-generated. Since G has one class of order 7, we must have $(7A)^2 = 7A$. The group G will become $(3C,3C,7A)$-generated. □

$(3, 5, r)$**-generations**

Proposition 3.2.7. *The group G is*

(i) not $(3X, 5A, 5A)$-generated for all $X \in \{A, B\}$,

(ii) $(3C, 5A, 5A)$-generated.

Proof. (i) If G is a $(3A, 5A, 5A)$-generated group, then we must have $c_{3A} + c_{5A} + c_{5A} \leq 30$. Since by Table 3.3 we have $c_{3A} + c_{5A} + c_{5A} = 16 + 8 + 8 = 32 > 30$ and by Ree's Theorem we conclude that G is not $(3A, 5A, 5A)$-generated group.

By Table 3.4 we see that only four maximal subgroups of G have each an element of order 5, namely, H_1, H_2, H_3 and H_7. The non-empty intersection with all the conjugacy classes for these four maximal subgroups does not contain elements of order 5. Also non-empty intersection with all the conjugacy classes for any three maximal subgroups of G does not have elements of order 5.

We have $2 \times S_6$ (4-copies), $2^4{:}S_5$, $S_3 \times S_5$ and $2 \times S_5$ are the only groups having elements of order 5 resulted when taking non-empty intersection with all the conjugacy classes for any two maximal subgroups of G.

We found that $h(5A, 2^4{:}S_5) = 6$, $h(5A, H_1) = h(5A, H_3) = h(5A, 2 \times S_6) = h(5A, 2 \times S_5) = 3$ and $h(5A, H_2)H_2 = h(5A, H_7) = h(5A, S_3 \times S_5) = 1$.

By Table A.6 we have $\Delta_G(3B, 5A, 5A) = 30$. Out of the above subgroups having elements of order 5, only H_1 and H_7 meet the $3B$, $5A$ classes of G. The maximal subgroup H_7 has no contributions because its structure constant is zero. We obtained that $\sum_{H_1}(3a, 5a, 5a) = 10$. We obtain $\Delta^*_G(3B, 5A, 5A) = \Delta_G(3B, 5A, 5A) - 3 \cdot \sum_{H_1}(3a, 5a, 5a) = 30 - 3(10) = 0$, proving that G is not $(3B,5A,5A)$-generated.

(ii) By Table A.7 we have $\Delta_G(3C,5A,5A) = 690$. Out of the above subgroups having elements of order 5, only H_1, H_2, H_3, H_7, $2 \times S_6$ and $S_3 \times S_5$ meet the $3C$, $5A$ classes of G. The relevant structure constant for the group $S_3 \times S_5$ is zero. We obtained that $\sum_{H_1}(3c,5a,5a) = 105$, $\sum_{H_2}(3b,5a,5a) = 135$, $\sum_{H_3}(3b,5a,5a) = 120$, $\sum_{H_7}(3c,5a,5a) = 15$ and $\sum_{2\times S_6}(3b,5a,5a) = 15$. We get $\Delta^*_G(3C,5A,5A) \geq \Delta_G(3C,5A,5A) - 3\cdot\sum_{H_1}(3c,5a,5a) - \sum_{H_2}(3b,5a,5a) - \sum_{H_7}(3c,5a,5a) + 4\times 3\cdot\sum_{2\times S_6}(3a,5b,5a) = 690 - 3(105) - 135 - 120 - 15 + 4(3)(15) = 285 > 0$, proving (ii). □

Proposition 3.2.8. *The group G is*

(i) not $(3A,5A,7A)$-generated

(ii) $(3X,5A,7A)$-generated, where $X \in \{B,C\}$.

Proof. (i) The groups H_2, H_4, H_5, H_8, $2^3{:}PSL_3(2)$, $PSL_3(2){:}2$, $PSL_3(2)$ and 7:6 have elements of order 7 and the group H_2 will contribute here because is the only one have elements of order 5. We obtained that $\sum_{H_2}(3a,5a,7a) = 7$ and we have $h(7A,H_2) = 1$. Since by Table A.5 we have $\Delta_G(3A,5A,7A) = 7$, we then obtain that $\Delta^*_G(3A,5A,7A) = \Delta_G(3A,5A,7A) - \sum_{H_2}(3a,5a,7a) = 7 - 7 = 0$. Hence, the group G is not $(3A,5A,7A)$-generated.

(ii) By Table A.6 we have $\Delta_G(3B,5A,7A) = 77$. Although the group H_2 is the only one having elements of order, it will not have any contributions because its elements of order 3 do not meet the class $3B$ of G. We obtained that $\Delta^*_G(3B,5A,7A) = \Delta_G(3B,5A,7A) = 77 > 0$, proving that G is $(3B,5A,7A)$-generated.

By Table A.7 we have $\Delta_G(3C,5A,7A) = 441$. Although H_2 is the only maximal subgroup meeting the classes $3C$, $5A$ and $7A$ of G so that $\sum_{H_2}(3b,5a,7a) = 77$. We obtain that $\Delta^*_G(3C,5A,7A) = \Delta_G(3C,5A,7A) - \sum_{H_2}(3b,5a,7a) = 441 - 77 = 368 > 0$. Thus, the group G is $(3C, 5A, 7A)$-generated. □

$(3,7,r)$-generations

In this subsection we discuss the cases $(3,7,r)$-generations. It follows that we will end up with 3 cases, namely $(3A, 7A, 7A)$-, $(3B, 7A, 7A)$- and $(3C, 7A, 7A)$-generation.

Proposition 3.2.9. *The group G is $(3X, 7A, 7A)$-generated for all $X \in \{A, B, C\}$.*

Proof. The subgroups H_2, H_4, H_5, H_8, $2^3{:}PSL_3(2)$, $PSL_3(2){:}2$, $PSL_3(2)$ and 7:6 are the only ones having elements of order 7. By Table A.5 we have $\Delta_G(3A, 7A, 7A) = 133$. Only H_2 has a contribution because it meets the $3A$, $7A$ classes of G. We obtained that $\sum_{H_2}(3a, 7a, 7a) = 42$ and $h(7A, H_2) = 1$. We then obtain that $\Delta^*_G(3A, 7A, 7A) \geq \Delta_G(3A, 7A, 7A) - \sum_{H_2}(3a, 7a, 7a) = 133 - 42 = 91 > 0$. This shows that the group G is $(3A, 7A, 7A)$-generated.

By Table A.6 we have $\Delta_G(3B, 7A, 7A) = 245$. Out of all the subgroups of G having elements of order 7, only H_4 and H_8 meet the $3B$, $7A$ classes of G. We obtained that $\sum_{H_4}(3a, 7a, 7a) = 7$ and $\sum_{H_8}(3a, 7a, 7a) = 21$. We then obtain $\Delta^*_G(3B, 7A, 7A) \geq \Delta_G(3B, 7A, 7A) - \sum_{H_4}(3a, 7a, 7a) - \sum_{H_8}(3a, 7a, 7a) = 245 - 7 - 21 = 217 > 0$, proving that G is $(3B, 7A, 7A)$-generated.

By Table A.7 we have $\Delta_G(3C, 7A, 7A) = 2289$. All these subgroups H_2, H_4, H_5, H_8, $2^3{:}PSL_3(2)$, $PSL_3(2){:}2$, $PSL_3(2)$ and 7:6 meet the $3C$, $7A$ classes of G. Although H_8 and 7:6 meet the $3C$, $7A$ classes of G, they will not have any contributions because their relevant structure constants are all zero. We obtained that $\sum_{H_2}(3b, 7a, 7a) = 294$, $\sum_{H_4}(3b, 7a, 7a) = 189$, $\sum_{H_5}(3a, 7x, 7y) = \Delta_{H_5}(3a, 7a, 7a) + \Delta_{H_5}(3a, 7a, 7b) + \Delta_{H_5}(3a, 7b, 7b) = 112 + 112 + 112 = 336$, $\sum_{2^3:PSL_3(2)}(3a, 7x, 7y) = \Delta_{2^3:PSL_3(2)}(3a, 7a, 7a) + \Delta_{2^3:PSL_3(2)}(3a, 7a, 7b) + \Delta_{2^3:PSL_3(2)}(3a, 7b, 7b)$ $= 28+28+28 = 84$, $\sum_{PSL_3(2):2}(3a, 7a, 7a) = 14$ and $\sum_{PSL_3(2)}(3a, 7x, 7y) = \Delta_{PSL_3(2)}(3a, 7a, 7a) +$

$\Delta_{PSL_3(2)}(3a,7a,7b) + \Delta_{PSL_3(2)}(3a,7b,7b) = 7+7+7 = 21$. The value of h for each contributing subgroups is 1. We then get $\Delta^*_G(3C,7A,7A) \geq \Delta_G(3C,7A,7A) - \sum_{H_2}(3b,7a,7a) - \sum_{H_4}(3b,7a,7a) - \sum_{H_5}(3a,7x,7y) + \sum_{2^3:PSL_3(2)}(3a,7x,7y) + \sum_{PSL_3(2):2}(3a,7a,7a) + 2\times\sum_{PSL_3(2)}(3a,7x,7y) - \sum_{PSL_3(2)}(3a,7x,7y) = 2289-294-189-112+84+14+2(21)-21 = 1813 > 0$. Hence G is $(3C,7A,7A)$-generated. □

3.2.3 Other results

In this subsection we handle all the remaining cases, namely the $(5,q,r)$-and $(7,q,r)$-generations. This will end up with four cases, namely $(5A,5A,5A)$-, $(5A,5A,7A)$-, $(5A,7A,7A)$- and $(7A,7A,7A)$-generation.

$(5,5,r)$-generations

We have to check the generation of G through the triples $(5A,5A,5A)$ and $(5A,5A,7A)$.

Proposition 3.2.10. *The group G is $(5A,5A,5A)$-generated.*

Proof. As in Proposition 3.2.7, the groups having elements of order 5 are H_1, H_2, H_3, H_7, $2\times S_6$, $2^4{:}S_5$, $S_3\times S_5$ and $2\times S_5$. We already have $h(5A,2^4{:}S_5)=6$, $h(5A,H_1)=h(5A,H_3)=h(5A,2\times S_6)=h(5A,2\times S_5)=3$ and $h(5A,H_2)H_2=h(5A,H_7)=h(5A,S_3\times S_5)=1$.

By Table A.8 we have $\Delta_G(5A,5A,5A)=3998$. We obtained that $\sum_{H_1}(5a,5a,5a)=1163$, $\sum_{H_2}(5a,5a,5a)=173$, $\sum_{H_3}(5a,5a,5a)=488$, $\sum_{H_7}(5a,5a,5a)=53$, $\sum_{2\times S_6}(3b,5a,5a)=53$, $\sum_{2^4:S_5}(5a,5a,5a)=128$, $\sum_{S_3\times S_5}(5a,5a,5a)=8$ and $\sum_{2\times S_5}(5a,5a,5a)=8$. We get $\Delta^*_G(5A,5A,5A) \geq \Delta_G(5A,5A,5A) - 3\cdot\sum_{H_1}(5a,5a,5a) - \sum_{H_2}(5a,5a,5a) - \sum_{H_3}(5a,5a,5a) - \sum_{H_7}(5a,5a,5a) + 4\times 3\cdot\sum_{2\times S_6}(3a,5b,5a) + 6\cdot\sum_{2^4:S_5}(5a,5a,5a) + \sum_{S_3\times S_5}(5a,5a,5a) + 3\cdot$

$\sum_{2\times S_5}(5a,5a,5a) = 3998-3(1163)-173-3(488)-53+4(3)(53)+6(128)+8+3(8) = 255 > 0$, proving that the group G is $(5A,5A,5A)$-generated.

□

Proposition 3.2.11. *The group G is $(5A,5A,7A)$-generated.*

Proof. By Proposition 3.2.3, G is $(2D,5A,7A)$-generated. It follows by Theorem 2.2.3 that G is $(5A,5A,(7A)^2)$-generated. Since G has one class of order 7, we must have $(7A)^2 = 7A$. The group G will become $(5A,5A,7A)$-generated. □

$(5,7,r)$**- and** $(7,7,r)$**-generations**

Proposition 3.2.12. *The group G is $(5A,7A,7A)$-generated.*

Proof. The subgroups of G having elements of order 7 are H_2, H_4, H_5, H_8, $2^3{:}PSL_3(2)$, $PSL_3(2){:}2$, $PSL_3(2)$ and 7:6. Only H_2 has elements of order 5. We obtained that $\sum_{H_2}(5a,7a,7a) = 378$. Since by Table A.8 we have $\Delta_G(5A,7A,7A) = 7483$, we then obtain that $\Delta^*_G(5A,7A,7A) \geq \Delta_G(5A,7A,7A) - \sum_{H_2}(5a,7a,7a) = 7483 - 378 = 7105 > 0$. Hence the group G is $(5A,7A,7A)$-generated. □

We conclude our investigation on the (p,q,r)-generation of the symplectic group $Sp(6,2)$ by considering the $(7A,7A,7A)$-generations.

Proposition 3.2.13. *The group G is $(7A,7A,7A)$-generated.*

Proof. By Proposition 3.2.4, G is $(2C,7A,7A)$-generated. By the application of Theorem 2.2.3, it follows that G is $(7A,7A,(7A)^2)$-generated. Since $(7A)^2 = 7A$, the group G becomes

$(7A, 7A, 7A)$-generated. □

3.3. The conjugacy classes ranks of the Symplectic Group $Sp(6,2)$.

Now we study the ranks of G with respect to the various conjugacy classes of all its non-identity elements. We start our investigation on the ranks of the non-trivial classes of G by looking at the four classes of involutions $2A$, $2B$, $2C$ and $2D$. It is well known that the rank of any of these involutions classes will be at least 3.

Proposition 3.3.1. *$rank(G : 2A) = 7$.*

Proof. For G with four disjoint cycles and acting on $n = 28$ points, we get $n(s-2)+2 = 58$. If G is $(2A, 2A, 2A, pX)$-generated group, then we must have $c_{2A} + c_{2A} + c_{2A} + c_{pX} \leq 58$, where pX is a non-identity conjugacy class whose order divides the order of G. By Table 4.1 we have $c_{2A} + c_{2A} + c_{2A} + c_{pX} = 3(22) + c_{pX} > 58$ and by Ree's Theorem [49] it follows that G is not $(2A, 2A, 2A, pX)$-generated group. Same applies to G with five disjoint cycles acting on $n = 28$ points, we get $n(s-2)+2 = 86$. By Table 4.1 we have $c_{2A} + c_{2A} + c_{2A} + c_{2A} + c_{pX} = 4(22) + c_{pX} > 86$ and it follows that G is not $(2A, 2A, 2A, 2A, pX)$-generated group. Same applies to G with six disjoint cycles acting on $n = 28$ points, we get $n(s-2)+2 = 114$. By Table 4.1 we have $c_{2A} + c_{2A} + c_{2A} + c_{2A} + c_{2A} + c_{pX} = 5(22) + c_{pX} > 114$ and it follows that G is not $(2A, 2A, 2A, 2A, 2A, pX)$-generated group for $pX \notin \{7A, 8B, 9A, 12C, 15A\}$. With reference to [5, Lemma 4] and [15, Remark 1], simple computation shows that the structure constant $\Delta_G(2A, 2A, 2A, 2A, 2A, nX) = 0$, this shows that the group G is not $(2A, 2A, 2A, 2A, 2A, nX)$-generated for each $nX \in \{7A, 8B, 9A, 12C, 15A\}$. The group G with seven disjoint cycles acting on $n = 28$ points, we get $n(s-2)+2 = 142$. By Table 4.1 we have $c_{2A} + c_{2A} + c_{2A} + c_{2A} + c_{2A} + c_{2A} + c_{pX} = 6(22) + c_{pX} > 142$ and it follows that G is not $(2A, 2A, 2A, 2A, 2A, 2A, pX)$-generated group for $pX \notin \{2A, 2B, 2C, 3A, 4C, 6A, 6D\}$. By Gap, we have $\Delta_G(2A, 2A, 2A, 2A, 2A, 2A, nX) = 0$, proving that the group G is not $(2A, 2A, 2A, 2A, 2A, 2A, nX)$-generated for each $nX \in \{8A, 12A, 12B\}$. By Gap, we have $\Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 3B) = 9720$. Subgroups fusing to $2A$ and $3B$ have all their relevant structure constant zero except the maximal subgroup H_1. Since $\sum_{H_1}(2a, 2a, 2a, 2a, 2a, 2a, 3a) = 9720$ and the value of h is 1, we then obtain that $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 3B) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 3B) - \sum_{H_1}(2a, 2a, 2a, 2a, 2a, 2a, 3a) = 9720 - 9720 = 0$. Similarly, we obtain the following results, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 3C) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 3C) - \sum_{H_1}(2a, 2a, 2a, 2a, 2a, 2a, 3c) - 3 \cdot \sum_{H_2}(2a, 2a, 2a, 2a, 2a, 2a, 3b) + 3 \cdot \sum_{2\times S_6}(2b, 2b, 2b, 2b, 2b, 2b, 3b) = 9450 - 70470 - 3(3960) + 3(24300) = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 4A) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 4A) - 4 \cdot \sum_{H_1}(2c, 2c, 2c, 2c, 2c, 2c, 4a) - 3 \cdot \sum_{H_3}(2f, 2f, 2f, 2f, 2f, 2f, 4d) - 6 \cdot \sum_{H_6}(2j, 2j, 2j, 2j, 2j, 2j, 4f) = 1246080 - 4(168960) - 3(128640) - 6(30720) = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 4B) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 4B) - 4 \cdot \sum_{H_2}(2b, 2b, 2b, 2b, 2b, 2b, 4b) - 2 \cdot \sum_{H_3}(2f, 2f, 2f, 2f, 2f, 2f, 4h) = 52480 - 4(5120) - 2(16000) = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 4E) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 4E) - \sum_{H_2}(2b, 2b, 2b, 2b, 2b, 2b, 4c) = 43008 - 43008 = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 5A) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 5A) - 3 \cdot \sum_{H_1}(2c, 2c, 2c, 2c, 2c, 2c, 5a) - \sum_{H_2}(2b, 2b, 2b, 2b, 2b, 2b, 5a) - 3 \cdot \sum_{H_3}(2f, 2f, 2f, 2f, 2f, 2f, 5a) - \sum_{H_7}(2c, 2c, 2c, 2c, 2c, 2c, 5a) + 2 \cdot 3 \cdot \sum_{2\times S_6}(2b, 2b, 2b, 2b, 2b, 2b, 5a) = 573750 - 3(126875) - 68125 - 3(93750) - 31250 + 2(3)(31250) = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 6B) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 6B) - 4 \cdot \sum_{H_1}(2c, 2c, 2c, 2c, 2c, 2c, 6c) - 3 \cdot \sum_{H_3}(2f, 2f, 2f, 2f, 2f, 2f, 6b) - 6 \cdot \sum_{H_6}(2j, 2j, 2j, 2j, 2j, 2j, 6g) = 1017800 - 4(126875) - 3(111780) - 6(29160) = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 6C) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 6C) - \sum_{H_1}(2c, 2c, 2c, 2c, 2c, 2c, 6d) = 77760 - 77760 = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 6E) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 6E) - \sum_{H_3}(2f, 2f, 2f, 2f, 2f, 2f, 6f) = 43740 - 43740 = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 6F) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 6F) - \sum_{H_1}(2c, 2c, 2f, 2c, 2c, 2c, 6e) = 23328 - 23328 = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 6G) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 6G) - \sum_{H_2}(2b, 2b, 2b, 2b, 2b, 2b, 6c)$

$= 7776 - 7776 = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 7A) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 7A) - \sum_{H_2}(2b, 2b, 2b, 2b, 2b, 2b, 7a) = 16807 - 16807 = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 8B) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 8B) - \sum_{H_3}(2f, 2f, 2f, 2f, 2f, 2f, 8b) = 40960 - 40960 = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 9A) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 9A) - \sum_{H_1}(2c, 2c, 2c, 2c, 2c, 2c, 9a) = 59049 - 59049 = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 10A) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 10A) - \sum_{H_3}(2f, 2f, 2f, 2f, 2f, 2f, 10a) = 31250 - 31250 = 0$, $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 12C) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 12C) - \sum_{H_1}(2c, 2c, 2c, 2c, 2c, 2c, 12a) = 41472 - 41472 = 0$ and $\Delta^*_G(2A, 2A, 2A, 2A, 2A, 2A, 15A) = \Delta_G(2A, 2A, 2A, 2A, 2A, 2A, 15A) - \sum_{H_2}(2b, 2b, 2b, 2b, 2b, 2b, 15a) = 2625 - 2625 = 0$. The group G is not $(2A, 2A, 2A, 2A, 2A, 2A, nX)$-generated for each $nX \in \{3B, 3C, 4A, 4B, 4E, 5A, 6B, 6C, 6E, 6F, 6G, 7A, 8B, 9A, 10A, 12C, 15A\}$. We then conclude that $rank(G : 2A) \notin \{2, 3, 4, 5, 6\}$. By Proposition 3.2.4, the group G is $(2A, 7A, 7A)$-generated. By Corollary 2.2.2, $rank(G : 2A) \leq 7$ and it follows that $rank(G : 2A) = 7$. □

Proposition 3.3.2. $rank(G : 2B) = 4$.

Proof. By Table 3.2, the group G acts on a 15-dimensional irreducible complex module $\mathbb{V}$ and we have $d_{2B} + d_{2B} + d_{2B} + d_{nX} = 3 \times 4 + d_{nX} < 2 \times 15$ for all nX where nX is any conjugacy class dividing $|Sp(6, 2)|$. By applying Scott's Theorem [50], we conclude that G is not $(2B, 2B, 2B, nX)$-generated. We then conclude by the above result and the result of Proposition 3.3.1 that $rank(G : 2B) \neq 3$. Direct computations show that the structure constant $\Delta_G(2B, 2B, 2B, 2B, 9A) = 4617$. (See [5, Lemma 4] and [15, Remark 1]). We see that only two maximal subgroups of G have each an element of order 9 viz. H_1 and H_8. The intersection of conjugacy classes of these two maximal subgroups do not contain elements of order 9. Only the maximal subgroup H_1 meets the classes $2B$ and $9A$ of G. We obtained that $\Sigma_{H_1}(2B, 2B, 2B, 2B, 9A) = 243$ and $h(9A, H_1) = 1$. It follows that $\Delta^*_G(2B, 2B, 2B, 2B, 9A) = 4617 - 243 = 4374$, proving that the group G is $(2B, 2B, 2B, 2B, 9A)$-generated. Hence the result. □

Proposition 3.3.3. $rank(G : 2C) = 4$.

Proof. To show that G is not generated by only three elements from class $2C$, we use Scott's Theorem. If G is $(2C, 2C, 2C, nX)$-generated group for any non-trivial class nX of G, then we must have $d_{2C} + d_{2C} + d_{2C} + d_{nX} \geq 2 \times 7$. However, it is clear from Table 4.2 that $3 \times d_{2C} + d_{nX} < 14$, for each nX of G. Thus G is not $(2C, 2C, 2C, nX)$-generated group and it follows that $rank(G : 2C) \neq 3$. We obtained that $\Delta_G(2C, 4D, 15A) = 45$ and only two maximal subgroups of G have each an element of order 15, namely, H_2 and H_7. The group formed by non-empty intersection with all the conjugacy classes for these two maximal subgroups having elements of order 15 is isomorphic to $S_3 \times S_5$. The subgroups H_2, H_7 and $S_3 \times S_5$ having elements of order 7 will not have any contributions because their elements of order 4 do not fuse to the class $4D$ of G. Since the is no contribution from any of the three group, we then have $\Delta^*_G(2C, 4D, 15A) = \Delta_G(2C, 4D, 15A) = 45 > 0$. This shows that the group G is $(2C, 4D, 15A)$-generated. By the application of Lemma 2.2.1, we then obtain that the group G is $(2C, 2C, 2C, 2C, (15A)^4)$-generated. Since the is only one class of order 15 in G, we then have $(15A)^4 = 15A$ so that G becomes $(2C, 2C, 2C, 2C, 15A)$-generated. Hence $rank(G : 2C) = 4$. □

Proposition 3.3.4. $rank(G : 2D) = 3$.

Proof. Using GAP, by taking

$$a = (3,4)(6,22)(7,21)(8,19)(9,23)(10,17)(11,12)(13,24)(15,20)(16,26)(18,28)(25,27) \in 2D,$$

$$b = (1,3,25)(2,8,26)(4,24,10)(5,6,20)(7,14,16)(9,13,18)(12,23,19)(15,17,22)(21,28,27) \in 3C.$$

Then $\langle a, b \rangle = G$ with

$$ab = (1,3,24,18,27)(2,8,12,11,23,13,10,22,20,17,4,25,21,14,16)(5,6,15)(7,28,9,19,26) \in 15A.$$

Thus, the group G is $(2D, 3C, 15A)$-generated. The result follows from the application of Theorem 2.2.3. □

Proposition 3.3.5. $rank(G : 3A) = 4$.

Proof. To show that G cannot be generated by only two (or three) elements from class $3A$, we use Scott's Theorem. If G is $(3A, 3A, nX)$-generated group (or $(3A, 3A, 3A, nX)$-generated group) for any non-trivial class nX of G, then we must have $d_{3A} + d_{3A} + d_{nX} \geq 2 \times 7$ (or $d_{3A} + d_{3A} + d_{3A} + d_{nX} \geq 2 \times 7$). However, it is clear from Table 4.2 that $2 \times d_{3A} + d_{nX} < 14$ (or $3 \times d_{3A} + d_{nX} < 14$), for each nX of G. Thus G is neither $(3A, 3A, nX)$- nor $(3A, 3A, 3A, nX)$-generated group and it follows that $rank(G : 3A) \notin \{2, 3\}$. By direct computations show that $\Delta_G(3A, 3A, 3A, 3A, 9A) = 229797$ and we see that only two maximal subgroups of G have each an element of order 9, namely, H_1 and H_8. The subgroup 9:6 of G arises from taking non-empty intersection with all the conjugacy classes for these two maximal subgroups, has elements of order 9. The subgroups 9:6 and H_8 will not have any contributions because their elements of 3 do not fuse to the class $3A$ of G. Only H_1 meets the classes $3A$ and $9A$ of G. We obtained that $\Sigma_{H_1}(3b, 3b, 3b, 3b, 9a) = 118989$ and $h(9A, H_1) = 1$. We then obtain that $\Delta^*_G(3A, 3A, 3A, 3A, 9A) \geq \Delta_G(3A, 3A, 3A, 3A, 9A) - \Sigma_{H_1}(3b, 3b, 3b, 3b, 9a) = 229797 - 118989 = 110808$, proving that G is $(3A, 3A, 3A, 3A, 9A)$-generated. Hence the result. □

Remark 3.3.1. An alternative way to show that G is not $(3A, 3A, nX)$-generated group for any non-trivial class nX of G, we note that the direct computations yield $\Delta_G(3A, 3A, nX) = 0$ for all non-trivial classes nX of G except for $nX \in \{2C, 3A, 3C, 4A, 5A, 6B\} := T$. For $nX \in T$, we have $\Delta_G(3A, 3A, 2C) = 32 < 1536 = |C_G(2C)|$, $\Delta_G(3A, 3A, 3A) = 46 < 2160 = |C_G(3A)|$, $\Delta_G(3A, 3A, 3C) = 2 < 108 = |C_G(3C)|$, $\Delta_G(3A, 3A, 4A) = 16 < 384 = |C_G(4A)|$, $\Delta_G(3A, 3A, 5A) = 5 < 30 = |C_G(5A)|$ and $\Delta_G(3A, 3A, 6B) = 6 < 144 = |C_G(6B)|$. Then using Lemma 2.1.3 we deduce that G is not $(3A, 3A, nX)$-generated group for $nX \in T$ and thus

$rank(G:3A) \neq 2$.

Proposition 3.3.6. $rank(G:3B) = 3$.

Proof. To show that G is not $(3B,3B,nX)$-generated group for any non-trivial class nX of G, we note that the direct computations yield $\Delta_G(3B,3B,nX) = 0$ for all non-trivial classes nX of G except for $nX \in \{2C,3A,3B,3C,4A,4D,5A,6C,6D,7A,9A,15A\}$.

It was proved in Proposition 3.2.6 that the group G is not $(3B,3B,7A)$-generated.

Direct computations show that $\Delta_G(3B,3B,9A) = 9$. Subgroups meeting the classes $3B$ and $9A$ of G are 9:6, H_1 and H_8. The subgroups 9:6 and H_1 will not have any contributions because their relevant structure constants are all zeros. The direct computations show that $\Sigma_{H_8}(3a,3a,9x) = \Delta_{H_8}(3a,3a,9a)+\Delta_{H_8}(3a,3a,9b)+\Delta_{H_8}(3a,3a,9c) = 0+0+9 = 9$. We found that $h(9A,H_8) = 1$. It follows that $\Delta^*_G(3B,3B,9A) = \Delta_G(3B,3B,9A) - \Sigma_{H_8}(3a,3a,9x) = 9-9 = 0$, showing the non-generation of G by the triple $(3B,3B,9A)$.

Let $T := \{2C,3A,3B,3C,4A,4D,5A,6C,6D,15A\}$. For $nX \in T$, we have $\Delta_G(3B,3B,2C) = 64 < 1536 = |C_G(2C)|$, $\Delta_G(3B,3B,3A) = 40 < 2160 = |C_G(3A)|$, $\Delta_G(3B,3B,3B) = 28 < 648 = |C_G(3B)|$, $\Delta_G(3B,3B,3C) = 20 < 108 = |C_G(3C)|$, $\Delta_G(3B,3B,4A) = 16 < 384 = |C_G(4A)|$, $\Delta_G(3B,3B,4D) = 16 < 128 = |C_G(4D)|$, $\Delta_G(3B,3B,5A) = 10 < 30 = |C_G(5A)|$, $\Delta_G(3B,3B,6C) = 12 < 72 = |C_G(6C)|$, $\Delta_G(3B,3B,6D) = 8 < 48 = |C_G(6D)|$ and $\Delta_G(3B,3B,15A) = 5 < 15 = |C_G(15A)|$. Then using Lemma 2.1.3 we deduce that G is not $(3B,3B,nX)$-generated group for $nX \in T$ and thus $rank(G:3B) \neq 2$.

Direct computations show that $\Delta_G(3B,3C,15A) = 25$. As in Proposition 3.3.3, subgroups having elements of order 15 are $S_3 \times S_5$, H_2 and H_7. The subgroups $S_3 \times S_5$ and H_2 will not have any contributions because their elements of 3 do not fuse to the class $3B$ of G. Only H_7

meets the classes $3B$, $3C$ and $15A$ of G. The direct computations show that $\sum_{H_7}(3e,3x,15a) = \Delta_{H_7}(3e,3c,15a) + \Delta_{H_7}(3e,3d,15a) = 5+5 = 10$ and we found that $h(15A,H_7) = 1$. We then obtain that $\Delta^*_G(3B,3C,15A) \geq \Delta_G(3B,3C,15A) - \sum_{H_7}(3e,3x,15a) = 25-10 = 15 > 0$, proving that G is $(3B,3C,15A)$-generated. By the application of Lemma 2.2.1, we then obtain that the group G is $(3B,3B,3B,(15A)^3)$-generated. Since the class $15A$ is only one of order 15 in G, we then have $(15A)^3 = 15A$ so that G becomes $(3B,3B,3B,15A)$-generated. Hence $rank(G:3B) = 3$. □

Proposition 3.3.7. $rank(G:3C) = 2$.

Proof. We have proved in Proposition 3.2.1 that the group G is generated by $(2D,3C,7A)$. By Corollary 2.2.4, $rank(G:3C) = 2$. □

Proposition 3.3.8. $rank(G:4A) = 3$.

Proof. To show that G cannot be generated by only two elements from class $4A$, we use Scott's Theorem. If G is $(4A,4A,nX)$-generated group for any non-trivial class nX of G, then we must have $d_{4A}+d_{4A}+d_{nX} \geq 2\times 7$. However, it is clear from Table 3.1 that $2\times d_{4A}+d_{nX} = 2(4)+d_{nX} < 14$, for each $nX \in \{2B,2C,2D,3A,3C,4A,4B,4E,5A,6B,6D\}$ of G. Thus G is not $(4A,4A,nX)$-generated group, for each $nX \in \{2B,2C,2D,3A,3C,4A,4B,4E,5A,6B,6D\}$. The group G is not $(4A,4A,nX)$-, for each $nX \in \{2B,2C,2D,3A,3C,4A,4B,4E,5A,6B,6D\}$. Let $T_1 := \{2A,3B,4C,4D,6A,6E,6F,6G,8A,10A,12A,12B,15A\}$. By Gap, we have $\Delta_G(4A,4A,nX) = 0$, so that the group G is not $(4A,4A,nX)$-generated for all $nX \in T_1$. We obtained that $\Delta_G(4A,4A,12C) = 11 < 12 = |C_G(12C)|$, showing that the group G is not $(4A,4A,12C)$-generated. By Gap, we have $\Delta_G(4A,4A,6C) = 114$, $\sum_{H_1}(4a,4a,6d) = 24$, $\sum_{H_4}(4b,4b,6b) = 14$, $\sum_{H_6}(4d,4d,6a) = 18$ and $h(4A,H_4) = 3$. We then obtain that $\Delta^*_G(4A,4A,6C) = \Delta_G(4A,4A,6C) - \sum_{H_1}(4a,4a,6d) - 3\cdot\sum_{H_4}(4b,4b,6b) - \sum_{H_6}(4d,4d,6a) =$

$114 - 24 - 3 \times 14 - 18 = 30 < 72 = |C_G(6C)|$. Again by Gap, we have $\Delta_G(4A, 4A, 7A) = 7$. Subgroups fusing to $4A$ and $7A$ have all their relevant structure constant zero except the maximal subgroup H_4. Since $\sum_{H_4}(4c, 4c, 7a) = 7$ and $h = 1$, we then obtain that $\Delta^*_G(4A, 4A, 7A) = \Delta_G(4A, 4A, 7A) - \sum_{H_4}(4c, 4c, 7a) = 7 - 7 = 0$. Similarly we obtain the following results, $\Delta^*_G(4A, 4A, 8B) = \Delta_G(4A, 4A, 8B) - \sum_{H_3}(4f, 4f, 8b) - \sum_{H_5}(4a, 4a, 8b) = 36 - 32 - 8 = 0$, $\Delta^*_G(4A, 4A, 9A) = \Delta_G(4A, 4A, 9A) - \sum_{H_1}(4a, 4a, 9a) = 9 - 9 = 0$ and $\Delta^*_G(4A, 4A, 6C) = \Delta_G(4A, 4A, 6C) - \sum_{H_1}(4c, 4c, 6a) = 27 - 27 = 0$. These show that the group G is not $(4A, 4A, 6C)$-, $(4A, 4A, 7A)$-, $(4A, 4A, 8B)$-and $(4A, 4A, 9A)$-generated. Thus, $rank(G : 4A) \neq 2$. Easy computations show that $\Delta_G(4A, 4A, 4A, 9A) = 47385$. The subgroups having elements of order 9 are H_1, H_8 and 9:6. The subgroups H_8 and 9:6 do not have elements of order 4. Only H_1 meets the classes $4A$ and $9A$ of the group G. We obtained that $\sum_{H_1}(4a, 4a, 4a, 9a) = 5832$. We then obtain that $\Delta^*_G(4A, 4A, 4A, 9A) \geq \Delta_G(4A, 4A, 4A, 9A) - \sum_{H_1}(4a, 4a, 4a, 9a) = 47385 - 5832 = 41553 > 0$, proving that G is $(4A, 4A, 4A, 9A)$-generated. Hence the result. □

Proposition 3.3.9. *$rank(G : 4B) = 3$.*

Proof. To show that G cannot be generated by only two elements from class $4B$, we use Scott's Theorem. If G is $(4B, 4B, nX)$-generated group for any non-trivial class nX of G, then we must have $d_{4B} + d_{4B} + d_{nX} \geq 2 \times 7$. However, it is clear from Table 3.1 that $2 \times d_{4A} + d_{nX} = 2(4) + d_{nX} < 14$, for each $nX \in \{2B, 2C, 2D, 3A, 3C, 4A, 4B, 4E, 5A, 6B, 6D\}$ of G. Thus G is not $(4A, 4A, nX)$-generated group, for each $nX \in \{2B, 2C, 2D, 3A, 3C, 4A, 4B, 4E, 5A, 6B, 6D\}$. Let $T_1 := \{2A, 3B, 4C\}$. By Gap, we have $\Delta_G(4B, 4B, nX) = 0$, so that the group G is not $(4A, 4A, nX)$-generated for all $nX \in T_1$. We obtained that $\Delta_G(4B, 4B, 4D) = 64 < 192 = |C_G(4D)|$, $\Delta_G(4B, 4B, 6A) = 36 < 144 = |C_G(6A)|$, $\Delta_G(4B, 4B, 6C) = 18 < 72 = |C_G(6C)|$, $\Delta_G(4B, 4B, 6E) = 18 < 36 = |C_G(6E)|$, $\Delta_G(4B, 4B, 8A) = 8 < 16 = |C_G(8A)|$,

$\Delta_G(4B,4B,12A) = 12 < 24 = |C_G(12A)|$, $\Delta_G(4B,4B,12B) = 12 < 24 = |C_G(12B)|$ and $\Delta_G(4B,4B,12C) = 3 < 12 = |C_G(12C)|$ so that the group G is not $(4B,4B,nX)$-generated for each $nX \in \{4D,6A,6C,6E,8A,12A,12B,12C\}$. It follows that $rank(G:4B) \neq 2$. Easy computations show that $\Delta_G(4B,3B,9A) = 9$. Although the subgroup H_1 meets the classes $3B$, $4B$ and $9A$ of the group G, it will not have any contribution because its relevant structure is zero. We then obtain that $\Delta^*_G(4B,3B,9A) \geq 27 > 0$, proving that G is $(4B,3B,9A)$-generated. Hence the result. □

Proposition 3.3.10. $rank(G:4C) = 3$.

Proof. Let $nX \in \{2A,2B,2C,2D,3A,3B,3C,4A,4B,4C,4D,4E,5A,6A,6B,6D,6E,10A\}$. If G is a $(4C,4C,nX)$-generated group, then we must have $c_{4C} + c_{4C} + c_{nX} \leq 30$. Since by Table 3.3 we have $c_{4C}+c_{4C}+c_{nX} = 12+12+c_{nX} > 30$ and by Ree's Theorem we conclude that G is not $4C,4C,nX)$-generated group. By Gap, we have $\Delta_G(4C,4C,6F) = \Delta_G(4C,4C,6G) = \Delta_G(4C,4C,8A) = \Delta_G(4C,4C,12A) = \Delta_G(4C,4C,12B) = \Delta_G(4C,4C,15A) = 0$, so that the group G is not $(4C,4C,6G)$-, $(4C,4C,6G)$-, $(4C,4C,8A)$-, $(4C,4C,12A)$-, $(4C,4C,12B)$- and $(4C,4C,15A)$-generated. By Gap, we have $\Delta_G(4C,4C,6C) = 162$. We obtained that $\sum_{H_1}(4c,4c,6d) = 162$ and $h = 1$. Subgroups fusing to $4C$ and $6C$ have all their relevant structure constant zero except the maximal subgroup H_1. Similarly we obtain the following results, $\Delta^*_G(4C,4C,7A) = \Delta_G(4C,4C,7A) - \sum_{H_2}(4a,4a,7a) = 7-7 = 0$. $\Delta^*_G(4C,4C,8B) = \Delta_G(4C,4C,8B) - \sum_{H_3}(4f,4f,8b) - \sum_{H_5}(4a,4a,8b) = 40-32-8 = 0$, $\Delta^*_G(4C,4C,9A) = \Delta_G(4C,4C,9A) - \sum_{H_1}(4c,4c,9a) = 81-81 = 0$ and $\Delta^*_G(4C,4C,12C) = \Delta_G(4C,4C,12C) - \sum_{H_1}(4c,4c,12a) = 27-27 = 0$. These show that the group G is not $(4C,4C,nX)$-generated for each $nX \in \{6C,7A,8B,9A,12C\}$. Easy computations show that $\Delta_G(4C,3B,9A) = 27$. Although the subgroup H_1 meets the classes $3B$, $4C$ and $9A$ of the group G, it will not have any contribution because its relevant structure is zero. We then obtain that $\Delta^*_G(4C,3B,9A) \geq$

$27 > 0$, proving that G is $(4C, 3B, 9A)$-generated. Hence the result. □

Proposition 3.3.11. $rank(G : 4D) = 2$.

Proof. It was proved in Proposition 3.3.3 that the group G is $(2C, 4D, 15A)$-generated. By the application of Corollary 2.2.4, it follows that $rank(G : 4D) = 2$. □

Proposition 3.3.12. $rank(G : 6A) = 3$.

Proof. Let $nX \in \{2A, 2B, 2C, 2D, 3A, 3B, 3C, 4A, 4B, 4C, 4D, 4E, 5A, 6A, 6B, 6D, 6E, 10A\}$. If G is a $(6A, 6A, nX)$-generated group, then we must have $c_{6A} + c_{6A} + c_{nX} \leq 30$. Since by Table 3.3 we have $c_{6A} + c_{6A} + c_{nX} = 12 + 12 + c_{nX} > 30$ and by Ree's Theorem we conclude that G is not $(6A, 6A, nX)$-generated group. By Gap, we have $\Delta_G(6A, 6A, 12A) = 8 < 24 = |C_G(12A)|$, so that the group G is not $(6A, 6A, 12A)$-generated. By Gap, we have $\Delta_G(6A, 6A, 6C) = 144$. Subgroups fusing to $6A$ and $6C$ have all their relevant structure constant zero except the maximal subgroup H_1. Since $\sum_{H_1}(6b, 6b, 6d) = 144$ and the value of h is 1, we then obtain that $\Delta^*_G(6A, 6A, 6C) = \Delta_G(6A, 6A, 6C) - \sum_{H_1}(4a, 4a, 7a) = 144 - 144 = 0$. Similarly we obtain the following results, $\Delta^*_G(6A, 6A, 6F) = \Delta_G(6A, 6A, 6F) - \sum_{H_1}(6b, 6b, 6e) = 72 - 72 = 0$, $\Delta^*_G(6A, 6A, 6G) = \Delta_G(6A, 6A, 6G) - \sum_{H_2}(6d, 6d, 6a) = 24 - 24 = 0$, $\Delta^*_G(6A, 6A, 7A) = \Delta_G(6A, 6A, 7A) - \sum_{H_2}(6d, 6d, 7a) = 63 - 63 = 0$, $\Delta^*_G(6A, 6A, 8A) = \Delta_G(6A, 6A, 8A) - \sum_{H_1}(6a, 6a, 8a) = 32 - 32 = 0$, $\Delta^*_G(6A, 6A, 8B) = \Delta_G(6A, 6A, 8B) - \sum_{H_2}(6d, 6d, 8a) = 64 - 64 = 0$, $\Delta^*_G(6A, 6A, 9A) = \Delta_G(6A, 6A, 9A) - \sum_{H_1}(6b, 6b, 9a) = 81 - 81 = 0$, $\Delta^*_G(6A, 6A, 12B) = \Delta_G(6A, 6A, 12B) - \sum_{H_2}(6b, 6b, 6a) = 32 - 32 = 0$, $\Delta^*_G(6A, 6A, 12C) = \Delta_G(6A, 6A, 12C) - \sum_{H_1}(6b, 6b, 12a) = 48 - 48 = 0$ and $\Delta^*_G(6A, 6A, 15A) = \Delta_G(6A, 6A, 15A) - \sum_{H_2}(6d, 6d, 15a) = 25 - 25 = 0$. Let $nX \in T := \{6C, 6F, 6G, 7A, 8A, 8B, 9A, 12B, 12C, 15A\}$. Thus G is not $(6A, 6A, nX)$-generated group and it follows that $rank(G : 6A) \neq 2$. Easy computations show that $\Delta_G(6A, 3B, 9A) = 27$. Although the subgroup H_1 meets the classes $3B$,

$6A$ and $9A$ of the group G, it will not have any contribution because its relevant structure is zero. We then obtain that $\Delta^*_G(6A,3B,9A) \geq 27 > 0$, proving that G is $(6A,3B,9A)$-generated. Hence the result. $\square$

Proposition 3.3.13. *$rank(G:6B) = 2$.*

Proof. The structure constant give us $\Delta_G(6B,6B,8B) = 40$ and only six maximal subgroups of G have each an element of order 8, namely, H_1, H_2, H_3, H_4, H_5 and H_6. Let T be the set of all maximal subgroups of G having elements of order 8. The non-empty intersection of conjugacy classes from any 6 or 5 or 4 or 3 maximal subgroups of T does not contain elements of order 8. The groups formed when taking the non-empty intersection with all the conjugacy classes for any two maximal subgroups having elements of order 8 are isomorphic to $PSL_3(2){:}2$, $((((2^2 \times 2^4){:}2){:}2){:}3){:}2$ (3-copies), $2^4{:}S_5$, $(3^2{:}3){:}QD_{16}$, $(((2^3{:}2^2){:}3){:}2){:}$ (4-copies) and $(S_4 \times S_4){:}2$. Out of all subgroups having elements of order 8, only M_2, M_3 and M_6 meet the classes $6B$ and $8B$ of G. None of them will have any contributions because their relevant structure constants are all zeros. We then obtain that $\Delta^*_G(6B,6B,8B) \geq \Delta_G(6B,6B,8B) = 40$, proving that G is $(6B,6B,8B)$-generated. Hence the result follows. $\square$

Proposition 3.3.14. *Let $nX \in T_1 := \{4E,5A,6C,6D,6E,6F,6G,7A,8A,8B,9A,10A,12A,$ $12B,12C,15A\}$ then $rank(G:nX) = 2$*

Proof. The maximal subgroups, H_2 and H_7 are the only ones containing elements of order 15. The group $S_3 \times S_5$ has elements of order 15 and it is arising from taking non-empty intersection with all the conjugacy classes for these two maximal subgroups of G.

We use Table 3.6 in getting the results of this Proposition. In the same Table 3.6 we give required information needed to calculate $\Theta_G(nX,nX,15A)$ where $nX \in T_1$. The value of h

for these contributing groups is 1. We give some information about $\Delta_G(nX,nX,15A)$, h , $\sum_{H_2}(nx,nx,15a)$, $\sum_{H_7}(nx,nx,15a)$ and $\sum_{S_3\times S_5}(nx,nx,15a)$. The last column $\Theta_G(nX,nX,15A) = \Delta_G(nX,nX,15A) - h\cdot\sum_{H_2}(nx,nx,15b) - h\cdot\sum_{H_7}(nx,nx,15b) +$ $h\cdot\sum_{S_3\times S_5}(nx,nx,15a)$ establishes each generation of G by its triples $(nX,nX,15A)$ because $\Delta^*_G(nX,nX,15A) \geq \Theta_G(nX,nX,15A)$ as it appears in Equation 2.4. Looking at Table 3.6, we see that $\Delta^*_G(nX,nX,15A) > 0$. It follows that G is $(nX,nX,15A)$-generated where $nX \in T_1$. This proves that $rank(G:nX) = 2$ for all $nX \in T_1$. □

Table 3.6: Some information on the $nX \in T_1$

nX	$\Delta_G(nX,nX,15A)$	h	$h\cdot\sum_{H_2}(nx,nx,15a)$	$h\cdot\sum_{H_7}(nx,nx,15a)$	$h\cdot\sum_{S_3\times S_5}(nx,nx,15a)$	$\Theta_G(nX,nX,15A)$
$4E$	1290	1	270	60	45	1005
$5A$	645	1	45	0	0	600
$6C$	280	1	-	40	-	240
$6D$	845	1	155	125	20	585
$6E$	1260	1	510	0	-	750
$6F$	1180	1	35	55	5	1095
$6G$	8580	1	510	120	-	7950
$7A$	28605	1	1620	-	-	26985
$8A$	5100	1	-	-	-	5100
$8B$	5100	1	1140	-	-	3960
$9A$	15645	1	-	-	-	15645
$10A$	15864	1	789	159	24	14940
$12A$	1490	1	-	20	-	1470
$12B$	2450	1	540	20	15	1905
$12C$	10920	1	-	-	-	10920
$15A$	5933	1	308	53	8	5580

The rank for each conjugacy class of elements for the symplectic group $Sp(6,2)$ will be summarized as follows:

- $rank(G:2A) = 7$, see the proof of Proposition 3.3.1,
- $rank(G:2X) = rank(G:3A) = 4$ for $X \in \{B,C\}$, results follow by Propositions 3.3.2, 3.3.3 and 3.3.5,

- $rank(G:2D) = rank(G:3B) = rank(G:4A) = rank(G:4B) = rank(G:6A) = 3$, results follow by Propositions 3.3.4, 3.3.6, 3.3.8, 3.3.9 and 3.3.12,
- $rank(G:nX) = 2$ for all $nX \notin \{1A, 2A, 2B, 2C, 2D, 3A, 3B, 4A, 4B, 4C, 6A\}$, results follow by Propositions 3.3.7, 3.3.11, 3.3.13 and 3.3.14.

CHAPTER 4

The Mathieu sporadic simple group M_{23}

In this chapter, we determine all the generation of the Mathieu group M_{23} by the triples (lX, mY, nZ), where l, m and n are primes that divide the order of M_{23}, that is, $l, m, n \in \{2, 3, 5, 7, 11, 23\}$. The triple generation of this group will be investigated in Section 4.2. As an application of Theorem 2.1.2, the group M_{23} is $(lX, mY, 7A)$-generated if and only if it is $(lX, mY, 7B)$-generated, is also $(lX, mY, 11A)$-generated if and only if it is $(lX, mY, 11B)$-generated and is $(lX, mY, 23A)$-generated if and only if it is $(lX, mY, 23B)$-generated. Therefore, it is sufficient to check the $(lX, mY, 7A)$-, $(lX, mY, 11A)$- and $(lX, mY, 23A)$-generations of M_{23}. The result on the (p, q, r)-generations of M_{23} can be summarized in the following theorem.

Theorem 4.0.1. *The sporadic simple group M_{23} is generated by all the triples (lX, mY, nZ), where l, m and n are primes dividing $|M_{23}|$, except for the cases $(lX, mY, nZ) \in \{(2A, 3A, nZ),$ $(2A, 5A, 5A), (2A, 5A, 7M), (3A, 3A, 5A), (3A, 3A, 7M)\}$ for $M \in \{A, B\}$.*

We also determine the ranks for all the non-identity conjugacy classes of the elements of the group M_{23} in Section 4.3. The main result of the conjugacy classes ranks are summarized by the following Theorem 4.0.2.

Theorem 4.0.2. *For the Mathieu sporadic simple group G, we have*

(i) $rank(G : 2A) = 3$,

(ii) $rank(G : nX) = 2$ if $nX \notin \{1A, 2A\}$ and where nX is a conjugacy class of G.

Remark 4.0.1. From Theorem 4.0.2, we noticed that $rank(G : nX) = 2$ for the non-identity conjugacy classes nX of G, except for the conjugacy class $2A$.

4.1. Introduction

According to [20], there are 26 sporadic simple groups which are members of the family of finite simple groups. The recognition of the sporadic simple groups depends on how they were constructed. The groups of *rank* 3 refer to those acting transitively on a set such that the stabilizer of a point has 3 orbits. So, the sporadic simple groups which were constructed by the centralizers of involutions, can be characterized by such centralizers and those of rank 3 are characterized by their point stabilizers. Thus, the sporadic simple groups may be roughly sorted as the Mathieu groups, the Leech groups, the Fischer's 3-transposition groups, the Monster centralizers and the other six groups that are sub-quotients of the Monster group.

For about a hundred years, the only known sporadic simple groups were the five Mathieu groups, described by **Emil Mathieu** as highly transitive permutation groups. Mathieu investigated multiply-transitive permutation groups on n points. The highest transitivity, found in a simple group is 5-transitive and was discovered by Mathieu. Thus, the 5-transitive permutation groups on 12 points and 24 points are M_{12} and M_{24} respectively. The other Mathieu groups arose as subgroups of these. So, M_{24} is the largest Mathieu sporadic simple group and contains all the other Mathieu sporadic simple groups as subgroups.

The group M_{23} was discovered by Mathieu. The Mathieu group M_{23} is best studied as the point stabilizer in the largest Mathieu group M_{24}. Thus, M_{23} is the automorphism group of the Steiner system $S(4,7,23)$, whose 253 heptads arise from the octads of $S(5,8,24)$ containing the fixed point. The sporadic simple group M_{23} has order $10200960 = 2^7 \times 3^2 \times 5 \times 7 \times 11 \times 23$. By the Atlas of finite groups [20], the group M_{23} has exactly 17 conjugacy classes of its elements and 7 conjugacy classes of its maximal subgroups. Representatives of the conjugacy classes of maximal subgroups are as follows:

$K_1 = M_{22}$	$K_2 = L_3(4){:}2_2$	$K_3 = 2^4{:}A_7$	$K_4 = A_8$
$K_5 = M_{11}$	$K_6 = 2^4{:}(3 \times A_5){:}2$	$K_7 = 23{:}11$	

Throughout Chapter 4 we let $G = M_{23}$, unless otherwise stated. From the electronic Atlas of finite group representations [55], we can see that G has a permutation representation on 23 points. Generators g_1 and g_2 can be taken as follows:

$$g_1 = (1,2)(3,4)(7,8)(9,10)(13,14)(15,16)(19,20)(21,22),$$

$$g_2 = (1,16,11,3)(2,9,21,12)(4,5,8,23)(6,22,14,18)(13,20)(15,17),$$

with $o(g_1) = 2,\ o(g_2) = 4$ and $o(g_1g_2) = 23$.

In Table 4.1, we list the values of the cyclic structure for each conjugacy of G together with the values of both c_i and d_i obtained from Ree and Scotts theorems, respectively.

Table 4.2 gives all the values of d_{nX} for classes nX of prime order for the G with $\dim(\mathbb{V}) = 22$.

In Table 4.3 we list the representatives of classes of the maximal subgroups together with the orbits lengths of M_{23} on these groups and the permutation characters except for the smallest maximal subgroup of M_{23}.

Table 4.4 gives us partial fusion maps of classes of maximal subgroups into the classes of M_{23}.

These will be used in our computations.

Table 4.1: Cycle structures of conjugacy classes of G

nX	Cycle Structure	c_i	d_i
$1A$	1^{23}	23	0
$2A$	$1^7 2^8$	15	8
$3A$	$1^5 3^6$	11	12
$4A$	$1^3 2^2 4^4$	9	14
$5A$	$1^3 5^4$	7	16
$6A$	$1^1 2^2 3^2 6^2$	7	16
$7A$	$1^2 7^3$	5	18
$7B$	$1^2 7^3$	5	18
$8A$	$1^1 2^1 4^1 8^2$	5	18
$11A$	$1^1 11^2$	3	20
$11B$	$1^1 11^2$	3	20
$14A$	$2^1 7^1 14^1$	3	20
$14B$	$2^1 7^1 14^1$	3	20
$15A$	$3^1 5^1 15^1$	3	20
$15B$	$3^1 5^1 15^1$	3	20
$23A$	23^1	1	22
$23B$	23^1	1	22

Table 4.2: $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, nX is a non-trivial class of G and $\dim(\mathbb{V}) = 22$.

nX	$2A$	$3A$	$5A$	$7A$	$7B$	$11A$	$11B$	$23A$	$23B$
Cycle Structure	$1^7\ 2^8$	$1^5\ 3^6$	$1^3\ 5^4$	$1^2\ 7^3$	$1^2\ 7^3$	$1^1\ 11^2$	$1^1\ 11^2$	23^1	23^1
c_i	15	11	7	5	5	3	3	1	1
d_{nX}	8	12	16	18	18	20	20	22	22

Table 4.3: Maximal subgroups of M_{23}

Maximal Subgroup	Order	Orbit Lengths	Character
K_1	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	[1,22]	$1a + 22a$
K_2	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	[2,21]	$1a + 22a + 230a$
K_3	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	[7,16]	$1a + 22a + 230a$
K_4	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	[8,15]	$1a + 22a + 230a + 253$
K_5	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	[11,12]	$1a + 22a + 230a + 1035a$
K_6	$2^7 \cdot 3^2 \cdot 5$	[3,20]	$1a + 22a + 230a + 253a + 1035a$
K_7	$11 \cdot 23$	[23]	

Table 4.4: The partial fusion maps into M_{23}

K_1-class	2a	3a	5a	7a	7b	11a	11b					
$\to$ G	2A	3A	5A	7B	7A	11B	11A					
h				2	2	1	1					
K_2-class	2a	2b	3a	5a	7a	7b						
$\to$ G	2A	2A	3A	5A	7B	7A						
h					1	1						
K_3-class	2a	2b	3a	3b	5a	7a	7b					
$\to$ G	2A	2A	3A	3A	5A	7A	7B					
h						1	1					
K_4-class	2a	2b	3a	3b	5a	7a	7b					
$\to$ G	2A	2A	3A	3A	5A	7B	7A					
h						2	2					
K_5-class	2a	3a	5a	11a	11b							
$\to$ G	2A	3A	5A	11B	11A							
h				1	1							
K_6-class	2a	2b	2c	3a	3b	3c	5a					
$\to$ G	2A	2A	2A	3A	3A	3A	5A					
h							1					
K_7-class	11a	11b	11c	11d	11e	11f	11g	11h	11i	11j	23a	23b
$\to$ G	11A	11B	11A	11A	11A	11B	11B	11B	11A	11B	23A	23B
h	1	1	1	1	1	1	1	1	1	1	1	1

4.2. The (p, q, r)-generations of M_{23}

Let pX, $p \in \{2, 3, 5, 7, 11, 23\}$ be a conjugacy class of G and c_i be the number of disjoint cycles in a representative of pX. Using the result of Ree's theorem 2.1.4, we have $\sum_{i=1}^{s} c_i \leq (s-2)n+2$. For the Mathieu sporadic simple group $G = M_{23}$ and from the Atlas of finite group representations [55] we have G acting on 23 points, so that $n = 23$ and since our generation is triangular, we have $s = 3$. Hence if G is (l, m, n)-generated, then $\sum c_i \leq 25$.

4.2.1 $(2, q, r)$-generations

Now the $(2, q, r)$-generations of G comprises the cases $(2, 3, r)$-, $(2, 5, r)$-, $(2, 7, r)$-, $(2, 11, r)$- and $(2, 23, r)$- generations.

$(2, 3, r)$-generations

Proposition 4.2.1. *The group G is not $(2A, 3A, 7X)$, $(2A, 3A, 11X)$ and $(2A, 3A, 23X)$-generated for $X \in \{A, B\}$.*

Proof. The condition $\frac{1}{2} + \frac{1}{3} + \frac{1}{r} < 1$ shows that $r > 6$. Therefore we have to consider the cases $(2A, 3A, 7X)$, $(2A, 3A, 11X)$ and $(2A, 3A, 23X)$ for all $X \in \{A, B\}$. Theorem 1.1 of [52] implies that G is not a Hurwitz group and hence G is not a $(2A, 3A, 7X)$-generated for $X \in \{A, B\}$. Generally, if G is $(2A, 3A, r)$-generated group, then we must have $c_{2A} + c_{3A} + c_p \leq 25$. From Table 4.1 we see that $c_{2A} + c_{3A} + c_r = 15 + 11 + c_p > 25$ for $p \in \{7A, 7B, 11A, 11B, 23A, 23B\}$. Now using Ree's Theorem [49], it follows that G is not $(2A, 3A, r)$-generated. □

Remark 4.2.1. The above results can be deduced by Scott's Theorem [50], as from Table 4.2 we can see that $d_{2A}+d_{3A}+d_{nX} = 8+12+d_{nX} < 2\times 22$ for $nX \in \{7A, 7B, 11A, 11B, 23A, 23B\}$.

$(2, 5, r)$-generations

The condition $\frac{1}{2}+\frac{1}{5}+\frac{1}{r} < 1$ shows that $r > \frac{10}{3}$. Thus we have to consider the cases $(2A, 5A, 5A)$, $(2A, 5A, 7X)$, $(2A, 5A, 11X)$ and $(2A, 5A, 23X)$ for $X \in \{A, B\}$.

Proposition 4.2.2. *The group G is neither $(2A, 5A, 5A)$- nor $(2A, 5A, 7X)$-generated for $X \in \{A, B\}$.*

Proof. If G is a $(2A, 5A, 5A)$-generated group, then we must have $c_{2A} + c_{5A} + c_{5A} \leq 25$. From Table 4.1 we see that $c_{2A} + c_{5A} + c_{5A} = 15 + 7 + 7 = 29 > 25$. Now using Ree's Theorem, it follows that G is not $(2A, 5A, 5A)$-generated.

By the same Table 4.1 we see that $c_{2A} + c_{5A} + c_{7A} = 15 + 7 + 5 = 27 > 25$. Again by Ree's Theorem, it follows that G is not $(2A, 5A, 7A)$-generated. Since the same holds for $(2A, 5A, 7B)$, it follows that G is not $(2A, 5A, 7X)$-generated for $X \in \{A, B\}$ and the proof is complete. □

Proposition 4.2.3. *The group G is $(2A, 5A, 11X)$-generated for $X \in \{A, B\}$.*

Proof. By Table 4.4, we see that K_1, K_5 and K_7 are the maximal subgroups having elements of order 11.

The intersection of the conjugacy classes these three maximal subgroups do not contain elements of order 11. Considering all various pairwise intersections of the conjugacy classes for these three maximal subgroups, we found that the only candidate having elements of order 11 is isomorphic to the group $PSL_2(11)$.

The maximal subgroup K_7 will not have any contributions because it does not contain elements of orders 2 and 5. We obtained that $\sum_{K_1}(2a, 5a, 11b) = 176$, $\sum_{K_5}(2a, 5a, 11b) = 33$ and $\sum_{PSL_2(11)}(2a, 5x, 11b) = \Delta_{PSL_2(11)}(2a, 5a, 11b) + \Delta_{PSL_2(11)}(2a, 5b, 11b) = 11 + 11 = 22$. By [31, 58], we have $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. Since by Table A.9 we have $\Delta_G(2A, 5A, 11A) = 253$, we then obtain that $\Delta^*_G(2A, 5A, 11A) \geq \Delta_G(2A, 5A, 11A) - \sum_{K_1}(2a, 5a, 11b) - \sum_{K_5}(2a, 5a, 11b) + \sum_{PSL_2(11)}(2a, 5x, 11b) = 253 - 176 - 33 + 22 = 66 > 0$. Hence G is $(2A, 5A, 11A)$-generated. Since the same holds for $(2A, 5A, 11B)$ (see Remark 2.1.2), it follows that G is $(2A, 5A, 11X)$-generated, for all $X \in \{A, B\}$. □

Proposition 4.2.4. *The group G is $(2A, 5A, 23X)$-generated for $X \in \{A, B\}$.*

Proof. By Table 4.4, we see the maximal subgroup K_7 is the only one have elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of orders 2 and 5. Since by Table A.9, we have $\Delta_G(2A, 5A, 23A) = 138$, we then deduce that $\Delta^*_G(2A, 5A, 23A) = \Delta_G(2A, 5A, 23A) = 138 > 0$. Thus G is $(2A, 5A, 23A)$-generated. Since the same holds for $(2A, 5A, 23B)$, it follows that G is a $(2A, 5A, 23X)$-generated group, for $X \in \{A, B\}$. □

$(2, 7, r)$-generations

We check for the generation of G through the triples $(2A, 7X, 7Y)$, $(2A, 7X, 11Y)$ and $(2A, 7X, 23Y)$ for all $X, Y \in \{A, B\}$.

Proposition 4.2.5. *The group G is $(2A, 7X, 7Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4.4 we see that the maximal subgroups of G whose orders are divisible by 7 are K_1, K_2, K_3 and K_4.

The intersection of conjugacy classes from these four maximal subgroups do not contain elements of order 7. The intersection of the conjugacy classes from any three maximal subgroups do not contain elements of order 7. Considering all various intersections of the conjugacy classes for pairwise of these three maximal subgroups, we noticed that the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ are the only ones having elements of order 7.

The group $PSL_3(2)$ has its relevant structure constant zero and as such it will not have any contributions. We obtained that $\sum_{PSL_3(4)}(2a, 7a, 7a) = 42$, $\sum_{A_7}(2a, 7a, 7a) = 7$,

$\sum_{2^3:PSL_3(2)}(2a, 7b, 7b) = 7$ and $h(7A, PSL_3(4)) = h(7A, A_7) = h(7A, 2^3{:}PSL_3(2)) = 2$.

For the contributing maximal subgroups, we have $\sum_{K_1}(2a, 7b, 7b) = 147$, $\sum_{K_2}(2x, 7b, 7b) = \Delta_{K_2}(2a, 7b, 7b) + \Delta_{K_2}(2b, 7b, 7b) = 0 + 42 = 42$, $\sum_{K_3}(2a, 7a, 7a) = 7$, $\sum_{K_4}(2x, 7b, 7b) = \Delta_{K_4}(2a, 7b, 7b) + \Delta_{K_4}(2b, 7b, 7b) = 14 + 28 = 42$ and found that $h(7A, K_2) = h(7A, K_3) = 1$ and $h(7A, K_1) = h(7A, K_4) = 2$. Since by Table A.9 we have $\Delta_G(2A, 7A, 7A) = 301$, we then obtain that $\Delta^*_G(2A, 7A, 7A) \geq \Delta_G(2A, 7A, 7A) - 2 \cdot \sum_{K_1}(2a, 7b, 7b) - \sum_{K_2}(2x, 7b, 7b) - \sum_{K_3}(2a, 7a, 7a) - 2 \cdot \sum_{K_4}(2x, 7b, 7b) + 2 \cdot \sum_{PSL_3(4)}(2a, 7b, 7b) + 2 \cdot 2 \cdot \sum_{A_7}(2a, 7a, 7a) + 2 \cdot 2 \cdot \sum_{2^3:PSL_3(2)}(2a, 7b, 7b) = 301 - 2(147) - 42 - 7 - 2(42) + 2(42) + 2(2)(7) + 2(2)(7) = 14 > 0$ and it follows that $(2A, 7A, 7A)$ is a generating triple for G. Since the same holds for $(2A, 7B, 7B)$, it follows that the group G is $(2A, 7X, 7X)$-generated, for all $X \in \{A, B\}$.

We now investigate the $(2A, 7A, 7B)$- generations for G. From the intersections, we noticed that the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ will all contribute here. We obtained that $\sum_{PSL_3(4)}(2a, 7a, 7b) = 63$, $\sum_{A_7}(2a, 7a, 7b) = 28$, $\sum_{2^3:PSL_3(2)}(2a, 7b, 7a) = 14$, $\sum_{PSL_3(2)}(2a, 7a, 7b) = 7$ and $h(7B, PSL_3(4)) = h(7B, A_7) = h(7B, 2^3{:}PSL_3(2)) = h(7B, PSL_3(2)) = 2$.

The maximal subgroup K_3 will not have any contributions because its relevant structure constant is zero. For the contributing maximal subgroups, we have $\sum_{K_1}(2a, 7b, 7a) = 224$, $\sum_{K_2}(2x, 7b, 7a) = \Delta_{K_2}(2a, 7b, 7a) + \Delta_{K_2}(2b, 7b, 7a) = 0 + 63 = 63$, $\sum_{K_4}(2x, 7b, 7a) = \Delta_{K_4}(2a, 7b, 7a) + \Delta_{K_4}(2b, 7b, 7a) = 21 + 42 = 63$ and found that $h(7B, K_2) = 1$ and $h(7B, K_1) = h(7B, K_4) = 2$. Since by Table A.9 we have $\Delta_G(2A, 7A, 7B) = 462$, we then obtain that $\Delta^*_G(2A, 7A, 7B) = \Delta_G(2A, 7A, 7B) - 2 \cdot \sum_{K_1}(2a, 7b, 7a) - \sum_{K_2}(2x, 7b, 7a) - 2 \cdot \sum_{K_4}(2x, 7b, 7a) + 2 \cdot \sum_{PSL_3(4)}(2a, 7a, 7b) + 2 \cdot \sum_{A_7}(2a, 7b, 7a) + 2 \cdot \sum_{2^3:PSL_3(2)}(2a, 7b, 7a) + 2 \cdot \sum_{PSL_3(2)}(2a, 7b, 7a) = 462 - 2(224) - 63 - 2(21) + 2(63) + 2(2)(28) + 2(2)(14) + 2(7) = 217 > 0$. Therefore G is $(2A, 7A, 7B)$-generated. $\square$

Proposition 4.2.6. *The group G is $(2A, 7X, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.2.3, we see that $PSL_2(11)$ is the only group having elements of order 11. This group $PSL_2(11)$ will not have any contributions because it does not contain elements of order 7. With regard to maximal subgroups having elements of order 11, by Table 4.4 we see that the maximal subgroup K_1 of G is the only one whose order is divisible by 7 and 11. We obtained that $\sum_{K_1}(2a, 7x, 11y) = 176$ and $h(11Z, K_1) = 1$ for $Z \in \{A, B\}$. By Table A.9 we have $\Delta_G(2A, 7X, 11Y) = 308$ so that $\Delta_G^*(2A, 7X, 11Y) \geq \Delta_G(2A, 7X, 11Y) - \sum_{K_1}(2a, 7x, 11y) = 308 - 176 = 132 > 0$, implies that G is $(2A, 7X, 11Y)$-generated for all $X, Y \in \{A, B\}$. □

Proposition 4.2.7. *The group G is $(2A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup K_7 does not have elements of order 7. By Table A.9 we have $\Delta_G(2A, 7X, 23Y) = 184$. Since there are no contributions from any of the maximal subgroups of G, we then have $\Delta_G^*(2A, 7X, 23Y) = \Delta_G(2A, 7X, 23Y) = 184 > 0$, proving that G is $(2A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$. □

$(2, 11, r)$**-generations**

Also here we check for the generation of G through the triples $(2A, 11A, 11A)$-, $(2A, 11A, 11B)$-, $(2A, 11A, 23A)$-, $(2A, 11A, 23B)$-, $(2A, 11B, 11B)$-, $(2A, 11B, 23A)$- and $(2A, 11B, 23B)$-generation. For this we have the following theorems:

Proposition 4.2.8. *The group G is $(2A, 11X, 11Y)$-generated for $X, Y \in \{A, B\}$.*

Proof. Looking at discussions in Proposition 4.2.3 on the intersections, we see that the group $PSL_2(11)$ may be involved. By Table 4.4 we see that the maximal subgroups of G containing elements of orders 2 and 11 are K_1 and K_5. The groups K_1, K_5 and $PSL_2(11)$ have elements of orders 2 and 11. We obtained that $\sum_{K_1}(2a, 11x, 11x) = 99$, $\sum_{K_5}(2a, 11x, 11x) = 11$ and $\sum_{PSL_2(11)}(2a, 11x, 11x) = 11$. We found that $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. Since by Table A.9 we have $\Delta_G(2A, 11X, 11X) = 341$, so that $\Delta^*_G(2A, 11X, 11X) \geq \Delta_G(2A, 11X, 11X) - \sum_{K_1}(2a, 11x, 11x) - \sum_{K_5}(2a, 11x, 11x) + \sum_{PSL_2(11)}(2a, 11x, 11x) = 341 - 147 - 11 + 11 = 194 > 0$, proving that G is $(2A, 11X, 11X)$-generated for $X \in \{A, B\}$.

Finally, we show that G is $(2A, 11A, 11B)$-generated. We obtained that $\sum_{K_1}(2a, 11b, 11a) = 132$ and $\sum_{K_5}(2a, 11b, 11a) = 11$.The group $PSL_2(11)$ will not have any contributions because its relevant structure constant is zero. Since by Table A.9 we have $\Delta_G(2A, 11A, 11B) = 341$, so that $\Delta^*_G(2A, 11A, 11B) = \Delta_G(2A, 11A, 11B) - \sum_{K_1}(2a, 11b, 11a) - \sum_{K_5}(2a, 11b, 11a)$ $= 341 - 224 - 11 = 106 > 0$, implies that G is $(2A, 11A, 11B)$-generated. We conclude that G is $(2A, 11Y, 11Z)$-generated for all $Y, Z \in \{A, B\}$. □

Proposition 4.2.9. *The group G is $(2A, 11X, 23Y)$-generated for $X, Y \in \{A, B\}$.*

Proof. By Table 4.4 we see the K_7 is the only maximal subgroup of G containing elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 2. By Table A.9 we have $\Delta_G(2A, 11X, 23Y) = 391$, so that $\Delta^*(2A, 11X, 23Y) = \Delta_G(2A, 11X, 23Y) = 391 > 0$. Hence the group G is $(2A, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$. □

$(2, 23, r)$-generations

In here we check for the generation of G through the triples $(2A, 23A, 23A)$, $(2A, 23A, 23B)$ and $(2A, 23B, 23B)$. For these we have the following theorems:

Proposition 4.2.10. *The group G is $(2A, 23X, 23Y)$-generated for $X, Y \in \{A, B\}$.*

Proof. By Table 4.4 we see the K_7 is the only maximal subgroup of G containing elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 2. By Table A.9 we have $\Delta_G(2A, 23X, 23X) = 161$ and $\Delta_G(2A, 23A, 23B) = 230$ for $X \in \{A, B\}$. Since there is no contributing group here, we then obtain that $\Delta^*_G(2A, 23X, 23X) = \Delta_G(2A, 23X, 23X) = 161 > 0$ and $\Delta^*_G(2A, 23A, 23B) = \Delta_G(2A, 23A, 23B) = 230 > 0$ for all $X \in \{A, B\}$. Hence, the group G is a $(2A, 23X, 23Y)$-generated for $X, Y \in \{A, B\}$. □

4.2.2 The $(3, q, r)$-generations

The condition $\frac{1}{3} + \frac{1}{3} + \frac{1}{r} < 1$ shows that $r > 3$. We then handle all the possible $(3, q, r)$-generations, namely $(3A, 3A, 5A)$-, $(3A, 3A, 7X)$-, $(3A, 3A, 11X)$-, $(3A, 3A, 23X)$-, $(3A, 5A, 5A)$-, $(3A, 5A, 7X)$-, $(3A, 5A, 11X)$-, $(3A, 5A, 23X)$-, $(3A, 7X, 7Y)$-, $(3A, 7X, 11Y)$-, $(3A, 7X, 23Y)$-, $(3A, 11X, 11Y)$-, $(3A, 11X, 23Y)$- and $(3A, 23X, 23Y)$-generations in this section.

$(3, 3, r)$-generations

Proposition 4.2.11. *The group G is neither $(3A, 3A, 5A)$- nor $(3A, 3A, 7X)$-generated group for $X \in \{A, B\}$.*

Proof. By Table 4.2, the group G acts on a 22-dimensional irreducible complex module $\mathbb{V}$. By Scott's Theorem applied to this module and using the Atlas of finite groups, we see that $d_{3A} = \dim(\mathbb{V}/C_{\mathbb{V}}(3A)) = \frac{2(22-4)}{3} = 12$, $d_{5A} = \dim(\mathbb{V}/C_{\mathbb{V}}(5A)) = \frac{4(22-2)}{5} = 16$ and $d_{7A} = d_{7B} = \dim(\mathbb{V}/C_{\mathbb{V}}(5A)) = \frac{6(22-1)}{7} = 18$. For the case $(3A, 3A, 5A)$, we get $d_{3A} + d_{3A} + d_{5A} = 2 \times 12 + 16 = 40 < 44$ showing that G is not $(3A, 3A, 5A)$-generated. We also get that $d_{3A} + d_{3A} + d_{7X} = 2 \times 12 + 16 = 42 < 44$ for $X \in \{A, B\}$ and by Scott's Theorem G is not $(3A, 3A, 7X)$-generated for all $X \in \{A, B\}$ and the proof is complete. □

Proposition 4.2.12. *The group G is $(3A, 3A, 11X)$-generated for $X \in \{A, B\}$.*

Proof. Looking at Proposition 4.2.3, we notice that the subgroups of G involved here are K_1, K_5 and $PSL_2(11)$ because both subgroups have their elements of respective orders 3 and 11 which fuse to the elements $3A$ and $11A$ (or $11B$) of the group G. We obtained that $\sum_{K_1}(3a, 3a, 11b) = 209$, $\sum_{K_5}(3a, 3a, 11b) = 11$ and $\sum_{PSL_2}(11)(3a, 3a, 11b) = 11$. We already have $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. Since by Table A.10 we have $\Delta_G(3A, 3A, 11A) = 275$, we then obtain that $\Delta^*_G(3A, 3A, 11A) \geq \Delta_G(3A, 3A, 11A) - \sum_{K_1}(3a, 3a, 11b) - \sum_{K_5}(3a, 3a, 11b) + \sum_{PSL_2(11)}(3a, 3a, 11b) = 275 - 209 - 11 + 11 = 66 > 0$, proving that G is $(3A, 3A, 11A)$-generated. Since the same holds for $(3A, 3A, 11B)$, it follows that G is $(3A, 3A, 11X)$-generated, for all $X \in \{A, B\}$. □

Proposition 4.2.13. *The group G is a $(3A, 3A, 23X)$-generated for $X \in \{A, B\}$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 3. By Table A.10 we have that $\Delta_G(3A, 3A, 23X) = 138$. Since there is no contributing group, we then obtain that $\Delta^*(3A, 3A, 23X) = \Delta_G(3A, 3A, 23X) = 138 > 0$, so that G is $(3A, 3A, 23X)$-generated for $X \in \{A, B\}$. □

$(3, 5, r)$**-generations**

Proposition 4.2.14. *The group G is $(3A, 5A, 5A)$-generated.*

Proof. Looking at Table 4.4 we see that all the maximal subgroups of G have elements of order 5 except for the seventh maximal subgroup. Let T be the set of all maximal subgroups of G except the seventh one. We look at various intersections of conjugacy classes for these maximal subgroups. We have the following:

- The groups arising from the intersections of conjugacy classes for any 4, 5 or 6 maximal subgroups in T do not contain elements of order 5.
- The group arising from intersections of the conjugacy classes for any three maximal subgroups in T having elements of orders 3 and 5 is S_5 (2-copies). We obtained that $\sum_{S_5}(3a, 5a, 5a) = 10$ and $h(5A, S_5) = 3$.
- The groups arising from intersections of the conjugacy classes for any two maximal subgroups in T having elements of orders 3 and 5 are $2^4{:}S_5$ (3-copies), $PSL_3(4)$, A_7 (2-copies), $2^4{:}A_6$, $PSL_2(11)$, $A_6{:}2$, A_5 and S_5 (2-copies). We obtained that $\sum_{2^4:S_5}(3a, 5a, 5a) = 160$, $\sum_{PSL_3(4)}(3a, 5x, 5y) = \Delta_{PSL_3(4)}(3a, 5a, 5a) + \Delta_{PSL_3(4)}(3a, 5a, 5b) + \Delta_{PSL_3(4)}(3a, 5b, 5b) = 445 + 445 + 445 = 1335$, $\sum_{A_7}(3x, 5a, 5a) = \Delta_{A_7}(3a, 5a, 5a) + \Delta_{A_7}(3b, 5a, 5a) = 20+60 = 80$, $\sum_{2^4:A_6}(3x, 5y, 5z) = \Delta_{2^4:A_6}(3a, 5a, 5a)+\Delta_{2^4:A_6}(3a, 5a, 5b)+\Delta_{2^4:A_6}(3a, 5b, 5b)+\Delta_{2^4:A_6}(3b, 5a, 5a)+\Delta_{2^4:A_6}(3b, 5a, 5b)+\Delta_{2^4:A_6}(3b, 5b, 5b) = 80+160+80+20+40+20 = 400$, $\sum_{PSL_2(11)}(3a, 5x, 5y) = \Delta_{PSL_2(11)}(3a, 5a, 5a)+\Delta_{PSL_2(11)}(3a, 5a, 5b)+\Delta_{PSL_2(11)}(3a, 5b, 5b) = 20 + 20 + 20 = 60$, $\sum_{A_6:2}(3a, 5a, 5a) = 30$, $\sum_{A_5}(3a, 5x, 5y) = \Delta_{A_5}(3a, 5a, 5a)+\Delta_{A_5}(3a, 5a, 5b)+\Delta_{A_5}(3a, 5b, 5b) = 5+5+5 = 15$ and $\sum_{S_5}(3a, 5a, 5a) = 10$. We found that the value of h for each of these eight groups is 3.

By Table A.10 we have $\Delta_G(3A, 5A, 5A) = 6550$. We obtained that $\sum_{K_1}(3a, 5a, 5a) = 2800$, $\sum_{K_2}(3a, 5a, 5a) = 910$, $\sum_{K_3}(3x, 5a, 5a) = \Delta_{K_3}(3a, 5a, 5a) + \Delta_{K_3}(3b, 5a, 5a) = 320+240 = 560$, $\sum_{K_4}(3x, 5a, 5a) = \Delta_{K_4}(3a, 5a, 5a) + \Delta_{K_4}(3b, 5a, 5a) = 25 + 135 = 160$, $\sum_{K_5}(3a, 5a, 5a) = 80$, $\sum_{K_6}(3x, 5a, 5a) = \Delta_{K_6}(3a, 5a, 5a) + \Delta_{K_6}(3b, 5a, 5a) + \Delta_{K_6}(3c, 5a, 5a) = 0+0+160 = 160$. The value of h for each maximal subgroup is 3 except for K_4 and K_6. The value of h is 1 for each of these maximal subgroups K_4 and K_6. It follows that $\Delta^*_G(3A, 5A, 5A) \geq \Delta_G(3A, 5A, 5A) - 3 \cdot \sum_{K_1}(3a, 5a, 5a) - 3 \cdot \sum_{K_2}(3a, 5a, 5a) - 3 \cdot \sum_{K_3}(3x, 5a, 5a) - \sum_{K_4}(3x, 5a, 5a) - 3 \cdot \sum_{K_5}(3a, 5a, 5a) - \sum_{K_6}(3x, 5a, 5a) - 2 \cdot 3 \cdot \sum_{S_5}(3a, 5a, 5a) + 3 \cdot 3 \cdot \sum_{2^4:S_5}(3a, 5a, 5a) + 3 \cdot \sum_{PSL_3(4)}(3a, 5x, 5y) + 2 \cdot 3 \cdot \sum_{A_7}(3x, 5a, 5a) + 3 \cdot \sum_{2^4:A_6}(3x, 5y, 5z) + 3 \cdot \sum_{PSL_2(11)}(3a, 5x, 5y) + 3 \cdot \sum_{A_6:2}(3a, 5a, 5a) + 3 \cdot \sum_{A_5}(3a, 5x, 5y) + 2 \cdot 3 \cdot \sum_{S_5}(3a, 5a, 5a) = 6550 - 3(2800) - 3(910) - 3(560) - 1(160) - 3(80) - 1(160) - 2(3)(10) + 3(3)(160) + 3(1335) + 2(3)(80) + 3(400) + 3(60) + 3(30) + 3(15) + 2(3)(10) = 620 > 0$. It follows that the group G is $(3A, 5A, 5A)$-generated. □

Proposition 4.2.15. *The group G is $(3A, 5A, 7X)$-generated for $X \in \{A, B\}$.*

Proof. As in Proposition 4.2.5, the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ may have contributions here.The groups $2^3{:}PLS_3(2)$ and $PSL_3(2)$ will not have any contributions because they do not have elements of order 5. We obtained that $\sum_{PSL_3(4)}(3a, 5x, 7b) = \Delta_{PSL_3(4)}(3a, 5a, 7b) + \Delta_{PSL_3(4)}(3a, 5b, 7b) = 441 + 441 = 882$, $\sum_{A_7}(3x, 5a, 7b) = \Delta_{A_7}(3a, 5a, 7b) + \Delta_{A_7}(3b, 5a, 7b) = 56 + 7 = 63$ and $h(7A, PSL_3(4)) = h(7A, A_7) = 2$.

The maximal subgroups K_1, K_2, K_3 and K_4 meet the $3A$, $5A$, $7A$ classes of G. We obtained that $\sum_{K_1}(3a, 5a, 7b) = 2464$, $\sum_{K_2}(3a, 5a, 7b) = 882$, $\sum_{K_3}(3x, 5a, 7a) = \Delta_{K_3}(3a, 5a, 7a) + \Delta_{K_3}(3b, 5a, 7a) = 112 + 224 = 336$, $\sum_{K_4}(3x, 5a, 7b) = \Delta_{K_4}(3a, 5a, 7b) + \Delta_{K_4}(3b, 5a, 7b) = 77 + 7 = 84$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table A.10 we have $\Delta_G(3A, 5A, 7A) = 5124$, we then obtain that $\Delta^*_G(3A, 5A, 7A) \geq \Delta_G(3A, 5A, 7A) - 2 \cdot \sum_{K_1}(3a, 5a, 7b) - \sum_{K_2}(3a, 5a, 7b) - \sum_{K_3}(3x, 5a, 7a) - 2 \cdot \sum_{K_4}(3a, 5a, 7b) + 2 \cdot \sum_{PSL_3(4)}(3a, 5x, 7b) + 2 \cdot 2 \cdot \sum_{A_7}(3x, 5a, 7b) = 5124 - 2(2464) - 882 - 336 - 2(84) + 2(882) + 2(2)(63) = 826 > 0$. Therefore, the group G is $(3A, 5A, 7A)$-generated. Since the same holds for $(3A, 5A, 7B)$, it follows that the group G is $(3A, 5A, 7X)$-generated for $X \in \{A, B\}$. □

Proposition 4.2.16. *The group G is $(3A, 5A, 11X)$-generated for $X \in \{A, B\}$.*

Proof. By Table 4.4 we see that the maximal subgroups of G containing elements of orders 3 and 11 are K_1 and K_5. The group $PSL_2(11)$contains elements of orders 3, 5 and 11. We obtained that $\sum_{K_1}(3a, 5a, 11b) = 2112$, $\sum_{K_5}(3a, 5a, 11a) = 99$ and $\sum_{PSL_2(11)}(3a, 5x, 11b) = \Delta_{PSL_2(11)}(3a, 5a, 11b) + \Delta_{PSL_2(11)}(3a, 5b, 11b) = 22 + 22 = 44$. We already have $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. Since by Table A.10 we have $\Delta_G(3A, 5A, 11A) = 4136$, we then have $\Delta^*_G(3A, 5A, 11A) \geq \Delta_G(3A, 5A, 11A) - \sum_{K_1}(3a, 5a, 11b) - \sum_{K_5}(3a, 5a, 11b) + \sum_{PSL_2(11)}(3a, 5x, 11b) = 4136 - 2112 - 99 + 44 = 1969 > 0$, so that G is $(3A, 5A, 11A)$-generated. Since the same holds for $(3A, 5A, 11B)$, it follows that the group G is $(3A, 5A, 11X)$-generated for $X \in \{A, B\}$. □

Proposition 4.2.17. *The group G is $(3A, 5A, 23X)$-generated group for $X \in \{A, B\}$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 5. By Table A.10 we have that $\Delta_G(3A, 5A, 23X) = 2438$. Since there is no contributing group, we then obtain that $\Delta^*_G(3A, 5A, 23X) = \Delta_G(3A, 5A, 23X) = 2438 > 0$, so that G is $(3A, 5A, 23X)$-generated for

$X \in \{A, B\}$. □

$(3, 7, r)$**-generations**

In this subsection we discuss the case $(3, 7, r)$-generations. It follows that we will end up with 11 cases, namely $(3A, 7A, 7A)$-, $(3A, 7A, 7B)$-, $(3A, 7A, 11A)$-, $(3A, 7A, 11B)$-, $(3A, 7A, 23A)$-, $(3A, 7A, 23B)$-, $(3A, 7B, 7B)$-, $(3A, 7B, 11A)$-, $(3A, 7B, 11B)$-, $(3A, 7B, 23A)$ and $(3A, 7B, 23B)$-generation.

Proposition 4.2.18. *The group G is $(3A, 7X, 7Y)$-generated for all $X, Y \in \{A, B\}$*

Proof. As in Proposition 4.2.5, the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ have elements of order 7. We obtained that $\sum_{PSL_3(4)}(3a, 7b, 7b) = 357$, $\sum_{A_7}(3x, 7b, 7b) = \Delta_{A_7}(3a, 7b, 7b)+\Delta_{A_7}(3b, 7b, 7b) = 56+21 = 77$, $\sum_{2^3:PSL_3(2)}(3a, 7b, 7b) = 28$, $\sum_{PSL_3(2)}(3a, 7a, 7a) = 7$ and $h(7A, PSL_3(4)) = h(7A, A_7) = h(7A, 2^3{:}PSL_3(2)) = h(7A, PSL_3(2)) = 2$.

The maximal subgroups K_1, K_2, K_3 and K_4 meet the $3A$, $7A$ classes of G. We obtained that $\sum_{K_1}(3a, 7b, 7b) = 1792$, $\sum_{K_2}(3a, 7b, 7b) = 357$, $\sum_{K_3}(3x, 7a, 7a) = \Delta_{K_3}(3a, 7a, 7a) + \Delta_{K_3}(3b, 7a, 7a) = 168 + 126 = 294$, $\sum_{K_4}(3x, 7b, 7b) = \Delta_{K_4}(3a, 7b, 7b) + \Delta_{K_4}(3b, 7b, 7b) = 147 + 21 = 168$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table A.10 we have $\Delta_G(3A, 7A, 7A) = 4886$, we then obtain that $\Delta^*_G(3A, 7A, 7A) \geq \Delta_G(3A, 7A, 7A)-2\cdot\sum_{K_1}(3a, 7b, 7b)-\sum_{K_2}(3a, 7b, 7b)-\sum_{K_3}(3x, 7a, 7a)-2\cdot\sum_{K_4}(3a, 7b, 7b)+2\cdot\sum_{PSL_3(4)}(3a, 7b, 7b)+2\cdot2\cdot\sum_{A_7}(3x, 7b, 7b)+2\cdot2\cdot\sum_{2^3:PSL_3(2)}(3a, 7b, 7b)+2\cdot\sum_{PSL_3(2)}(3a, 7a, 7a) = 4886-2(1792)-357-394-2(168)+2(357)+2(2)(77)+2(2)(28)+2(7) = 1363 > 0$. Therefore, the group G is $(3A, 7A, 7A)$-generated. Since the same holds for $(3A, 7B, 7B)$, it follows that the group G is $(3A, 7X, 7X)$-generated for $X \in \{A, B\}$.

We now prove that G is $(3A, 7A, 7B)$-generated. We obtained that $\sum_{PSL_3(4)}(3a, 7b, 7a) = 357$,

$\sum_{A_7}(3x,7b,7b) = \Delta_{A_7}(3a,7b,7b)+\Delta_{A_7}(3b,7b,7b) = 28+14 = 32$, $\sum_{2^3:PSL_3(2)}(3a,7b,7b) = 28$, $\sum_{PSL_3(2)}(3a,7a,7b) = 7$, $\sum_{K_1}(3a,7b,7a) = 1792$, $\sum_{K_2}(3a,7b,7a) = 357$, $\sum_{K_3}(3x,7a,7b) = \Delta_{K_3}(3a,7a,7b) + \Delta_{K_3}(3b,7a,7b) = 112 + 70 = 182$, $\sum_{K_4}(3x,7b,7a) = \Delta_{K_4}(3a,7b,7a) + \Delta_{K_4}(3b,7b,7a) = 21+147 = 168$, $\sum_{PSL_3(4)}(3a,7b,7a) = 357$, $\sum_{A_7}(3x,7b,7a) = \Delta_{A_7}(3a,7b,7a)+\Delta_{A_7}(3b,7b,7a) = 56+21 = 77$, $\sum_{2^3:PSL_3(2)}(3a,7b,7a) = 28$ and $\sum_{PSL_3(2)}(3a,7a,7b)$. Since by the same Table A.10 we have $\Delta_G(3A,7A,7B) = 4886$, so that $\Delta^*_G(3A,7A,7B) \geq \Delta_G(3A,7A,7B)$ $- 2\cdot\sum_{K_1}(3a,7b,7a) - \sum_{K_2}(3a,7b,7a) - \sum_{K_3}(3x,7a,7b) - 2\cdot\sum_{K_4}(3x,7a,7b) +$ $2\cdot\sum_{PSL_3(4)}(3a,7b,7a)+2\cdot2\cdot\sum_{A_7}(3x,7b,7a)+2\cdot2\cdot\sum_{2^3:PSL_3(2)}(3a,7b,7a)+2\cdot\sum_{PSL_3(2)}(3a,7a,7b)$ $= 4886 - 2(1792) - 357 - 182 - 2(168) + 2(357) + 2(2)(42) + 2(2)(28) + 2(7) = 1435 > 0$. This proves that G is $(3A,7A,7B)$-generated group. □

Proposition 4.2.19. *The group G is $(3A,7X,11Y)$-generated for all $X,Y \in \{A,B\}$.*

Proof. Looking at Proposition 4.2.3, K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The group $PSL_2(11)$ will not have any contributions because it does not have elements of order 7. Looking at Table 4.4, we see that K_1 is the only maximal subgroup of G having elements of orders 3, 7 and 11. We obtained that $\sum_{K_1}(3a,7x,11y) = 1760$ and $h(11X,K_1) = 1$ for $X \in \{A,B\}$. By Table A.10 we have $\Delta_G(3A,7X,11Y) = 4136$. We obtained that $\Delta^*_G(3A,7X,11Y) \geq \Delta_G(3A,7X,11Y) - \sum_{K_1}(3a,7x,11y) = 4136 - 1760 = 2376$ and so that the group G becomes is $(3A,7X,11Y)$-generated for all $X,Y \in \{A,B\}$. □

Proposition 4.2.20. *The group G is $(3A,7X,23Y)$-generated for all $X,Y \in \{A,B\}$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of orders 3 and 7. By Table A.10 we have that $\Delta_G(3A,7X,23Y) = 3312$. Since there is no contributing group, we then obtain that $\Delta_G^*(3A, 7X, 23Y) = \Delta_G(3A, 7X, 23Y) = 3312 > 0$, so that the group G is $(3A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$. □

$(3, 11, r)$-generations

In this subsection we discuss the case $(3, 11, r)$-generations. It follows that we will end up with 7 cases, namely $(3A, 11A, 11A)$-, $(3A, 11A, 11B)$-, $(3A, 11A, 23A)$-, $(3A, 11A, 23B)$-, $(3A, 11B, 11B)$-, $(3A, 11B, 23A)$-, $(3A, 11B, 23B)$-generation.

Proposition 4.2.21. *The group G is $(3A, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.2.3, K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The maximal subgroup K_7 will not have any contributions because it does not have elements of order 3. We obtained that $\sum_{K_1}(3a, 11b, 11b) = 1320$, $\sum_{K_5}(3a, 11b, 11b) = 22$ and $\sum_{PSL_2(11)}(3a, 11b, 11b) = 0$. The value of h for each group is 1. Since by Table A.10 we have $\Delta_G(3A, 11A, 11A) = 5126$, it follows that $\Delta_G^*(3A, 11A, 11A) \geq \Delta_G(3A, 11A, 11A) - \sum_{K_1}(3a, 11b, 11b) - \sum_{K_5}(3a, 11b, 11b) + \sum_{PSL_2(11)}(3a, 11b, 11b) = 5126 - 1320 - 22 + 0 = 3784 > 0$. Therefore, the group G is $(3A, 11A, 11A)$-generated. Since the same holds for $(3A, 11B, 11B)$,the group G becomes $(3A, 11X, 11X)$-generated for $X \in \{A, B\}$.

We now prove that G is $(3A, 11A, 11B)$-generated. We obtained that $\sum_{K_1}(3a, 11a, 11b) = 1276$, $\sum_{K_5}(3a, 11a, 11b) = 77$ and $\sum_{PSL_2(11)}(3a, 11a, 11b) = 22$. By the same Table A.10 we have $\Delta_G(3A, 11A, 11B) = 5379$. Then we obtain that $\Delta_G^*(3A, 11A, 11B) \geq \Delta_G(3A, 11A, 11B) - \sum_{K_1}(3a, 11b, 11a) - \sum_{K_5}(3a, 11b, 11a) + \sum_{PSL_2(11)}(3a, 11b, 11a) = 5379 - 1276 - 77 + 22 = 4048 > 0$, proving that G is $(3A, 11A, 11B)$-generated. Hence, the group G is $(3A, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$. □

Proposition 4.2.22. *The group G is $(3A, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 3. By Table A.10 we have that $\Delta_G(3A, 11X, 23Y) = 5129$. Since there is no contributing group, we then obtain that $\Delta^*(3A, 11X, 23Y) = \Delta_G(3A, 11X, 23Y) = 5129 > 0$, so that the group G is $(3A, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$. □

$(3, 23, r)$**-generations**

In this subsection we discuss the case $(3, 23, r)$-generations. It follows that we will end up with 3 cases, namely $(3A, 23A, 23A)$-, $(3A, 23A, 23B)$-, $(3A, 23B, 23B)$-generation which will be handled in the following Proposition 4.2.23.

Proposition 4.2.23. *The group G is $(3A, 23X, 23Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 3. By Table A.10 we have $\Delta_G(3A, 23A, 23A) = 3082$. Since there is no contributing group, we then obtain that $\Delta^*(3A, 23X, 23X) = \Delta_G(3A, 23A, 23A) = 3082 > 0$, so that the group G is $(3A, 23A, 23A)$-generated. Since the same holds for $(3A, 23B, 23B)$, the group G will be $(3A, 23B, 23B)$-generated. Similarly, $\Delta^*(3A, 23A, 23B) = \Delta_G(3A, 23A, 23B) = 2714 > 0$, so that the group G becomes $(3A, 23A, 23B)$-generated. □

4.2.3 Other results

In this section we handle all the remaining cases, namely the $(5, q, r)$-, $(7, q, r)$-, $(11, q, r)$- and $(23, q, r)$-generations.

$(5,5,r)$-generations

In this subsection we discuss the case $(5,5,r)$-generations. It follows that we will end up with 5 cases, namely $(5A,5A,5A)$-, $(5A,5A,11A)$-, $(5A,5A,11B)$-, $(5A,5A,23A)$-, $(5A,5A,23B)$-generation.

Proposition 4.2.24. *The group G is $(5A,5A,5A)$-generated.*

Proof. From Table 4.4 we see that all the maximal subgroups of G have elements of order 5 except for the seventh maximal subgroup. Let T be the set of all maximal subgroups of G except the seventh one. We look at various intersections of conjugacy classes for these maximal subgroups. We have the following:

- The groups arising from the intersections of conjugacy classes for any 4, 5 or 6 maximal subgroups in T do not contain elements of order 5.
- The groups arising from intersections of the conjugacy classes for any three maximal subgroups in T having elements of order 5 are S_5 (2-copies), D_{10} and 5:4. We obtained that $\sum_{S_5}(5a,5a,5a)=8$, $\sum_{D_{10}}(5x,5y,5z)=\Delta_{D_{10}}(5a,5a,5a)+\Delta_{D_{10}}(5a,5a,5b)+\Delta_{D_{10}}(5a,5b,5b)+\Delta_{D_{10}}(5b,5b,5b)=0+1+1+0=2$ and $\sum_{5:4}(5a,5a,5a)=3$. We found that the value of h for each of these three groups is 3.
- The groups arising from intersections of the conjugacy classes for any two maximal subgroups in T having elements of order 5 are $2^4{:}S_5$ (3-copies), $PSL_3(4)$, A_7 (2-copies), $2^4{:}A_6$, $PSL_2(11)$, $A_6{:}2$, A_5 and S_5 (2-copies). We obtained that $\sum_{2^4:S_5}(5a,5a,5a)=128$, $\sum_{PSL_3(4)}(3a,5x,5y)=\Delta_{PSL_3(4)}(5a,5a,5a)+\Delta_{PSL_3(4)}(5a,5a,5b)+\Delta_{PSL_3(4)}(5a,5b,5b)+\Delta_{PSL_3(4)}(5b,5b,5b)=845+781+781+845=3252$, $\sum_{A_7}(a,5a,5a)=\Delta_{A_7}(5a,5a,5a)=108$, $\sum_{2^4:A_6}(5x,5y,5z)=\Delta_{2^4:A_6}(5a,5a,5a)+\Delta_{2^4:A_6}(5a,5a,5b)+\Delta_{2^4:A_6}(5a,5b,5b)+$

$\Delta_{2^4:A_6}(5b,5b,5b) = 320+176+176+320 = 992$, $\sum_{PSL_2(11)}(5x,5y,5z) = \Delta_{PSL_2(11)}(5a,5a,5a)+\Delta_{PSL_2(11)}(5a,5a,5b)+\Delta_{PSL_2(11)}(5a,5b,5b)+\Delta_{PSL_2(11)}(5b,5b,5b) = 20+31+31+20 = 102$, $\sum_{A_6:2}(5a,5a,5a) = 53$, $\sum_{A_5}(5x,5y,5z) = \Delta_{A_5}(5a,5a,5a) + \Delta_{A_5}(5a,5a,5b)+\Delta_{A_5}(5a,5b,5b)+\Delta_{A_5}(5b,5b,5b) = 5+1+1+5 = 12$ and $\sum_{S_5}(5a,5a,5a) = 8$. We found that the value of h for each of these eight groups is 3.

By Table A.11 we have $\Delta_G(5A,5A,5A) = 61058$. We obtained that $\sum_{K_1}(5a,5a,5a) = 18368$, $\sum_{K_2}(5a,5a,5a) = 3188$, $\sum_{K_3}(5a,5a,5a) = 1728$, $\sum_{K_4}(5a,5a,5a) = 173$, $\sum_{K_5}(5a,5a,5a) = 378$, $\sum_{K_6}(5a,5a,5a) = 128$ The value of h for each maximal subgroup is 3 except for K_4 and K_6. The value of h is 1 for each of these maximal subgroups K_4 and K_6. It follows that $\Delta^*_G(5A,5A,5A) \geq \Delta_G(5A,5A,5A) - 3\cdot\sum_{K_1}(5a,5a,5a) - 3\cdot\sum_{K_2}(5a,5a,5a) - 3\cdot\sum_{K_3}(5a,5a,5a) - \sum_{K_4}(5a,5a,5a) - 3\cdot\sum_{K_5}(5a,5a,5a) - \sum_{K_6}(5a,5a,5a) - 2\cdot3\cdot\sum_{S_5}(5a,5a,5a) - 3\cdot\sum_{D_{10}}(5x,5y,5z) - 3\cdot\sum_{5:4}(5a,5a,5a) + 3\cdot3\cdot\sum_{2^4:S_5}(5a,5a,5a) + 3\cdot\sum_{PSL_3(4)}(5x,5y,5z) + 2\cdot3\cdot\sum_{A_7}(5a,5a,5a) + 3\cdot\sum_{2^4:A_6}(5x,5y,5z) + 3\cdot\sum_{PSL_2(11)}(5x,5y,5z) + 3\cdot\sum_{A_6:2}(5a,5a,5a) + 3\cdot\sum_{A_5}(5x,5y,5z) + 2\cdot3\cdot\sum_{S_5}(5a,5a,5a) = 61058 - 3(18368) - 3(3188) - 3(1728) - 1(173) - 3(378) - 1(128) - 2(3)(11) - 3(2) - 3(3) + 3(3)(128) + 3(3252) + 2(3)(108) + 3(992) + 3(102) + 3(53) + 3(12) + 2(3)(8) = 6499 > 0$. It follows that the group G is $(5A,5A,5A)$-generated. □

Proposition 4.2.25. *The group G is $(5A,5A,7X)$-generated for $X \in \{A,B\}$.*

Proof. As in Proposition 4.2.5, we observe that the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ may have contributions here.The groups $2^3{:}PLS_3(2)$ and $PSL_3(2)$ will not have any contributions because they do not have elements of order 5. We obtained that $\sum_{PSL_3(4)}(5x,5y,7b) = \Delta_{PSL_3(4)}(5a,5a,7b) + \Delta_{PSL_3(4)}(5a,5b,7b) + \Delta_{PSL_3(4)}(5b,5b,7b) = 819+819+819 = 2457$, $\sum_{A_7}(5a,5a,7b) = 84$ and $h(7A,PSL_3(4)) = h(7A,A_7) = 2$.

The maximal subgroups K_1, K_2, K_3 and K_4 meet the $5A$, $7A$ classes of G. We obtained that

$\sum_{K_1}(5a,5a,7b) = 17920$, $\sum_{K_2}(5a,5a,7b) = 3276$, $\sum_{K_3}(5a,5a,7a) = 1344$, $\sum_{K_4}(5a,5a,7b) = 91$. We found that $h(7A,K_1) = h(7A,K_4) = 2$ and $h(7A,K_2) = h(7A,K_3) = 1$.

Since by Table A.11 we have $\Delta_G(5A,5A,7A) = 54320$, we then obtain that $\Delta^*_G(5A,5A,7A) \geq \Delta_G(5A,5A,7A) - 2\cdot\sum_{K_1}(5a,5a,7b) - \sum_{K_2}(5a,5a,7b) - \sum_{K_3}(5a,5a,7a) - 2\cdot\sum_{K_4}(5a,5a,7b) + 2\cdot\sum_{PSL_3(4)}(5x,5y,7b) + 2\cdot 2\cdot\sum_{A_7}(5a,5a,7b) = 54320 - 2(17920) - 3276 - 1344 - 2(91) + 2(2457) + 2(2)(84) = 18928 > 0$. Therefore, the group G is $(5A,5A,7A)$-generated. Since the same holds for $(5A,5A,7B)$, it follows that the group G is $(5A,5A,7X)$-generated for $X \in \{A,B\}$. □

Proposition 4.2.26. *The group G is $(5A,5A,11X)$-generated for $X \in \{A,B\}$.*

Proof. By Proposition 4.2.3 we proved that the group G is $(2A,5A,11X)$-generated for $X \in \{A,B\}$. It follows by Theorem 2.2.3 that G is $(5A,5A,(11A)^2)$- and $(5A,5A,(11B)^2)$-generated. By GAP, we see that $(11A)^2 = 11B$ and $(11B)^2 = 11A$ and the results follow. □

Proposition 4.2.27. *The group G is $(5A,5A,23X)$-generated for $X \in \{A,B\}$.*

Proof. By Proposition 4.2.4 we proved that G is $(2A,5A,23X)$-generated for $X \in \{A,B\}$. It follows by Theorem 2.2.3 that the group G is $(5A,5A,(23A)^2)$- and $(5A,5A,(23B)^2)$-generated. Since by GAP we have $(23A)^2 = 23A$ and $(23B)^2 = 23B$, then the results follow. □

$(5,7,r)$-generations

In this subsection we discuss the case $(5,7,r)$-generations. It follows that we will end up with 11 cases, namely $(5A,7A,7A)$-, $(5A,7A,7B)$-, $(5A,7A,11A)$-, $(5A,7A,11B)$-, $(5A,7A,23A)$-, $(5A,7A,23B)$-, $(5A,7B,7B)$-, $(5A,7B,11A)$-, $(5A,7B,11B)$-, $(5A,7B,23A)$-, $(5A,7B,23B)$-generation.

Proposition 4.2.28. *The group G is $(5A, 7X, 7Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. As in Proposition 4.2.5, we observe that the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ may have contributions here.The groups $2^3{:}PLS_3(2)$ and $PSL_3(2)$ will not have any contributions because they both do not have elements of order 5. We obtained that $\sum_{PSL_3(4)}(5x, 7b, 7b) = \Delta_{PSL_3(4)}(5a, 7b, 7b) + \Delta_{PSL_3(4)}(5b, 7b, 7b) = 567 + 567 = 1134$, $\sum_{A_7}(5a, 7b, 7b) = 84$ and $h(7A, PSL_3(4)) = h(7A, A_7) = 2$.

The maximal subgroups K_1, K_2, K_3 and K_4 meet the $5A$, $7A$ classes of G. We obtained that $\sum_{K_1}(5a, 7b, 7b) = 12544$, $\sum_{K_2}(5a, 7b, 7b) = 1134$, $\sum_{K_3}(5a, 7a, 7a) = 672$, $\sum_{K_4}(5a, 7b, 7b) = 189$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table A.11 we have $\Delta_G(5A, 7A, 7A) = 52584$, we then obtain that $\Delta^*_G(5A, 7A, 7A) \geq \Delta_G(5A, 7A, 7A) - 2\cdot\sum_{K_1}(5a, 7b, 7b) - \sum_{K_2}(5a, 7b, 7b) - \sum_{K_3}(5a, 7a, 7a) - 2\cdot\sum_{K_4}(5a, 7b, 7b) + 2\cdot\sum_{PSL_3(4)}(5x, 7b, 7b) + 2\cdot 2\cdot\sum_{A_7}(5a, 7b, 7b) = 52584 - 2(12544) - 1134 - 672 - 2(189) + 2(1134) + 2(2)(84) = 27916 > 0$. Therefore, the group G is $(5A, 7A, 7A)$-generated. Since the same holds for $(5A, 7B, 7B)$, it follows that the group G is $(5A, 7X, 7X)$-generated for $X \in \{A, B\}$.

We now prove that the group G is $(5A, 7A, 7B)$-generated. We obtained that $\sum_{PSL_3(4)}(5x, 7b, 7a) = \Delta_{PSL_3(4)}(5a, 7b, 7a) + \Delta_{PSL_3(4)}(5b, 7b, 7a) = 567 + 567 = 1134$, $\sum_{A_7}(5a, 7b, 7b) = 84$, $\sum_{K_1}(5a, 7b, 7a) = 12544$, $\sum_{K_2}(5a, 7b, 7a) = 1134$, $\sum_{K_3}(5a, 7a, 7b) = 672$ and $\sum_{K_4}(5a, 7b, 7a) = 189$. Since by same Table A.11 we have $\Delta_G(5A, 7A, 7B) = 52584$, we then obtain that $\Delta^*_G(5A, 7A, 7B) \geq \Delta_G(5A, 7A, 7B) - 2\cdot\sum_{K_1}(5a, 7b, 7a) - \sum_{K_2}(5a, 7b, 7a) - \sum_{K_3}(5a, 7a, 7b) - 2\cdot\sum_{K_4}(5a, 7b, 7a) + 2\cdot\sum_{PSL_3(4)}(5x, 7b, 7a) + 2\cdot 2\cdot\sum_{A_7}(5a, 7b, 7a) = 52584 - 2(12544) - 1134 - 672 - 2(189) + 2(1134) + 2(2)(84) = 27916 > 0$, proving that the group G is $(5A, 7A, 7B)$-generated. □

Proposition 4.2.29. *The group G is $(5A, 7X, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.2.3, we see that K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The groups K_5, K_7 and $PSL_2(11)$ will not have any contributions because they both do not have elements of order 7. We obtained that $\sum_{K_1}(5a, 7x, 11y) = 12672$ and $h(11Z, K_1) = 1$ for $Z \in \{A, B\}$. By Table A.11 we have $\Delta_G(5A, 7X, 11Y) = 48576$ for all $X, Y \in \{A, B\}$. We then obtain that $\Delta^*_G(5A, 7X, 11Y) \geq \Delta_G(5A, 7X, 11Y) - \sum_{K_1}(5a, 7x, 11y) = 48576 - 12672 = 35904 > 0$, so that the group G becomes $(5A, 7X, 11Y)$-generated for all $X, Y \in \{A, B\}$. □

Proposition 4.2.30. *The group G is $(5A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of orders 5 and 7. By Table A.11 we have $\Delta_G(5A, 7X, 23Y) = 44160$ for all $X, Y \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*_G(5A, 7X, 23Y) = \Delta_G(5A, 7X, 23Y) = 44160 > 0$, so that the group G is $(5A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$. □

$(5, 11, r)$-generations

In this subsection we discuss the case $(5, 11, r)$-generations. It follows that we will end up with 7 cases, namely $(5A, 11A, 11A)$-, $(5A, 11A, 11B)$-, $(5A, 11A, 23A)$-, $(5A, 11A, 23B)$, $(5A, 11B, 11B)$-, $(5A, 11B, 23A)$- and $(5A, 11B, 23B)$,-generation.

Proposition 4.2.31. *The group G is $(5A, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.2.3, we see that K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The group K_7 will not have any contributions because it does not have elements of order 5. We obtained that $\sum_{K_1}(5a, 11b, 11b) = 8448$, $\sum_{K_5}(5a, 11b, 11b) =$

198 and $\sum_{PSL_2(11)}(5x, 11b, 11b) = \Delta_{PSL_2(11)}(5a, 11b, 11b) + \Delta_{PSL_2(11)}(5b, 11b, 11b) = 11 + 11 = 22$. By Table A.11, we have $\Delta_G(5A, 11A, 11A) = 62238$. We already have $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. We then have $\Delta^*_G(5A, 11X, 11X) \geq \Delta_G(5A, 11A, 11A) - \sum_{K_1}(5a, 11b, 11b) - \sum_{K_5}(5a, 11b, 11b) + \sum_{PSL_2(11)}(5a, 11b, 11b) = 62238 - 8448 - 198 + 22 = 53614 > 0$, showing that the group G is $(5A, 11A, 11A)$-generated. Since the same holds for $(5A, 11B, 11B)$ implies that the group G is $(5A, 11A, 11A)$-generated.

For the $(5A, 11A, 11B)$-generations, we obtained that $\sum_{K_1}(5a, 11b, 11a) = 8448$, $\sum_{K_5}(5a, 11b, 11a) = 99$ and $\sum_{PSL_2(11)}(5a, 11x, 11y) = \Delta_{PSL_2(11)}(5a, 11b, 11b) + \Delta_{PSL_2(11)}(5b, 11b, 11b) = 11 + 11 = 22$. Since by Table A.11 we have $\Delta_G(5A, 11A, 11B) = 61479$, we obtain that $\Delta^*_G(5A, 11A, 11B) \geq \Delta_G(5A, 11A, 11B) - \sum_{K_1}(5a, 11b, 11a) - \sum_{K_5}(5a, 11b, 11a) + \sum_{PSL_2(11)}(5a, 11b, 11a) = 61479 - 8448 - 99 + 22 = 52954 > 0$, proving that the group G is $(5A, 11A, 11B)$-generated. □

Proposition 4.2.32. *The group G is $(5A, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 5. By Table A.11 we have $\Delta_G(5A, 11X, 23Y) = 61893$ for all $X, Y \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*_G(5A, 11X, 23Y) = \Delta_G(5A, 11X, 23Y) = 61893 > 0$, so that G is $(5A, 11X, 23Y)$-generated group for all $X, Y \in \{A, B\}$. □

$(5, 23, r)$-generations

In this subsection we discuss the case $(5, 23, r)$-generations. It follows that we will end up with 3 cases, namely $(5A, 23A, 23A)$-, $(5A, 23A, 23B)$- and $(5A, 23B, 23B)$-generation.

Proposition 4.2.33. *The group G is $(5A, 23X, 23Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 5. By Table A.11 we have $\Delta_G(5A, 23X, 23Y) = 32706$ for all $X, Y \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*_G(5A, 23X, 23Y) = \Delta_G(5A, 23X, 23Y) = 32706 > 0$, so that the group G is $(5A, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$. □

$(7, 7, r)$-generations

In this subsection we discuss the case $(7, 7, r)$-generations. It follows that we will end up with 16 cases, namely $(7A, 7A, 7A)$-, $(7A, 7A, 7B)$-, $(7A, 7A, 11A)$-, $(7A, 7A, 11B)$, $(7A, 7A, 23A)$-, $(7A, 7A, 23B)$-, $(7A, 7B, 7B)$-, $(7A, 7B, 11A)$-, $(7A, 7B, 11B)$, $(7A, 7B, 23A)$-, $(7A, 7B, 23B)$, $(7B, 7B, 7B)$-, $(7B, 7B, 11A)$-, $(7B, 7B, 11B)$-, $(7B, 7B, 23A)$- and $(7B, 7B, 23B)$-generation.

Proposition 4.2.34. *The group G is $(7X, 7Y, 7Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.2.5 we proved that G is $(2A, 7A, 7X)$-generated for $X \in \{A, B\}$. It follows by Theorem 2.2.3 that the group G is $(7A, 7A, (7A)^2)$- and $(7A, 7A, (B)^2)$-generated. Since by the power maps, we have $(7A)^2 = 7A$ and $(7B)^2 = 7B$, the group G becomes $(7A, 7A, 7A)$- and $(7A, 7A, 7B)$-generated. Since G is $(7A, 7A, 7A)$-generated, the same will hold for $(7B, 7B, 7B)$.

We are left only to investigate of the $(7A, 7B, 7B)$ generation for the group G. As in Proposition 4.2.5, we observe that the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ have contributions. We obtained that $\sum_{PSL_3(4)}(7b, 7a, 7a) = 357$, $\sum_{A_7}(7b, 7a, 7a) = 36$, $\sum_{2^3:PSL_3(2)}(7b, 7a, 7a) = 8$, $\sum_{PSL_3(2)}(7a, 7b, 7b) = 1$ and $h(7A, PSL_3(4)) = h(7A, A_7) = h(7A, 2^3{:}PSL_3(2)) = h(7A, PSL_3(2)) = 2$.

The maximal subgroups K_1, K_2, K_3 and K_4 have elements of order 7. We obtained that $\sum_{K_1}(7b, 7a, 7a) = 8576$, $\sum_{K_2}(7b, 7a, 7a) = 379$, $\sum_{K_3}(7a, 7b, 7b) = 148$, $\sum_{K_4}(7b, 7a, 7a) = 379$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table A.11 we have $\Delta_G(7A, 7B, 7B) = 51948$, we then obtain that $\Delta^*_G(7A, 7B, 7B) \geq \Delta_G(7A, 7B, 7B) - 2\cdot\sum_{K_1}(7b, 7a, 7a) - \sum_{K_2}(7b, 7a, 7a) - \sum_{K_3}(7a, 7b, 7b) - 2\cdot\sum_{K_4}(7b, 7a, 7a) + 2\cdot\sum_{PSL_3(4)}(7b, 7a, 7a) + 2\cdot2\cdot\sum_{A_7}(7b, 7a, 7a) + 2\cdot2\cdot\sum_{2^3:PSL_3(2)}(7b, 7a, 7a) + 2\cdot\sum_{PSL_3(2)}(7a, 7b, 7b) = 51948 - 2(8576) - 379 - 148 - 2(379) + 2(379) + 2(2)(36) + 2(2)(8) + 2(1) = 34447 > 0$. Therefore, the group G is $(7A, 7B, 7B)$-generated. □

Proposition 4.2.35. *The group G is $(7X, 7Y, 11Z)$-generated for $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.2.6 we have proved that G is $(2A, 7X, 11Y)$-generated for all $X, Y \in \{A, B\}$. It follows by Theorem 2.2.3 that G is $(7X, 7X, (11Y)^2)$-generated. It follows that G is $(7X, 7X, (11A)^2)$- and $(7X, 7X, (11B)^2)$-generated for $X \in \{A, B\}$. Since by the power maps we have $(11A)^2 = 11B$ and $(11B)^2 = 11A$, it then follows that G is $(7X, 7X, 11B)$- and $(7X, 7X, 11A)$-generated group for $X \in \{A, B\}$.

We investigate the $(7A, 7B, 11X)$ generations of G, where $X \in \{A, B\}$. Looking at Proposition 4.2.3, we see that K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The groups K_5, K_7 and $PSL_2(11)$ will not have any contributions because they both do not have elements of order 7. We obtained that $\sum_{K_1}(7b, 7a, 11x) = 9856$ for $x \in \{a, b\}$. We already have $h(11X, K_1) = 1$ for $X \in \{A, B\}$. Since by Table A.11 we have $\Delta_G(7A, 7B, 11X) = 56496$ for $X \in \{A, B\}$, we then obtain that $\Delta^*_G(7A, 7B, 11X) \geq \Delta_G(7A, 7B, 11X) - \sum_{K_1}(7b, 7a, 11x) = 56496 - 9856 = 46640 > 0$, proving G is $(7A, 7B, 11X)$-generated for $X \in \{A, B\}$. □

Proposition 4.2.36. *The group G is $(7X, 7Y, 23Z)$-generated for $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.2.7 we have proved that G is $(2A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$. It follows by Theorem 2.2.3 that G is $(7X, 7X, (23Y)^2)$-generated. Since $(23A)^2 = 23A$ and $(23B)^2 = 23B$ then it follows that G is $(7X, 7X, 23A)$- and $(7X, 7X, 23B)$-generated for $X \in \{A, B\}$.

We prove that G is $(7A, 7B, 23X)$-generated for $X \in \{A, B\}$. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 7. By Table A.11 we have $\Delta_G(7A, 7B, 23X) = 45264$ for $X \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*(7A, 7B, 23X) = \Delta_G(7A, 7B, 23X) = 45264 > 0$, so that G is $(7A, 7B, 23X)$-generated group for $X \in \{A, B\}$. □

$(7, 11, r)$- and $(7, 23, 23)$-generations

In this subsection we discuss the cases $(7, 11, r)$- and $(7, 23, r)$-generations. It follows that we will end up with 20 cases, namely $(7A, 11A, 11A)$-, $(7A, 11A, 11B)$-, $(7A, 11A, 23A)$-, $(7A, 11A, 23B)$-, $(7A, 11B, 11B)$-, $(7A, 11B, 23A)$-, $(7A, 11B, 23B)$-, $(7B, 11A, 11A)$-, $(7B, 11A, 11B)$-, $(7B, 11A, 23A)$-, $(7B, 11A, 23B)$-, $(7B, 11B, 11B)$-, $(7B, 11B, 23A)$-, $(7B, 11B, 23B)$-, $(7A, 23A, 23A)$-, $(7A, 23A, 23B)$-, $(7A, 23B, 23B)$-$(7B, 23A, 23A)$-, $(7B, 23A, 23B)$- and $(7B, 23B, 23B)$-generation.

Proposition 4.2.37. *The group G is $(7X, 11Y, 11Z)$-generated for $X, Y, Z \in \{A, B\}$.*

Proof. Looking at Proposition 4.2.3, we see that K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The groups K_5, K_7 and $PSL_2(11)$ will not have any contributions because they both do not have elements of order 7. We obtained that $\sum_{K_1}(7X, 11Y, 11Z) = 5632$ and $h(11Z, K_1) = 1$ for all $X, Y, Z \in \{A, B\}$. By Table A.11 we have $\Delta_G(7X, 11Y, 11Z) =$

64416. We then obtained that $\Delta_G^*(7X, 11Y, 11Z) \geq \Delta_G(7X, 11Y, 11Z) - \sum_{K_1}(7X, 11Y, 11Z) = 64416 - 5632 = 58784 > 0$ and so G is $(7X, 11Y, 11Z)$-generated group for all $X, Y, Z \in \{A, B\}$. $\square$

Proposition 4.2.38. *The group G is $(7X, 11Y, 23Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 7. By Table A.11 we have $\Delta_G(7X, 11Y, 23Z) = 67712$ for all $X, Y, Z \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*(7X, 11Y, 23Z) = \Delta_G(7X, 11Y, 23Z) = 67712 > 0$, so that G is $(7X, 11Y, 23Z)$-generated group for all $X, Y, Z \in \{A, B\}$. $\square$

Proposition 4.2.39. *The group G is $(7X, 23Y, 23Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 7. By Table A.11 we have $\Delta_G(7X, 23Y, 23Z) = 32384$ for all $X, Y, Z \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*(7X, 23Y, 23Z) = \Delta_G(7X, 23Y, 23Z) = 32384 > 0$, so that the group G is $(7X, 23Y, 23Z)$-generated for all $X, Y, Z \in \{A, B\}$. $\square$

$(11, 11, r)$-generations

In this subsection we discuss the case $(11, 11, r)$-generations. It follows that we will end up with 10 cases, namely $(11A, 11A, 11A)$-, $(11A, 11A, 11B)$-, $(11A, 11A, 23A)$-, $(11A, 11A, 23B)$-, $(11A, 11B, 11B)$-, $(11A, 11B, 23A)$-, $(11A, 11B, 23B)$-, $(11B, 11B, 11B)$-, $(11B, 11B, 23A)$- and $(11B, 11B, 23B)$-generation.

Proposition 4.2.40. *The group G is $(11X, 11Y, 11Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.2.8 we have proved that G is $(2A, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$. Then by Theorem 2.2.3 it follows that G is $(11X, 11X, (11Y)^2)$-generated for all $X, Y, Z \in \{A, B\}$. Since $(11A)^2 = 11B$ and $(11B)^2) = 11A$ then it follows that G is $(11X, 11X, 11Y)$-generated for all $X, Y, Z \in \{A, B\}$.

We prove that G is $(11A, 11B, 11B)$-generated. Looking at Proposition 4.2.3, we see that K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The maximal subgroup K_7 have its relevant structure constant zero, so it will not have any contributions. We obtained that $\sum_{K_1}(11b, 11a, 11a) = 3632$, $\sum_{K_5}(11b, 11a, 11a) = 35$ and $\sum_{PSL_2(11)}(11b, 11a, 11a) = 2$. We have found that $h(11B, K_1) = h(11B, K_5) = h(11B, PSL_2(11)) = 1$. By Table A.12 we have $\Delta_G(11A, 11B, 11B) = 87485$, we then obtain $\Delta_G^*(11A, 11B, 11B) \geq \Delta_G(11A, 11B, 11B) - \sum_{K_1}(11b, 11a, 11a) - \sum_{K_5}(11b, 11a, 11a) + \sum_{PSL_2(11)}(11b, 11a, 11a) = 87485 - 3632 - 35 + 2 = 83820 > 0$, proving that the group G is $(11A, 11B, 11B)$-generated. □

Proposition 4.2.41. *The group G is $(11X, 11Y, 23Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.2.9 we have proved that G is $(2A, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$. Then by Theorem 2.2.3 it follows that G is $(11X, 11X, (23Y)^2)$-generated for all $X, Y \in \{A, B\}$. Since $(23A)^2 = 23A$ and $(23B)^2 = 23B$ we then obtained that G is $(11X, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$.

We still have to prove the $(11A, 11B, 23X)$-generations where $X \in \{A, B\}$. By Table 4.4 we see that K_7 is the only maximal subgroup having elements of orders 11 and 23. We then obtain that $\sum_{K_7}(11x, 11y, 23z) = \Delta_{K_7}(11a, 11j, 23z) + \Delta_{K_7}(11c, 11h, 23z) + \Delta_{K_7}(11d, 11g, 23z) + \Delta_{K_7}(11e, 11f, 23z) + \Delta_{K_7}(11i, 11b, 23x) = 23 + 23 + 23 + 23 + 23 = 115$ for $z \in \{a, b\}$. We have found that $h(23X, K_7) = 1$ for $X \in \{A, B\}$. Since by Table A.12 we have $\Delta_G(11A, 11B, 23X) = 79994$, then we obtained that $\Delta_G^*(11A, 11B, 23X) \geq \Delta_G(11A, 11B, 23Z) - \sum_{K_7}(11x, 11y, 23z) =$

$79994 - 115 = 79879 > 0$ for all $Z \in \{A, B\}$. Hence G is $(11A, 11B, 23X)$-generated group for $X \in \{A, B\}$. $\square$

$(11, 23, r)$**-generations**

We will be looking at the cases $(11A, 23A, 23A)$-, $(11A, 23A, 23B)$-, $(11A, 23B, 23B)$-, $(11B, 23A, 23A)$-, $(11B, 23A, 23B)$- and $(11B, 23B, 23B)$-generation.

Proposition 4.2.42. *The group G is $(11X, 23Y, 23Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of orders 11 and 23. This maximal subgroup K_7 will not have any contributions because its relevant structure constants are all zero. By Table A.12 we have $\Delta_G(11X, 23Y, 23Z) = 42067$ for all $X, Y, Z \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*_G(11X, 23Y, 23Z) = \Delta_G(11X, 23Y, 23Z) = 42067 > 0$, showing that G is $(11X, 23Y, 23Z)$-generated group for all $X, Y, Z \in \{A, B\}$. $\square$

$(23, 23, r)$**-generations**

We conclude our investigation on the (p, q, r)-generations of the Mathieu sporadic simple group G by considering the $(23, 23, 23)$-generations. We will be looking at the cases $(23A, 23A, 23A)$-, $(23A, 23A, 23B)$-, $(23A, 23B, 23B)$- and $(23B, 23B, 23B)$-generation.

Proposition 4.2.43. *The group G is $(23X, 23Y, 23Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.2.10 we have proved that G is $(2A, 23X, 23Y)$-generated for all $X, Y \in \{A, B\}$. Then by Theorem 2.2.3 it follows that G is $(23X, 23X, (23Y)^2)$-generated for all $X, Y \in$

$\{A, B\}$. Since $(23A)^2 = 23A$ and $(23B)^2 = 23B$ then it follows that G is $(23X, 23X, 23Y)$-generated for all $X, Y \in \{A, B\}$. We now check the $(23A, 23B, 23B)$-generation of G. By Table 4.4, K_7 is the only maximal subgroup having elements of order 11. We obtained that $\sum_{K_7}(23a, 23b, 23b) = 5$ and $h(23B, K_7) = 1$. Since by Table A.12 we have $\Delta_G(23A, 23B, 23B) = 17646$, then we obtained that $\Delta^*_G(23A, 23B, 23B) \geq \Delta_G(23A, 23B, 23B) - \sum_{K_7}(23a, 23b, 23b) = 17646 - 5 = 17641 > 0$. Hence the group G is $(23A, 23B, 23B)$-generated. □

4.3. The conjugacy class ranks of the sporadic simple group M_{23}.

Now we study the ranks of M_{23} with respect to the various conjugacy classes of all its non-identity elements.

Proposition 4.3.1. *$rank(G : 2A) = 3$.*

Proof. The rank of any involution class will be at least 3. Thus, $rank(G : 2A) \neq 2$. Direct computation shows that the structure constant $\Delta_G(2A, 2A, 2A, 23A) = 3174$. More information on direct computation see [5, Lemma 4] and [15, Remark 1]. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 2. Since there is no contributing group, we then obtain that $\Delta^*_G(2A, 2A, 2A, 23A) \geq 3174$, proving that G is $(2A, 2A, 2A, 23A)$-generated. Hence the result follows. □

Proposition 4.3.2. *Let $nX \in T := \{3A, 4A, 5A, 6A, 7A, 7B, 8A, 14A, 14B, 15A, 15B, 23A, 23B\}$ then $rank(G : nX) = 2$.*

Proof. By Table 4.4, K_7 is the only maximal subgroup having elements of order 23. We use Table 4.5 in getting the results of this Proposition. In the same Table 4.5 we give required information needed to calculate the minimum value of $\Delta_G^*(nX, nX, 23A)$ where $nX \in T$. Since $\Delta_G^*(nX, nX, 23A) > 0$, it follows that G is $(nX, nX, 23A)$-generated where $nX \in T$. This proves that $rank(G : nX) = 2$ for all $nX \in T$. □

Proposition 4.3.3. *$rank(G : 11A) = 2 = rank(G : 11B)$.*

Proof. We have proved in Proposition 4.2.9 that the group G is $(2A, 11X, 23A)$-generated for all $X \in \{A, B\}$. By applying Theorem 2.2.3, we see that the group G is $(11A, 11A, (23A)^2)$-generated. Since $(23A)^2 = 23A$, the group G becomes $(11A, 11A, 23A)$-generated. Since the same follows for $(11B, 11B, 23A)$, we then have $rank(G : 11X) = 2$ for $X \in \{A, B\}$. □

The following Table 4.5 gives information on partial structure contants of G computed using GAP and the relevant information required to calculate $\Theta_G(nX, nX, 23A)$. We give some information about $\Delta_G(nX, nX, 23A)$, $h(23A, K_7)$ and $\sum_{K_7}(nx, nx, 23b)$. The last column $\Theta_G(nX, nX, 23A) = \Delta_G(nX, nX, 23A) - h\sum_{K_7}(nx, nx, 23b)$ establishes each generation of G by its triples $(nX, nX, 23A)$ because $\Delta_G^*(nX, nX, 23A) \geq \Theta_G(nX, nX, 23A)$, that is $\Delta_G^*(nX, nX, 23A) > 0$ then the group G is $(nX, nX, 23A)$-generated.

The rank for each conjugacy class of elements for the sporadic simple group M_{23} will be summarized as follows:

- $rank(G : 2A) = 3$, the result follows by 4.3.1.
- Let $nX \in T := \{3A, 4A, 5A, 6A, 7A, 7B, 8A, 11A, 11B, 14A, 14B, 15A, 15B, 23A, 23B\}$ then $rank(G : nX) = 2$. The results follow by the proofs of Propositions 4.3.2 and 4.3.3.

Table 4.5: Some information on the $nX \in T$

nX	$\Delta_G(nX, nX, 23A)$	$h(23A, K_7)$	$h\sum_{K_7}(nx, nx, 23b)$	$\Theta_G(nX, nX, 23A)$
$3A$	138	1	-	138
$4A$	7866	1	-	7866
$5A$	37582	1	-	37582
$6A$	72588	1	-	72588
$7A$	52992	1	-	52992
$7B$	52992	1	-	52992
$8A$	154376	1	-	154376
$14A$	52992	1	-	52992
$14B$	52992	1	-	52992
$15A$	41998	1	-	41998
$15B$	41998	1	-	41998
$23A$	17646	1	5	17641
$23B$	17646	1	6	17640

CHAPTER 5

The alternating group A_{11}

In this chapter, we will establish all the (p, q, r)-generations together with the ranks of the conjugacy classes of the alternating group A_{11}. The result on the (p, q, r)-generations of A_{11} can be summarized in the following theorem.

Theorem 5.0.1. *With the notation being as in the Atlas [20], the alternating group A_{11} is generated by all the triples (lX, mY, nZ), l, m and n primes dividing $|A_{11}|$, except for the cases* $(lX, mY, nZ) \in \{(2M, 3V, 7A),\ (2M, 3N, 11O),\ (2A, 5B, 5B),\ (2M, 5A, 5N),\ (2M, 5N, 7A),$ $(2M, 5A, 11N),\ (2M, 7A, 7A),\ (2A, 7A, 11M),\ (3V, 3W, 5M),\ (3V, 3W, 7A),\ (3A, 3V, 11M),$ $(3B, 3B, 11M),\ (3V, 5A, 5M),\ (3A, 5B, 5B),\ (3M, 5N, 7A),\ (3C, 5A, 7A),\ (3M, 5A, 11N),$ $(3V, 7A, 7A),\ (3A, 7A, 11M),\ (5A, 5A, 5M),\ (5A, 5M, 7A),\ (5A, 5A, 11M),\ (5A, 7A, 7A)\}$, *for all* $M, N, O \in \{A, B\}$ *and* $V, W \in \{A, B, C\}$.

The main result on the ranks in this thesis can summarized by Theorem 5.0.2 as follows.

Theorem 5.0.2. *For the alternating group G, we have*

(i) $rank(G : 2A) = rank(G : 3A) = 5$,

(ii) $rank(G : 2B) = rank(G : 3B) = rank(G : 4A) = rank(G : 5A) = rank(G : 6B) = 3$,

(iii) $rank(G : nX) = 2$ *if* $nX \notin \{1A, 2A, 2B, 3A, 3B, 4A, 5A, 6B\}$, *where nX is a conjugacy*

class of G.

5.1. Introduction

The *Symmetric group* denoted by S_X is defined to be the group of all permutations $\rho{:}X \rightarrow X$, where X is a non-empty set. Suppose $|X| = n$, thus S_X will be denoted by S_n. In general, for $n \geq 2$, the number of even permutations in the symmetric group S_n is the same as number of odd permutations. So, S_n splits equally into odd and even permutations. It is well known that an alternating group A_n is a normal subgroup of S_n. The group A_{11} is defined to be the alternating group of degree 11. The group A_{11} is a normal subgroup of S_{11}, the symmetric group on a set of size 11. The Schur multiplier and the outer automorphism group of A_{11} are both 2.

The group A_{11} is a simple group of order $19958400 = 2^7 \times 3^4 \times 5^2 \times 7 \times 11$. By the Atlas of finite groups [20], the group A_{11} has exactly 31 conjugacy classes of its elements and 7 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups can be taken as follows:

$M_1 = A_{10}$ $\qquad M_2 = S_9$ $\qquad M_3 = (A_8 \times 3){:}2$

$M_4 = (A_7 \times A_4){:}2$ $\qquad M_5 = (A_6 \times A_5){:}2$ $\qquad M_6 = M_{11}$

$M_7 = M_{11}$.

Throughout this Chapter, by G we mean the alternating group A_{11}, unless stated otherwise. From the electronic Atlas of finite group representations [55], we see that G can be generated in terms of permutations on 11 points. Generators g_1 and g_2 can be taken as follows:

$$g_1 = (1, 2, 3)$$
$$g_2 = (3, 4, 5, 6, 7, 8, 9, 10, 11),$$

with $o(g_1) = 3,\ o(g_2) = 9$ and $o(g_1g_2) = 11$.

In Table 5.1, we list the values of the cyclic structure for each conjugacy of G together with the values of both c_i and d_i obtained from Ree and Scotts theorems, respectively.

Table 5.2 gives the partial fusion maps of classes of maximal subgroups into the classes of G. These will be used in our computations.

In Table 5.3, we have the order of each maximal subgroup, listed the representatives of classes of the maximal subgroups together with the orbits lengths of G on these groups and the permutation characters.

Table 5.1: Cycle structures of conjugacy classes of G

nX	Cycle Structure	c_i	d_i
$2A$	$1^7 2^2$	9	2
$2B$	$1^3 2^4$	7	4
$3A$	$1^8 3^1$	9	2
$3B$	$1^5 3^2$	7	4
$3C$	$1^2 3^3$	5	6
$4A$	$1^5 2\ 3$	7	4
$4B$	$1^2 4^2$	5	6
$4C$	$1^1 2^3 4^1$	5	6
$5A$	$1^6 5^1$	7	4
$5B$	$1^1 5^2$	3	8
$6A$	$1^4 3^1$	5	6
$6B$	$1^4 2^2 3^1$	7	4
$6C$	$1^2 2^2 3^2$	5	6
$6D$	$1^3 2^1 6^1$	5	6
$6E$	$2^1 3^1 6^1$	3	8
$7A$	$1^4 7^1$	5	6
$8A$	$1^2 2^1 8^1$	3	8
$9A$	$1^2 9^1$	3	8
10	$1^2 2^2 5^1$	5	6
$11A$	11^1	1	10
$11B$	11^1	1	10
$12A$	$3^1 4^2$	3	8
$12B$	$1^2 2^1 3^1 4^1$	5	6
$12C$	$1^1 4^1 6^1$	3	8
$14A$	$2^2 7^1$	3	8

Table 5.1 continued

nX	Cycle Structure	c_i	d_i
$15A$	$1^3 3^1 5^1$	5	6
$15B$	$3^2 5^1$	5	6
$20A$	$2^1 4^1 5^1$	3	8
$20B$	$2^1 4^1 5^1$	3	8
$21A$	$1^1 3^1 5^1$	3	8
$21B$	$1^1 3^1 5^1$	3	8

Table 5.2: The partial fusion maps into G

M_1-class	2a	2b	3a	3b	3c	5a	5b	7a						
$\rightarrow$ G	2A	2B	3A	3B	3C	5A	5B	7A						
h						6	1	4						
M_2-class	2a	2b	2c	2d	3a	3b	3c	5a	7a					
$\rightarrow$ G	2A	2A	2B	2B	3A	3C	3B	5A	7A					
h								15	6					
M_3-class	2a	2b	2c	2d	3a	3b	3c	3d	3e	5a	7a			
$\rightarrow$ G	2A	2B	2A	2B	3A	3A	3B	3B	3C	5A	7A			
h										20	4			
M_4-class	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e	5a	7a		
$\rightarrow$ G	2A	2A	2A	2B	2B	3A	3A	3B	3B	3C	5A	7A		
h											15	1		
M_5-class	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e	5a	5b	5c	5d
$\rightarrow$ G	2A	2A	2A	2B	2B	3A	3A	3B	3B	3C	5A	5A	5B	5B
h											1	6	1	1
M_6-class	2a	3a	5a	11a	11b									
$\rightarrow$ G	2B	3C	5B	11A	11B									
h			5	1	1									
M_7-class	2a	3a	5a	11a	11b									
$\rightarrow$ G	2B	3C	5B	11A	11B									
h			5	1	1									

Table 5.3: Maximal subgroups of G

Maximal Subgroup	Order	Orbit Lengths	Character
M_1	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	[1,10]	$1a + 10a$
M_2	$2^7 \cdot 3^4 \cdot 5 \cdot 7$	[2,9]	$1a + 10a + 44a$
M_3	$2^7 \cdot 3^3 \cdot 5 \cdot 7$	[3,8]	$1a + 10a + 44a + 110a$
M_4	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	[7,4]	$1a + 10a + 44a + 110a + 165a$
M_5	$2^6 \cdot 3^3 \cdot 5^2$	[5,6]	$1a + 10a + 44a + 110a$ +132a + $165a$

Table 5.3 continued

Maximal Subgroup	Order	Orbit Lengths	Character
M_6	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	[11]	$1a + 132a + 462a + 825a + 1100a$
M_7	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	[11]	$1a + 132a + 462a + 825a + 1100a$

5.2. (p, q, r)-generations of A_{11}

Let tX, $p \in \{2, 3, 5, 7, 11\}$ be a conjugacy class of G and c_i be the number of disjoint cycles in a representative of tX. The group G is not $(2Y, 2Z, tX)$-generated, for if G is $(2Y, 2Z, tX)$-generated, then G is a dihedral group and thus is not simple for all $Y, Z \in \{A, B\}$. Also we know that if G is (lX, mY, nZ)-generated with $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \geq 1$ and G is simple, then $G \cong A_5$, but $G \cong A_{11}$ and $A_{11} \ncong A_5$. Hence if G is (lX, mY, nZ)-generated, then we must have $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.

5.2.1 $(2, q, r)$-generations

Now the $(2, q, r)$-generations of G comprises the cases $(2, 3, r)$-, $(2, 5, r)$-, $(2, 7, r)$- and $(2, 11, r)$-generations.

$(2, 3, r)$-generations

The condition $\frac{1}{2} + \frac{1}{3} + \frac{1}{r} < 1$ shows that we must have $r > 6$. Thus we have to consider the cases $(2X, 3Y, 7A)$ and $(2X, 3Y, 11Z)$ for all $X, Z \in \{A, B\}$ and $Y \in \{A, B, C\}$.

Proposition 5.2.1. *The group G is not $(2X, 3Y, 7A)$-generated where $X \in \{A, B\}$, $Y \in \{A, B, C\}$.*

Proof. If the group G is $(2X, 3Y, 7A)$-generated then we must have $c_{2X} + c_{3Y} + c_{7A} \leq 13$ where

$X \in \{A, B\}$ and $Y \in \{A, B, C\}$. By Table 5.1 we see that $c_{2X} \in \{7, 9\}$ and $c_{3Y} \in \{5, 7, 9\}$, it follows that $c_{2X} + c_{3Y} + c_{7A} = c_{2X} + c_{3X} + 5 > 13$, for $X \in \{A, B\}$ and $Y \in \{A, B, C\}$. Now using Ree's Theorem [49], it follows that G is not $(2X, 3Y, 7A)$-generated where $X \in \{A, B\}$ and $Y \in \{A, B, C\}$. □

Proposition 5.2.2. *The group G is*

(i) neither $(2X, 3Y, 11Z)$*- nor* $(2A, 3C, 11Z)$*-generated for all* $X, Y, Z \in \{A, B\}$*,*

(ii) $(2B, 3C, 11X)$*-generated for* $X \in \{A, B\}$*.*

Proof. (i) Since by Table A.13, we have $\Delta_G(2A, 3A, 11X) = \Delta_G(2A, 3B, 11X) = \Delta_G(2A, 3C, 11X) = \Delta_G(2B, 3A, 11X) = \Delta_G(2B, 3B, 11X) = 0$, Lemma 2.1.3 implies that the group G is neither $(2X, 3Y, 11Z)$- nor $(2A, 3C, 11Z)$-generated for all $X, Y, Z \in \{A, B\}$.

(ii) From Table 5.2 we see H_6 (or H_7) (two non-conjugate copies) is the only maximal subgroup containing elements of orders 2, 3 and 11. The non-empty intersection of the conjugacy classes for H_6 with the conjugacy classes for H_7 which has elements of order 11 is isomorphic to the group 11:5. This subgroup 11:5 of G has no elements of orders 2 and 3, as such it will not have any contributions here. We obtained that $\sum_{H_6}(2a, 3a, 11x) = 11$ and $h(11X, H_6) = 1$ (see [31, 58]). Since by Table A.13 we have $\Delta_G(2B, 3C, 11X) = 110$, we then obtained that $\Delta^*_G(2B, 3C, 11X) \geq \Delta_G(2B, 3C, 11X) - \sum_{H_6}(2a, 3a, 11x) - \sum_{H_7}(2a, 3a, 11x) = 110 - 11 - 11 = 88 > 0$ for $X \in \{A, B\}$. This proves that the group G is $(2B, 3C, 11X)$-generated for $X \in \{A, B\}$, proving (ii). □

$(2, 5, r)$-generations

The condition $\frac{1}{2} + \frac{1}{5} + \frac{1}{r} < 1$ shows that we must have $r > \frac{10}{3}$. We have to consider the following the cases $(2X, 5Y, 5Z)$, $(2X, 5Y, 7A)$ and $(2X, 5Y, 11Z)$ for all $X, Y, Z \in \{A, B\}$.

Proposition 5.2.3. *The group G is*

(i) neither $(2A, 5B, 5B)$- nor $(2X, 5A, 5Y)$-generated for all $X, Y \in \{A, B\}$,

(ii) $(2B, 5B, 5B)$-generated.

Proof. (i) If G is $(2A, 5B, 5B)$-generated group, then we must have $c_{2A} + c_{5B} + c_{5B} \leq 13$. For $r \in \{5A, 5B\}$, then by Table 5.1 we have $c_r \in \{3, 7\}$ and it follows that $c_{2A} + c_{5A} + c_r = 9 + 7 + c_r > 13$, $c_{2A} + c_{5B} + c_r = 9 + 3 + c_r > 13$ and $c_{2B} + c_{5A} + c_r = 7 + 7 + c_r > 13$. Now using Ree's Theorem [49], it follows that G is not $(2A, 5B, 5B)$-generated. Same applies to $(2X, 5A, 5Y)$ for all $X, Y \in \{A, B\}$. Thus G is neither $(2A, 5B, 5B)$- nor $(2X, 5A, 5Y)$-generated for all $X, Y \in \{A, B\}$, proving (i).

(ii) Looking at Table 5.2 we see that all the maximal subgroups of G have elements of order 5. Let T be the set of all maximal subgroups of G. We look at various non-empty intersections of conjugacy classes for these maximal subgroups. We have the following:

- The subgroups arising from the non-empty intersections of conjugacy classes for any 6 or 7 maximal subgroups in T do not contain elements of order 5.
- The subgroups arising from non-empty intersections of the conjugacy classes for any 5 maximal subgroups in T having elements of order 5 are S_5 and A_6. Both subgroups S_5 and A_6 will not have any contributions because none of their elements of order 5 fuse to the class $5B$ of the group G.
- The subgroups arising from non-empty intersections of the conjugacy classes for any four maximal subgroups in T having elements of order 5 are S_7, $2 \times S_6$, $2 \times S_5$, $A_6{:}2$, A_6 and A_7. No contributions from any of these subgroups of G because they both do not meet the classes $2B$ and $5B$ of G.

- The subgroups arising from non-empty intersections of the conjugacy classes for any three maximal subgroups in T having elements of order 5 are S_7, $2 \times S_6$, $2 \times S_5$, $A_6{:}2$, A_6, $A_6{:}S_3$ and A_8. No contributions from any of these subgroups of G because they both do not meet the classes $2B$ and $5B$ of G.

- The subgroups arising from non-empty intersections of the conjugacy classes for any two maximal subgroups in T having elements of order 5 are S_8, $2 \times S_6$, $A_6{:}S_4$, A_9, S_7, $A_6{:}S_3$, $A_5{:}S_5$, $A_6{:}2$, $S_4 \times S_5$, S_5, $S_6 \times S_3$, $3{:}S_7$, $2 \times S_7$, $11{:}5$ and $4{:}5$. Only S_5 and $4{:}5$ meet the classes $2B$ and $5B$ of G. Although $4{:}5$ meets the classes $2B$ and $5B$ of G, it will not have any contributions because its relevant structure constant is zero.

By Table A.13 we have $\Delta_G(2B, 5B, 5B) = 825$. We obtained that $\sum_{S_5}(2x, 5a, 5a) = \Delta_{S_5}(2a, 5a, 5a) + \Delta_{M_5}(2b, 5a, 5a) = 5 + 0 = 5$ and $h(5B, S_5) = 5$. We see that M_1, M_5 and M_6 (or M_7) are the only maximal subgroups having their elements of orders 2 and 5 fusing to respective classes $2B$ and $5B$ of the group G. We also obtained that $\sum_{M_1}(2b, 5b, 5b) = 225$, $\sum_{M_5}(2x, 5y, 5z) = \Delta_{M_5}(2d, 5c, 5c) + \Delta_{M_5}(2d, 5c, 5d) + \Delta_{M_5}(2d, 5d, 5d) + \Delta_{M_5}(2e, 5c, 5c) + \Delta_{M_5}(2e, 5c, 5d) + \Delta_{M_5}(2e, 5d, 5d) = 0 + 0 + 0 + 50 + 50 + 50 = 150$ and $\sum_{M_6}(2a, 5a, 5a) = 45$ (or $\sum_{M_7}(2a, 5a, 5a) = 45$). We found that $h(5B, M_1) = 1 = h(5B, M_5)$ and $h(5B, M_6) = 5$ ($h(5B, M_7) = 5$). It follows that $\Delta^*_G(2B, 5B, 5B) \geq \Delta_G(2B, 5B, 5B) - \sum_{M_1}(2b, 5b, 5b) - \sum_{M_5}(2x, 5y, 5z) - 5 \cdot \sum_{M_6}(2a, 5a, 5a) - 5 \cdot \sum_{M_7}(2a, 5a, 5a) + 5 \cdot \sum_{S_5}(2x, 5a, 5a) = 825 - 1(225) - 1(150) - 5(45) - 5(45) + 5(5) = 25 > 0$, proving that $(2B, 5B, 5B)$ is a generating triple for the group G. □

Proposition 5.2.4. *The group G is not $(2X, 5Y, 7A)$-generated for all $X, Y \in \{A, B\}$.*

Proof. If G is $(2X, 5Y, 7A)$-generated group, then we must have $c_{2X} + c_{5Y} + c_{7A} \leq 13$ for all $X, Y \in \{A, B\}$. From Table 5.1 we see that $c_{2A} + c_{5A} + c_{7A} = 9 + 7 + 5 > 13$, $c_{2A} + c_{5B} + c_{7A} =$

$9+3+5>13$, $c_{2B}+c_{5A}+c_{7A}=7+7+5>13$ and $c_{2B}+c_{5B}+c_{7A}=7+3+5>13$. It follows by Ree's Theorem that G is not $(2X,5Y,7A)$-generated for all $X,Y\in\{A,B\}$. □

Proposition 5.2.5. *The group G is*

(i) not $(2X,5A,11Y)$-generated for all $X,Y\in\{A,B\}$,

(ii) $(2X,5B,11Y)$-generated for all $X,Y\in\{A,B\}$.

Proof. (i) By Table A.13 we see that $\Delta_G(2X,5A,11Y)=0$ and by Lemma 2.1.3, G is not $(2X,5A,11Y)$-generated for all $X,Y\in\{A,B\}$ and (i) is complete.

(ii) From Table 5.2 we see that M_6 (or M_7) (two non-conjugate copies) is the only maximal subgroup containing elements of orders 2, 5 and 11. As stated in Proposition 5.2.2, the non-empty intersection of the conjugacy classes for H_6 with the conjugacy classes for H_7 which has elements of order 11 is isomorphic to the group 11:5. This subgroup 11:5 of G has no elements of order 2, as such it will not have any contributions. No element of order 2 from the maximal subgroup M_6 (or M_7) fuses to the class $2A$ of G. By Table A.13 we then obtain that $\Delta^*_G(2A,5B,11X)\geq\Delta_G(2A,5B,11X)=44>0$ for $X\in\{A,B\}$. This proves that the group G is $(2A,5B,11X)$-generated for $X\in\{A,B\}$.

We obtained that $\sum_{M_6}(2a,5a,11x)=33$ and found that $h(11X,M_6)=1$ (or $h(11X,M_7)=1$). Since by Table A.13 we have $\Delta_G(2B,5B,11X)=660$, we then obtained that $\Delta^*_G(2B,5B,11X)\geq\Delta_G(2B,5B,11X)-\sum_{M_6}(2a,5a,11x)-\sum_{M_7}(2a,5a,11x)=660-33-33=594>0$ for $X\in\{A,B\}$. This proves that the group G is $(2B,5B,11X)$-generated for $X\in\{A,B\}$. □

$(2, 7, r)$**-generations**

Here we have to check the generation of G through the triples $(2A, 7A, 7A)$-, $(2A, 7A, 11A)$-, $(2A, 7A, 11B)$-, $(2B, 7A, 7A)$-, $(2B, 7A, 11A)$- and $(2B, 7A, 11B)$-generation

Proposition 5.2.6. *The group G is not $(2X, 7A, 7A)$-generated for $X \in \{A, B\}$.*

Proof. We start by counting the $(2A, 7A, 7A)$-subgroups in A_8.

Consider the following sub-chains of lattice of subgroups in A_8, starting at the bottom and gradually working our way up.

(1) $L_3(2) < A_7 < A_8$: We compute $\sum^*_{L_3(2)}(2A, 7A, 7A) = 7$, $\sum_{A_7}(2A, 7A, 7A) = 35$ and $\sum_{A_8}(2A, 7A, 7A) = 70$. Since the normalizer of $L_3(2)$ in A_7 is 7 we obtain that a fixed element $z \in 7A$ lies in a unique A_7-conjugate copy of $L_3(2)$. As there are two non-conjugate copies of $L_3(2)$, we have $\sum^*_{A_7}(2A, 7A, 7A) = 35 - 2(7) = 21$.

(2) $L_3(2) < 2^3{:}L_3(2) < A_8$: In this case, we calculate $\sum_{2^3:L_3(2)}(2A, 7A, 7A) = 14$. By looking at maximal subgroups of $2^3{:}L_3(2)$ we observe that $(2^3{:}7){:}3$ and $L_3(2)$ (two non conjugate copies) are the only maximal subgroups that might be $(2A, 7A, 7A)$-generated. As the $2A \cap ((2^3{:}7){:}3) = \emptyset$, we obtain $\sum^*_{2^3:L_3(2)}(2A, 7A, 7A) = 14 - 2(7) = 0$. That is, there is no contribution from groups $2^3{:}L_3(2)$ and $(2^3{:}7){:}3$ to $\sum_{A_8}(2A, 7A, 7A)$.

As there are three non-conjugate copies of $L_3(2)$ in A_8, we obtain $\sum^*_{A_8}(2A, 7A, 7A) = \sum_{A_8}(2A, 7A, 7A) - 3\sum^*_{L_3(2)}(2A, 7A, 7A) - \sum^*_{A_7}(2A, 7A, 7A) = 70 - 3(7) - 21 = 28$. Further, we see that $N_G(A_8) = 3{:}S_8$, $N_G(A_7) = ((2^3{:}7){:}3)$ and $N_{A_{11}}(L_3(2)) = A_4 \times L_3(2)$. Thus, a fixed $z \in 7A$ in G is contained in three, one and two conjugates of A_8, A_7 and $L_3(2)$, respectively. Since there are two non-conjugate copies of $L_3(2)$ in G. We obtain that $\Delta^*_G(2A, 7A, 7A) \leq$

$\Delta_G(2A,7A,7A) - 3\sum^*_{A_8}(2A,7A,7A) - 4\sum^*_{L_3(2)}(2A,7A,7A) - \sum^*_{A_7}(2A,7A,7A) = 175-4(7)-3(28)-21 = 42 < 84 = |C_G(7A)|$. This shows that $(2A,7A,7A)$ is not a generating triple of A_{11}.

By proving that G is not $(2B,7A,7A)$ generated, we first compute the structure constant $\Delta_G(2B,7A,7A) = 644$. The only maximal subgroups of G that can potentially contribute to the structure constant G are isomorphic to A_{10}, S_9, $(A_8 \times 3){:}2$ and $(A_7 \times A_4){:}2$. We calculate now contribution from each these maximal subgroups to G.

First, we consider the group $\Sigma_{(A_7\times A_4):2}$. The $2B$-class of G does not meet the group A_7. We have $\Sigma_{A_7} = 0$. Further, as $\Sigma_{(A_7\times A_4):2} = \Sigma_{A_7}$ we have $\Sigma^*_{(A_7\times A_4):2} = 0$. This means the maximal subgroup $(A_7\times A_4){:}2$ does not contribute to $\Delta_{A_{11}}$.
For the group $(A_8\times 3){:}2$, we calculate $\Sigma_{(A_8\times 3):2} = \Sigma_{A_8} = 35$. Up to isomorphism, A_7 and $2^3{:}L_3(2)$ (two non-conjugate copies) are the only maximal subgroups of A_8. From above case, we know that $\Sigma_{A_7} = 0$. Next consider the subchain of groups $2^3{:}7 < (2^3{:}7){:}3 < 2^3{:}L_3(2)$. We compute that $\Sigma^*_{2^3:7} = \Sigma_{2^3:7} = 7$, $\Sigma_{(2^3:7):3} = 7 = \Sigma_{2^3:L_3(2)}$. As $|N_{(2^3:7):3}(2^3{:}7)| = (2^3{:}7){:}3 = N_{2^3:L_3(2)}(2^3{:}7)$, we obtain that a fixed $z \in 7A$ is contained in a unique copy of each of $(2^3{:}7){:}3$-conjugate of $2^3{:}7$ and $2^3{:}L_3(2)$ groups. Thus we obtain $\Sigma^*_{(2^3:7):3} = \Sigma_{(2^3:7):3} - \Sigma_{2^3:7} = 7-7 = 0$ and $\Sigma^*_{(2^3:L_3(2))} = \Sigma_{2^3:L_3(2)} - \Sigma_{2^3:7} = 7-7 = 0$. Observe that, the only contribution toward Σ_{A_8} so far is coming from a unique conjugate of $2^3{:}7$. As there are two non-conjugate copies of $2^3{:}L_3(2)$, we compute

$$\Sigma^*_{A_8} = \Sigma_{A_8} - 2\,\Sigma_{2^3:7} = 35 - 2(7) = 21.$$

Next, we treat the maximal group S_9. We compute $\Sigma_{S_9} = \Sigma_{A_9.2} = \Sigma_{A_9} = 154$. From the list of maximal subgroups of A_9, observe that the $(2B,7A,7A)$-generated proper subgroups of A_{11} are contained in the subgroups isomorphic to S_7, A_8 or $2^3{:}L_2(8)$ (two non-conjugate copies). From above, we have $\Sigma_{S_7} = \Sigma_{A_7.2} = 0$ as $A_7 \cap 2B = \emptyset$. Also $\Sigma^*_{A_8} = 21$. We investigate contribution from $L_2(8){:}3$ to $\Delta_{A_{11}}$. We calculate $\Sigma_{L_2(8):3} = \Sigma_{L_2(8)} = 28$ and $\Sigma^*_{2^3:7} = 7$. Since $2^3{:}7 < L_2(8)$ and a fixed element $z \in 7A$ lies in two $L_2(8)$-conjugates of $2^3{:}7$, we have

$$\Sigma^*_{L_2(8)} = \Sigma_{L_2(8)} - 2\Sigma_{2^3:7} = 28 - 2(7) = 14.$$

We now collect the total contribution coming from A_9 to G. Note that a fixed element z of order 7 (in A_{11}) lies in two, two and four A_9-conjugates of groups A_8, $L_2(8)$ and $2^3{:}7$, respectively. We obtain $\Sigma^*_{A_9} = \Sigma_{A_9} - 2\ \Sigma^*_{A_8} - 2\ \Sigma^*_{L_2(8)} - 4\ \Sigma^*_{2^3:7} = 154 - 2(28) - 2(14) - 4(7) = 56$.

Finally, it remains to compute contribution from the group A_{10}. We calculate $\Sigma(A_{10}) = 644$. From the list of maximal subgroups of A_{10}, the groups that may contain $(2B, 7A, 7A)$-generated proper subgroups, up to isomorphism, are A_9, S_8 and $(A_7{\times}3){:}2$. In fact, we have already have contributions from these groups as $\Sigma_{S_8} = \Sigma_{A_8.2} = \Sigma_{A_8}$, $\Sigma^*_{A_9} = 56$ and $\Sigma_{(A_7\times 3):2} = \Sigma_{A_7} = 0$. As, $N_{A_{10}}(A_8) = S_8$, $N_{A_{10}}(L_2(8)) = 3{:}L_2(8)$, $N_{A_{10}}(2^3{:}7) = (2^3{:}7) : 3$ and A_9 is self normalized in A_{10} being maximal in A_{10}. A fixed element $z \in 7B$ is contained in three, three, six and six A_{10}-conjugates of groups A_9, A_8, $L_2(8)$ and $2^3{:}7$, respectively. We calculate that $\Sigma^*_{A_{10}} = \Sigma_{A_{10}} - 3\ \Sigma^*_{A_9} - 3\ \Sigma^*_{A_8} - 6\ \Sigma^*_{L_2(8)} - 6\ \Sigma_{2^3:7} = 357 - 3(56) - 3(21) - 6(14) - 6(7) = 0$.

To summarize, the only proper $(2B, 7A, 7A)$-subgroups of A_{11} are A_9, A_8, $2^3{:}7$ and $L_2(8)$. As the respective numbers of A_{11}-conjugates of these subgroups containing a fixed element $z \in 7A$ are six, four, six and twelve, we obtain $\Delta^*_G \leq \Delta_{A_{11}} - 6\ \Sigma^*_{A_9} - 4\ \Sigma^*_{A_8} - 6\ \Sigma_{2^3:7} - 12\ \Sigma^*_{L_2(8)} = 644 - 6(56) - 4(21) - 6(7) - 12(14) = 14 < 84 = |C_{A_{11}}(7A)|$, which establishes that G is not $(2B, 7A, 7A)$-generated. □

Proposition 5.2.7. *The group G is*

(i) not $(2A, 7A, 11X)$-generated for $X \in \{A, B\}$,

(ii) $(2B, 7A, 11X)$-generated for $X \in \{A, B\}$.

Proof. (i) Table A.13 gives that $\Delta_G(2A, 7A, 11X) = 0$ for $X \in \{A, B\}$ and thus the result.

(ii)None of these subgroups M_{11} and M_6 (or M_7) of G contain elements of order 7. By Table A.13 we obtained that $\Delta^*_G(2B, 7A, 11X) \geq \Delta_G(2B, 7A, 11X) = 55 > 0$ for $X \in \{A, B\}$. Hence the result. $\square$

$(2, 11, r)$-generations

Also here we have to check for the generation of G through the triples $(2A, 11A, 11A)$-, $(2A, 11A, 11B)$- $(2A, 11B, 11B)$-, $(2B, 11A, 11A)$-, $(2B, 11A, 11B)$- and $(2B, 11B, 11B)$-generation. We handle all these cases in the following Proposition.

Proposition 5.2.8. *The group G is $(2X, 11Y, 11Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. As in Proposition 5.2.5, the subgroups M_6 (or M_7) and 11:5 contain elements of order 11. None of these groups meet the classes $2A$, $11A$ and $11B$ of G. Thus $\Delta^*_G(2A, 11A, 11B) = \Delta_G(2A, 11A, 11B) = 220 > 0$ and $\Delta^*_G(2A, 11X, 11X) = \Delta_G(2A, 11X, 11X) = 110 > 0$ for $X \in \{A, B\}$. This proves that G is $(2A, 11X, 11Y)$-generated for $X, Y \in \{A, B\}$.

We obtained that $\sum_{M_6}(2a, 11x, 11y) = 11$ and we have $h(11X, M_6) = 1$ for all $x, y \in \{a, b\}$. By Table A.13 we have $\Delta_G(2B, 11A, 11B) = 1320$ and $\Delta_G(2B, 11X, 11X) = 2145$ for $X \in \{A, B\}$. It renders that $\Delta^*_G(2B, 11A, 11B) \geq \Delta_G(2B, 11A, 11B) - \sum_{M_6}(2a, 11a, 11b) - \sum_{M_7}(2a, 11a, 11b)$ $= 1320 - 11 - 11 = 1298 > 0$ and $\Delta^*_G(2B, 11X, 11X) \geq \Delta_G(2B, 11X, 11X) - \sum_{M_6}(2a, 11x, 11x) - \sum_{M_7}(2a, 11x, 11x) = 2145 - 11 - 11 = 2123 > 0$, proving that G is $(2B, 11X, 11Y)$-generated for $X, Y \in \{A, B\}$. $\square$

5.2.2 $(3, q, r)$-generations

In this section we handle all the possible $(3, q, r)$-generations, namely $(3X, 3Y, 5A)$-, $(3X, 3Y, 5B)$-, $(3X, 3Y, 7A)$-, $(3X, 3Y, 11A)$-, $(3X, 3Y, 11B)$-, $(3X, 5A, 5A)$-, $(3X, 5A, 5B)$-, $(3X, 5A, 7A)$-, $(3X, 5A, 11A)$-, $(3X, 5A, 11B)$-, $(3X, 5B, 5B)$-, $(3X, 5B, 7A)$-, $(3X, 5B, 11A)$-, $(3X, 5B, 11B)$-, $(3X, 7A, 7A)$-, $(3X, 7A, 11A)$-, $(3X, 7A, 11B)$-, $(3X, 11A, 11A)$-, $(3X, 11A, 11B)$- and $(3X, 11B, 11B)$-generations.

$(3, 3, r)$-generations

Proposition 5.2.9. *The group G is neither $(3X, 3Y, 5Z)$- nor $(3X, 3Y, 7A)$-generated group for all $X, Y \in \{A, B, C\}$ and $Z \in \{A, B\}$.*

Proof. The group G acts on a 10-dimensional irreducible complex module $\mathbb{V}$. Applying Scott's Theorem to the module $\mathbb{V}$ and using the Atlas of finite groups, we get that $d_{3A} = \dim(\mathbb{V}/C_{\mathbb{V}}(3A)) = \frac{2(10-7)}{3} = 2$, $d_{3B} = \dim(\mathbb{V}/C_{\mathbb{V}}(3B)) = \frac{2(10-4)}{3} = 4$, $d_{3C} = \dim(\mathbb{V}/C_{\mathbb{V}}(3C)) = \frac{2(10-1)}{3} = 6$, $d_{5A} = \dim(\mathbb{V}/C_{\mathbb{V}}(5A)) = \frac{4(10-5)}{5} = 4$, $d_{5B} = \dim(\mathbb{V}/C_{\mathbb{V}}(5B)) = \frac{4(10-0)}{5} = 8$ and $d_{7A} = \dim(\mathbb{V}/C_{\mathbb{V}}(7A)) = \frac{6(10-3)}{7} = 6$. For the cases $(3A, 3A, nX)$ we get $d_{3A} + d_{3A} + d_{nX} = 2 \times 2 + d_{nX} < 2 \times 10$ and hence by Scott's Theorem, G is not $(3A, 3A, nX)$-generated for all $nX \in \{5A, 5B, 7A\}$. We get non-generations when Scott's Theorem is applied to the following cases $(3A, 3B, nX)$, $(3A, 3C, nX)$, $(3B, 3B, nX)$, $(3B, 3C, nX)$ and $(3C, 3C, nX)$ for all $nX \in \{5A, 5B, 7A\}$. □

Proposition 5.2.10. *The group G is*

(i) neither $(3A, 3X, 11Y)$- nor $(3B, 3B, 11Y)$-generated for $X \in \{A, B, C\}$ and $Y \in \{A, B\}$,

(ii) $(3B, 3C, 11X)$- and $(3C, 3C, 11X)$-generated for $X \in \{A, B\}$.

Proof. (i) By Table A.14 we see that $\Delta_G(3A, 3X, 11Y) = 0 = \Delta_G(3B, 3B, 11Y)$ for $X \in \{A, B, C\}$ and $Y \in \{A, B\}$. Hence, G is neither $(3A, 3X, 11Y)$- nor $(3B, 3B, 11Y)$-generated for $X \in \{A, B, C\}$ and $Y \in \{A, B\}$.

(ii) Of the groups having elements of order 11 discussed above, none of them meet the classes $3B$, $3C$ and $11A$ or $11B$ of G. By Table A.13 we then obtained that $\Delta^*_G(3B, 3C, 11X) = \Delta_G(3B, 3C, 11X) = 66 > 0$, proving that G is $(3B, 3C, 11X)$-generated for $X \in \{A, B\}$. Now we prove that G is $(3C, 3C, 11X)$-generated for $X \in \{A, B\}$. By Proposition 5.2.2, we proved that G is $(2B, 3C, 11X)$-generated for $X \in \{A, B\}$. It follows by Theorem 2.2.3 that G is $(3C, 3C, (11A)^2)$- and $(3C, 3C, (11B)^2)$-generated. By GAP, we see that $(11A)^2 = 11B$ and $(11B)^2 = 11A$ and thus G is $(3C, 3C, 11X)$-generated for $X \in \{A, B\}$. □

$(3, 5, r)$**-generations**

Proposition 5.2.11. *The group G is*

(i) neither $(3X, 5A, 5Y)$- nor $(3A, 5B, 5B)$ generated for $X \in \{A, B, C\}$ and $Y \in \{A, B\}$, while it is

(ii) $(3X, 5B, 5B)$-generated for $X \in \{B, C\}$.

Proof. (i) If G is $(3X, 5A, 5Y)$-generated group, then we must have $c_{3X} + c_{5A} + c_{5Y} \leq 13$ where $X \in \{A, B, C\}$ and $Y \in \{A, B\}$. Since by Table 5.1 we have $c_{3X} \in \{5, 7, 9\}$ for $X \in \{A, B, C\}$, we then obtain by same Table 5.1 that

$$c_{3X} + c_{5A} + c_{5A} = c_{3X} + 7 + 7 > 13,$$

$$c_{3X} + c_{5A} + c_{5B} = c_{3X} + 7 + 3 > 13.$$

Now using Ree's Theorem, it follows that G is not $(3X, 5A, 5Y)$-generated for $X \in \{A, B, C\}$

and $Y \in \{A, B\}$. Again by Table 5.1, we have $c_{3A} + c_{5B} + c_{5B} = 9 + 3 + 3 > 13$ and by Ree's Theorem, the group G is not $(3A, 5B, 5B)$-generated.

(ii) Following the discussions in Proposition 5.2.3, the subgroups S_5 and 4:5 may have contributions here. We show that G is $(3X, 5B, 5B)$-generated for $X \in \{B, C\}$. We firstly consider the triple $(3B, 5B, 5B)$. By Table A.14 we have $\Delta_G(3B, 5B, 5B) = 1080$. We realize that none of them will have any contributions since the elements of order 3 for both S_5 and 4:5 do not fuse to the class $3B$ of G. We noticed that the elements of order 5 for each the following maximal subgroups M_2, M_3 and M_4 do not fuse to the class $5B$ of G. The maximal subgroup M_6 (or M_7) does not fuse to the class $3B$ of G. We obtained that $\sum_{M_1}(3b, 5b, 5b) = 650$ and $\sum_{M_5}(3x, 5y, 5z) = \Delta_{M_5}(3c, 5c, 5c) + \Delta_{M_5}(3c, 5c, 5d) + \Delta_{M_5}(3c, 5d, 5d) + \Delta_{M_5}(3d, 5c, 5c) + \Delta_{M_5}(3d, 5c, 5d) + \Delta_{M_5}(3d, 5d, 5d) = 5 + 10 + 5 + 75 + 75 + 75 = 245$. We then obtained that $\Delta^*_G(3B, 5B, 5B) \geq \Delta_G(3B, 5B, 5B) - \sum_{M_1}(3b, 5b, 5b) - \sum_{M_5}(3x, 5y, 5z) = 1080 - 1(650) - 1(245) = 185 > 0$.

We turn to the other case, namely the triple $(3C, 5B, 5B)$. In order to show that $(3C, 5B, 5B)$ is a generating triple of G, we consider its 10-dimensional irreducible representation over $\mathbb{F}_2$ (Wilson [55]). The group $G = \langle a, b \rangle$ is generated by its standard generators a and b, where a and b are 10×10 matrices over $\mathbb{F}_2$ with orders 3 and 9, respectively such that a is in class $3A$ and ab has order 11. Then via GAP, we produce $c = ab^3a^{-1}b^2(ba)^3b^3aba^{-1}$ and $d = ab^{-1}ab^2a^{-1}b^4ab^{-1}a^{-1}b^2ab^2$ such that c and d are in $5B$ and $cd \in 12A$. Set $y = c$ and $x = dc^{-1}$ then we see that $P = \langle x, y \rangle$ and such that $x \in 3C$, $y \in 5B$ and $xy \in 5B$. Moreover, there are elements of order 5, 7 and 11 in P. As G has no proper subgroup divisible by $5 \times 7 \times 11$, we have $G = \langle x, y \rangle = P$, as claimed. Hence (ii) is complete. □

Proposition 5.2.12. *The group G is*

(i) neither $(3X, 5Y, 7A)$*- nor* $(3C, 5A, 7A)$*-generated for all* $X, Y \in \{A, B\}$,

(ii) $(3C, 5B, 7A)$*-generated.*

Proof. (i) Since by Table A.14 we have $\Delta_G(3A, 5A, 7A) = 7 < 84 = |C_G(7A)|$ and $\Delta_G(3A, 5B, 7A) = 0$, it follows that G is not $(3A, 5X, 7A)$-generated for $X \in \{A, B\}$. By Proposition 5.2.9 we see that $d_{3B} = 4$, $d_{3C} = 6$, $d_{5A} = 4$, $d_{5B} = 8$ and $d_{7A} = 6$. Thus $d_{3B} + d_{5X} + d_{7A} = 4 + d_{5X} + 6 < 20$ and $d_{3C} + d_{5A} + d_{7A} = 6 + 4 + 6 < 20$ for $X \in \{A, B\}$. By Scott's Theorem the group G is not $(3B, 5X, 7A)$- and $(3C, 5A, 7A)$-generated for $X \in \{A, B\}$.

(ii) By Table A.14 we have $\Delta_G(3C, 5B, 7A) = 5376$. By Table 5.2, the maximal subgroups having elements of order 7 are M_1, M_2, M_3 and M_4. We look at various non-empty intersections of conjugacy classes for these maximal subgroups.

We got the following subgroups of G having elements of order 7:

- The subgroups arising from the non-empty intersections of conjugacy classes for these four maximal subgroups are A_7 and S_7.
- The subgroups arising from the non-empty intersections of the conjugacy classes for any three maximal subgroups are S_7 (4-copies) and A_8.
- The subgroups arising from the non-empty intersections of the conjugacy classes for any two maximal subgroups are S_7, $2 \times S_7$, $A_7{:}S_3$ (4-copies), S_8 (3-copies) and A_9.

Out of all these subgroups and maximal subgroups of G having elements of order 7, only M_1 meets the classes $3C$, $5B$ and $7A$ of G. We obtained that $\sum_{M_1}(3c, 5b, 7a) = 882$ and $h(7A, M_1) = 4$. It then follows that $\Delta_G^*(3C, 5B, 7A) \geq \Delta_G(3C, 5B, 7A) - \sum_{M_1}(3c, 5b, 7a) = 5376 - 4(882) = 1848 > 0$, proving (ii). □

Proposition 5.2.13. *The group G is*

(i) not $(3X, 5A, 11Y)$*-generated for all* $X, Y \in \{A, B\}$,

(ii) $(3C, 5A, 11Y)$*- and* $(3X, 5B, 11Y)$*-generated for* $X \in \{A, B, C\}$ *and* $Y \in \{A, B\}$.

Proof. (i) Since $c_{3X} \in \{7, 9\}$ by Table 5.1, it follows that $c_{3X} + c_{5A} + c_{11Y} = c_{3X} + 7 + 1 > 13$ for all $X, Y \in \{A, B\}$ and the result follows.

(ii) No element of order 5 from the maximal subgroups M_6 (or M_7) fuses the class $5A$ of G and the subgroup 11:5 does not have elements of order 3. By Table A.14, we have $\Delta_G(3C, 5A, 11X) = 22$ for $X \in \{A, B\}$. Since there is no contribution from any of the subgroups of G, we then have $\Delta^*_G(3C, 5A, 11X) \geq \Delta_G(3C, 5A, 11X) = 22 > 0$, proving that G is $(3C, 5A, 11X)$-generated for $X \in \{A, B\}$. Similarly we have $\Delta_G(3A, 5B, 11X) = 11$ for $X \in \{A, B\}$ and it follows that $\Delta^*_G(3A, 5B, 11X) \geq \Delta_G(3A, 5B, 11X) = 11 > 0$, proving that G is $(3A, 5B, 11X)$-generated for $X \in \{A, B\}$. By the same Table A.14, we have $\Delta_G(3B, 5B, 11X) = 704$ for $X \in \{A, B\}$. No element of order 3 from these subgroups fuses to the class $3B$ of G. Therefore we get $\Delta^*_G(3B, 5B, 11X) \geq \Delta_G(3B, 5B, 11X) = 704 > 0$, proving that G is $(3B, 5B, 11X)$-generated for $X \in \{A, B\}$.

For the other argument, the computations show that $\sum_{M_6}(3a, 5a, 11x) = \Delta_{M_6}(3a, 5a, 11a) + \Delta_{M_6}(3a, 5a, 11b) = 99 + 99 = 198$ and $h(11X, M_6) = 1$. Similarly $\sum_{M_7}(3a, 5a, 11x) = 198$. Since by Table A.14 we have $\Delta_G(3C, 5B, 11X) = 4928$, we obtain that $\Delta^*_G(3C, 5B, 11X) \geq \Delta_G(3C, 5B, 11X) - \sum_{M_6}(3a, 5a, 11x) - \sum_{M_7}(3a, 5a, 11x) = 4928 - 198 - 198 = 4532 > 0$ for $X \in \{A, B\}$. This proves that G is $(3C, 5B, 11X)$-generated for $X \in \{A, B\}$. □

$(3, 7, r)$- and $(3, 11, r)$-generations

In this subsection we discuss the cases $(3, 7, r)$- and $(3, 11, r)$-generations. This comprises of 18 cases : $(3A, 7A, 7A)$-, $(3A, 7A, 11A)$-, $(3A, 7A, 11B)$-, $(3B, 7A, 7A)$-, $(3B, 7A, 11A)$-,

$(3B, 7A, 11B)$-, $(3C, 7A, 7A)$-, $(3C, 7A, 11A)$-, $(3C, 7A, 11B)$-, $(3A, 11A, 11A)$-, $(3A, 11A, 11B)$-, $(3A, 11B, 11B)$-, $(3B, 11A, 11A)$-, $(3B, 11A, 11B)$-, $(3B, 11B, 11B)$-, $(3C, 11A, 11A)$-, $(3C, 11A, 11B)$- and $(3A, 11B, 11B)$-generation.

Proposition 5.2.14. *The group G is not $(3X, 7A, 7A)$-generated for $X \in \{A, B, C\}$.*

Proof. This is a direct application of Ree's Theorem, since by Table 5.1 we see that $c_{3X} \in \{5, 7, 9\}$, it then follows that $c_{3X} + c_{7A} + c_{7A} = c_{3X} + 5 + 5 > 13$, it follows that G is not $(3X, 7A, 7A)$-generated for $X \in \{A, B, C\}$. □

Proposition 5.2.15. *The group G is*
(i) not $(3A, 7A, 11X)$-generated for $X \in \{A, B\}$,
(ii) $(3Y, 7A, 11X)$-generated for $X \in \{A, B\}$ and $Y \in \{B, C\}$.

Proof. (i) Since by Table A.14 we have $\Delta_G(3A, 7A, 11X) = 0$, it follows that G is not $(3A, 7A, 11X)$-generated for $X \in \{A, B\}$. (ii) None of these groups M_{11} and M_6 (or M_7) contain elements of order 7. Therefore
$\Delta^*_G(3B, 7A, 11X) = \Delta_G(3B, 7A, 11X) = 33 > 0$ and $\Delta^*_G(3C, 7A, 11X) = \Delta_G(3C, 7A, 11X) = 990 > 0$. Hence the results. □

Proposition 5.2.16. *The group G is $(3X, 11Y, 11Z)$-generated for $X \in \{A, B, C\}$ and $Y, Z \in \{A, B\}$.*

Proof. None of these groups 11:5 and M_6 (or M_7) meet the classes $3A$, $11A$ and $11B$ of G. Then by Table A.14 we have $\Delta^*_G(3A, 11X, 11X) = \Delta_G(3A, 11X, 11X) = 110 > 0$ and $\Delta^*_G(3A, 11A, 11B) = \Delta_G(3A, 11A, 11B) = 55$, proving that G is $(3A, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$. Again, none of these groups M_{11} and M_6 (or M_7) meet the classes $3B$, $11A$ and $11B$ of G. Then by Table A.14 we obtained that $\Delta^*_G(3B, 11X, 11X) = \Delta_G(3B, 11X, 11X) =$

3212 and $\Delta^*_G(3B, 11A, 11B) = \Delta_G(3B, 11A, 11B) = 2332 > 0$, proving that G is $(3B, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$. By Table A.14 we have $\Delta(3C, 11X, 11Y) = 12760$ for all $X, Y \in \{A, B\}$. Two non-conjugate copies of M_6 (or M_7) is the only one meeting the classes $3C$, $11A$ and $11B$ of G. We obtained that $\sum_{M_6}(3a, 11a, 11b) = 22$, $\sum_{M_6}(3a, 11x, 11x) = 77$ and $h(11Y, M_6) = 1$. Similarly $\sum_{H_7}(3a, 11a, 11b) = 22$, $\sum_{M_7}(3a, 11x, 11x) = 77$ and $h(11Y, M_7) = 1$. Therefore $\Delta^*_G(3C, 11A, 11B) \geq \Delta_G(3C, 11A, 11B) - \sum_{M_6}(3a, 11a, 11b) - \sum_{M_7}(3a, 11a, 11b) = 12760 - 77 - 77 = 12529 > 0$ and $\Delta^*_G(3C, 11X, 11X) \geq \Delta_G(3C, 11X, 11X) - \sum_{M_6}(3a, 11x, 11x) - \sum_{M_7}(3a, 11x, 11x) = 12760 - 22 - 22 = 12716 > 0$, proving that G is $(3C, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$. Thus, G is $(3X, 11Y, 11Z)$-generated for $X \in \{A, B, C\}$ and $Y \in \{A, B\}$. □

5.2.3 Other results

In this section we handle all the remaining cases, namely the $(5, q, r)$-, $(7, q, r)$- and $(11, q, r)$-generations.

$(5, 5, r)$-generations

We have to check for the generation of G through the triples $(5A, 5A, 5A)$-, $(5A, 5A, 5B)$-, $(5A, 5A, 7A)$-, $(5A, 5A, 11A)$-, $(5A, 5A, 11B)$-, $(5A, 5B, 5B)$-, $(5A, 5B, 7A)$-, $(5A, 5B, 11A)$-, $(5A, 5B, 11B)$-, $(5B, 5B, 5B)$-, $(5B, 5B, 7A)$-, $(5B, 5B, 11A)$- and $(5A, 5A, 11B)$-generation.

Proposition 5.2.17. *The group G is*

(i) not $(5A, 5A, 5X)$-generated for $X \in \{A, B\}$,

(ii) $(5X, 5B, 5B)$-generated for $X \in \{A, B\}$.

Proof. (i) Since by Table A.15 we have that $\Delta_G(5A, 5A, 5A) = 428 < 1800 = |C_G(5A)|$ and

$\Delta_G(5A, 5A, 5B) = 2 < 25 = |C_G(5B)|$, it follows be Lemma 2.1.3 that G is not $(5A, 5A, 5X)$-generated for $X \in \{A, B\}$.

(ii) The subgroups S_5 and 4:5 arising from the intersections discussed in Proposition 5.2.3, are the only ones that may have contributions here. The subgroups M_2, M_3, M_4, M_6 (or M_7), S_5 and 4:5 will not have any contributions because their elements of order 5 do not fuse to the class $5A$ of G. The computations render $\sum_{M_1}(5a, 5b, 5b) = 316$ and $\sum_{M_5}(5x, 5c, 5y) = \Delta_{M_5}(5a, 5c, 5c) + \Delta_{M_5}(5a, 5c, 5d) + \Delta_{M_5}(5a, 5d, 5d) + \Delta_{M_5}(5b, 5c, 5c) + \Delta_{M_5}(5b, 5c, 5d) + \Delta_{M_5}(5b, 5d, 5d) = 6+2+6+31+22+31 = 98$. We found that $h(5B, M_1) = h(5B, M_5) = 1$. Since by Table A.15, we have $\Delta_G(5A, 5B, 5B) = 456$, we have $\Delta^*_G(5A, 5B, 5B) \geq \Delta_G(5A, 5B, 5B) - \sum_{M_1}(5a, 5b, 5b) - \sum_{M_5}(5x, 5c, 5y) = 456-316-98 = 42 > 0$. This proves that G is $(5A, 5B, 5B)$-generated.

Now we prove that G is $(5B, 5B, 5B)$-generated. By Proposition 5.2.3 we proved that G is $(2B, 5B, 5B)$-generated. It follows by Theorem 2.2.3 that G is $(5B, 5B, (5B)^2)$-generated. By GAP, we see that $(5B)^2 = 5B$ so that G becomes $(5B, 5B, 5B)$-generated as required. □

Proposition 5.2.18. *The group G is*

(i) not $(5A, 5X, 7A)$-generated for $X \in \{A, B\}$,

(ii) $(5B, 5B, 7A)$-generated.

Proof. (i) If G is $(5A, 5X, 7A)$-generated group, then we must have $c_{5A} + c_{5X} + c_{7A} \leq 13$ where $X \in \{A, B\}$. Since by Table 5.1 we have $c_{5X} \in \{3, 7\}$, we then obtained by same Table 5.1 that $c_{5A} + c_{5X} + c_{7A} = 7 + c_{5X} + 5 > 13$ for $X \in \{A, B\}$. By Ree's Theorem, G is not $(5A, 5X, 7A)$-generated for $X \in \{A, B\}$.

(ii) By Table A.15 we have that $\Delta_G(5B, 5B, 7A) = 32256$. Out of all the subgroups of G having elements of order 7 in Proposition 5.2.12, only M_1 meets the classes $5B$ and $7A$ of the group

G. We obtained that $\sum_{M_1}(5b, 5b, 7a) = 3654$ and $h(7A, M_1) = 4$. We have $\Delta^*_G(5B, 5B, 7A) \geq \Delta_G(5B, 5B, 7A) - 4 \cdot \sum_{M_1}(5b, 5b, 7a) = 32256 - 4(3654) = 17640$ and so G is $(5B, 5B, 7A)$-generated. □

Proposition 5.2.19. *The group G is*

(i) not $(5A, 5A, 11X)$-generated for $X \in \{A, B\}$,

(ii) $(5X, 5B, 11Y)$-generated for all $X, Y \in \{A, B\}$.

Proof. (i) By Table A.15 we have $\Delta_G(5A, 5A, 11X) = 0$. Hence the results follow.

(ii) As in Proposition 5.2.5, we see that two non-conjugate copies of M_6 (or M_7) and the group 11:5 contain elements of orders 5 and 11. None of these two groups meet the classes $5A$, $5B$ and $11A$ or $11B$ of G. It follows that $\Delta^*_G(5A, 5B, 11X) = \Delta_G(5A, 5B, 11X) = 440 > 0$, proving that G is $(5A, 5B, 11X)$-generated for $X \in \{A, B\}$.

We now prove that G is $(5B, 5B, 11X)$-generated for $X \in \{A, B\}$. By Proposition 5.2.5, the group G is $(2X, 5B, 11Y)$-generated, it follows by Theorem 2.2.3 that G is $(5B, 5B, 11X)$-generated for all $X, Y \in \{A, B\}$. □

Proposition 5.2.20. *The group G is*

(i) not $(5A, 7A, 7A)$-generated,

(ii) $(5B, 7A, 7A)$-generated.

Proof. (i) The group G acts on a 10-dimensional irreducible complex module $\mathbb{V}$. By Scott's Theorem [50] applied to the module $\mathbb{V}$ and using the Atlas of finite groups, we get $d_{5A} + d_{7A} + d_{7A} = 4 + 6 + 6 = 16 < 2 \times 10$ and hence by Scott's Theorem, G is not $(5A, 7A, 7A)$-generated.

(ii) By Table A.15 we have that $\Delta_G(5B, 7A, 7A) = 8736$. As in Proposition 5.2.18, the maximal subgroup M_1 will have contributions here because it is the only one meeting the classes

$5B$ and $7A$ of G. We have $\sum_{M_1}(5b, 7a, 7a) = 1974$ and $h(7A, M_1) = 4$. We then obtain $\Delta^*_G(5B, 7A, 7A) \geq \Delta_G(5B, 7A, 7A) - 4 \cdot \sum_{M_1}(5b, 7a, 7a) = 8736 - 4(1974) = 840 > 0$ and hence G is $(5B, 7A, 7A)$-generated group. □

Proposition 5.2.21. *The group G is $(5X, 7A, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. None of these subgroups 11:5 and M_6 (or M_7) meet the classes $5X$, $7A$ and $11Y$ of G for all $X, Y \in \{A, B\}$. Since there is no contribution from any of these subgroups of G, using Table A.15 we will have $\Delta^*_G(5A, 7A, 11X) = \Delta_G(5A, 7A, 11X) = 11 > 0$ and $\Delta^*_G(5B, 7A, 11X) = \Delta_G(5B, 7A, 11X) = 9504 > 0$ for $X \in \{A, B\}$. Hence G is $(5A, 7A, 11X)$- and $(5B, 7A, 11X)$-generated for $X \in \{A, B\}$. □

Proposition 5.2.22. *The group G is $(5X, 11Y, 11Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. As in Proposition 5.2.5, M_6 (or M_7) (two non-conjugate copies) and 11:5 are the only groups containing elements of order 11. Since none of them meet the classes $5A$, $11A$ and $11B$ of G, by Table A.15 we then obtain $\Delta^*_G(5A, 11A, 11B) = \Delta_G(5A, 11A, 11B) = 1804 > 0$ and $\Delta^*_G(5A, 11X, 11X) = \Delta_G(5A, 11X, 11X) = 1892 > 0$ for $X \in \{A, B\}$. Hence G is $(5A, 11A, 11B)$- and $(5A, 11X, 11X)$-generated for $X \in \{A, B\}$.

By the same Table A.15, we have $\Delta_G(5B, 11A, 11B) = 69696$ and $\Delta_G(5B, 11X, 11X) = 76032$ for $X \in \{A, B\}$. The subgroup 11:5 of G will not have any contributions because its relevant structure constants are all zeros. We obtained that $\sum_{M_6}(5a, 11a, 11b) = 99$ and $\sum_{M_7}(5a, 11a, 11b) = 99$. We also have $\sum_{M_6}(5a, 11x, 11x) = 198$ (or $\sum_{M_7}(5a, 11x, 11x) = 198$), $h(11x, M_6) = 1$ (or $h(11x, M_7) = 1$) for $x \in \{a, b\}$. It follows that $\Delta^*_G(5B, 11A, 11B) \geq \Delta_G(5B, 11A, 11B) - \sum_{M_6}(5a, 11a, 11b) - \sum_{M_7}(5a, 11a, 11b) = 69696 - 99 - 99 = 69498 > 0$ and $\Delta^*_G(5B, 11X, 11X) \geq \Delta_G(5B, 11X, 11X) - \sum_{M_6}(5a, 11x, 11x) - \sum_{M_6}(5a, 11x, 11x) =$

$76032-198-198 = 75636 > 0$ for $X \in \{A, B\}$. Hence G is $(5B, 11A, 11B)$- and $(5B, 11X, 11X)$-generated for $X \in \{A, B\}$. $\square$

$(7, 7, r)$**-generations**

Proposition 5.2.23. *The group G is $(7A, 7A, 7A)$-generated.*

Proof. We start by counting the $(7A, 7A, 7A)$-subgroups in A_8.

Consider the following sub-chains of lattice of subgroups in A_8, starting at the bottom and gradually working our way up.

(1) $L_3(2) < A_7 < A_8$: We compute $\sum^*_{L_3(2)}(7A, 7A, 7A) = 12$, $\sum_{A_7}(7A, 7A, 7A) = 180$ and $\sum_{A_8}(7A, 7A, 7A) = 1580$. Since the normalizer of $L_3(2)$ in A_7 is 7 we obtain that a fixed element $z \in 7A$ lies in a unique A_7-conjugate copy of $L_3(2)$. As there are two non-conjugate copies of $L_3(2)$, we have $\sum^*_{A_7}(7A, 7A, 7A) = 180 - 2(12) = 156$.

(2) $L_3(2) < 2^3{:}L_3(2) < A_8$: In this case, we calculate $\sum_{2^3:L_3(2)}(7A, 7A, 7A) = 96$. By looking at maximal subgroups of $2^3{:}L_3(2)$ we observe that $((2^3{:}2^2){:}3){:}2$ and $L_3(2)$ (two non conjugate copies) are the only maximal subgroups that might be $(2A, 7A, 7A)$-generated. As the $((2^3{:}2^2){:}3){:}2$ do not have elements of order 7, we obtain $\sum^*_{2^3:L_3(2)}(7A, 7A, 7A) = 96 - 2(12) = 72$. As there are three non-conjugate copies of $L_3(2)$ and two non-conjugate copies of $2^3{:}L_3(2)$ in A_8, we obtain $\sum^*_{A_8}(7A, 7A, 7A) = \sum_{A_8}(7A, 7A, 7A) - 3\sum^*_{L_3(2)}(7A, 7A, 7A) - \sum^*_{A_7}(7A, 7A, 7A) = 1580 - 3(12) - 2(96) = 1352$.

Further, we see that $N_{A_{11}}(A_8) = 3{:}S_8$, $N_{A_{11}}(A_7) = ((2^3{:}2^2){:}3){:}2$ and $N_{A_{11}}(L_3(2)) = A_4 \times L_3(2)$. Thus, a fixed $z \in 7A$ in A_{11} is contained in three, one and two conjugates of A_8, A_7 and $L_3(2)$, respectively. Since there are two non-conjugate copies of $L_3(2)$ in A_{11}. We obtain that $\Delta^*_{A_{11}}(7A,7A,7A) \geq \Delta_{A_{11}}(7A,7A,7A) - 3\sum^*_{A_8}(7A,7A,7A) - 4\sum^*_{L_3(2)}(7A,7A,7A) - \sum^*_{A_7}(7A,7A,7A) = 11996 - 3(1580) - 4(1352) - 156 = 1692 > 0$. This shows that $(7A,7A,7A)$ is a generating triple of A_{11}. □

Proposition 5.2.24. *The group G is $(7A,7A,11X)$-generated.*

Proof. By Proposition 5.2.7 we proved that G is $(2B,7A,11X)$-generated for $X \in \{A,B\}$. It follows by Theorem 2.2.3 that G is $(7A,7A,(11A)^2)$- and $(7A,7A,(11B)^2)$-generated. By GAP, we see that $(11A)^2 = 11B$ and $(11B)^2 = 11A$ so that G becomes $(7A,7A,11X)$-generated for $X \in \{A,B\}$. □

Proposition 5.2.25. *The group G is $(7A,11X,11Y)$-generated for all $X,Y \in \{A,B\}$.*

Proof. The subgroups M_6 (or M_7) (two non-conjugate copies) and 11:5 of G are the only ones whose order is divisible by 11 and they both do not have elements of order 7. By Table A.16 we have $\Delta^*_G(7A,11X,11Y) = \Delta_G(7A,11X,11Y) = 29700 > 0$, proving that G is $(7A,11X,11Y)$-generated for all $X,Y \in \{A,B\}$. □

$(11,11,r)$-generations

We conclude our investigation on the (p,q,r)-generation of the alternating group A_{11} by considering the $(11,11,11)$-generations. Thus we will be looking at the cases $(11A,11A,11A)$-, $(11A,11A,11B)$-, $(11A,11B,11B)$- and $(11B,11B,11B)$-generation.

Proposition 5.2.26. *The group G is $(11A,11A,11A)$-, $(11A,11A,11B)$-, $(11A,11B,11B)$- and $(11B,11B,11B)$-generated.*

Proof. The cases $(11A,11A,11A)$, $(11A,11A,11B)$ and $(11B,11B,11B)$ follow by Proposition

5.2.8 together with the applications of Theorem 2.2.3. By Proposition 5.2.2, 11:5 and M_6 (or M_7) (two non-conjugate copies) are the only groups of G whose order is divisible by 11. We have $\sum_{11:5}(11a, 11b, 11b) = 2$, $\sum_{M_6}(11a, 11b, 11b) = 35$, $h(11B, 11{:}5) = h(11B, M_6) = 1$. Since by Table A.16, we have $\Delta_G(11A, 11B, 11B) = 1476600$, we then obtained that $\Delta_G^*(11A, 11B, 11B) \geq \Delta_G(11A, 11B, 11B) - \sum_{M_6}(11a, 11b, 11b) - \sum_{M_7}(11a, 11b, 11b) + \sum_{11:5}(11a, 11b, 11b) = 1476600 - 35 - 35 + 2 = 1476532 > 0$, proving that G is $(11A, 11B, 11B)$-generated group. □

5.3. The conjugacy classes ranks of A_{11}

Now we study the ranks of G with respect to the various conjugacy classes of all its non-identity elements. We start our investigation on the ranks of the alternating group A_{11} by looking at two classes of involutions, namely $2A$ and $2B$. Since $G \ncong D_{2n}$, it follows that the lower bound of the rank of an involutry class of G will be 3.

Lemma 5.3.1. *rank*$(G : 2A) \notin \{3, 4\}$.

Proof. Now if G is $(2A, 2A, 2A, nX)$-generated, then by Scott's Theorem [50] we must have $d_{2A} + d_{2A} + d_{2A} + d_{nX} \geq 2 \times 10$. However, it is clear from Table 5.1 that $3 \times d_{2A} + d_{nX} = 3 \times 2 + d_{nX} < 20$ for each nX, where nX is a set of all the non-identity classes of G and therefore G is not $(2A, 2A, 2A, nX)$-generated, for any nX. We use the similar arguments to prove that G is not $(2A, 2A, 2A, 2A, nX)$-generated because $4 \times d_{2A} + d_{nX} = 4 \times 2 + d_{nX} < 20$ for any $nX \in T$. Hence *rank*$(G : 2A) \notin \{3, 4\}$. □

Proposition 5.3.2. *rank*$(G : 2A) = 5$.

Proof. It was proved in Proposition 5.2.5 that the group G is $(2A, 5B, 11X)$-generated for $X \in \{A, B\}$. Since G is $(2A, 5B, 11X)$-generated for $X \in \{A, B\}$, then by Corollary 2.2.2, we must have $rank(G:2A) \leq 5$. Since by Lemma 5.3.1, $rank(G:2A) \notin \{3,4\}$, it follows that $rank(G:2A) = 5$. □

Proposition 5.3.3. $rank(G:2B) = 3$.

Proof. It was proved in Proposition 5.2.2 that the group G is $(2B, 3C, 11X)$-generated for $X \in \{A, B\}$, then by Corollary 2.2.2, we must have $rank(G:2B) \leq 3$. It then follows that $rank(G:2B) = 3$. □

Proposition 5.3.4. $rank(G:3A) = 5$.

Proof. Now if G is $(3A, 3A, 3A, nX)$-generated, then by Scott's Theorem [50] we must have $d_{3A} + d_{3A} + d_{3A} + d_{nX} \geq 2 \times 10$. However, it is clear from Table 5.1 that $3 \times d_{3A} + d_{nX} = 3 \times 2 + d_{nX} < 20$ for each non-identity class of G and therefore G is not $(3A, 3A, 3A, nX)$-generated. We use similar arguments to prove that G is not $(3A, 3A, 3A, 3A, nX)$-generated because $4 \times d_{2A} + d_{nX} = 4 \times 2 + d_{nX} < 20$ for any non-identity nX of G. It was proved in Proposition 5.2.13 that the group G is $(3A, 5B, 11A)$-generated. By applying Lemma 2.2.1 above, it follows that G is $(3A, 3A, 3A, 3A, 3A, (11A)^5)$-generated. Using GAP, $(11A)^5 = 11A$ so that G becomes $(3A, 3A, 3A, 3A, 3A, 11A)$-generated. Since $rank(G:3A) \notin \{2,3,4\}$, it follows that
$rank(G:3A) = 5$. □

Proposition 5.3.5. $rank(G:3B) = 3$.

Proof. If the group G is $(3B, 3B, nX)$-generated then we must have $c_{3B} + c_{3B} + nX \leq 13$ where nX is any non-identity class of G. Since by Table 5.1 we have $c_{3B} + c_{3B} + c_{nX} = 7 + 7 + c_{nX} > 13$, using Ree's Theorem [49], it follows that G is not $(3B, 3B, nX)$-generated.

Thus $rank(G : 3B) \neq 2$. It was proved in Proposition 5.2.10 that the group G is $(3B, 3C, 11X)$-generated for $X \in \{A, B\}$. By applying Lemma 2.2.1 above, then we obtained that the group G is $(3B, 3B, 3B, (11X)^3)$-generated for all $X \in \{A, B\}$. It is easy to check with GAP that $(11A)^3 = 11A$ and $(11B)^3 = 11B$. Thus G becomes $(3B, 3B, 3B, 11X)$-generated for $X \in \{A, B\}$. Hence $rank(G : 3B) = 3$. □

Proposition 5.3.6. $rank(G : 3C) = 2$.

Proof. It was proved in Proposition 5.2.2 that the group G is $(2B, 3C, 11X)$-generated for $X \in \{A, B\}$, then by Corollary 2.2.4, it follows that $rank(G : 3C) = 2$. □

Proposition 5.3.7. $rank(G : 4A) = 3$.

Proof. If the group G is $(4A, 4A, nX)$-generated then we must have $c_{4A} + c_{4A} + c_{nX} \leq 13$ where nX is any non-identity class of G. Since by Table 5.1 we have $c_{3B} + c_{3B} + c_{nX} = 7 + 7 + c_{nX} > 13$, using Ree's Theorem [49], it follows that G is not $(3B, 3B, nX)$-generated. Thus $rank(G : 4A) \neq 2$. We see that no maximal subgroup of G meets the classes $3C$, $4A$ and $11A$ of G. We then obtained that $\Delta^*_G(4A, 3C, 11A) \geq \Delta_G(4A, 3C, 11A) = 132 > 0$, proving that G is $(4A, 3C, 11A)$-generated. By applying Lemma 2.2.1, we then obtain that the group G is $(4A, 4A, 4A, (11A)^3)$-generated. Since $(11A)^3 = 11A$, the group G will become $(4A, 4A, 4A, 11A)$-generated. Hence $rank(G : 4A) = 3$. □

Proposition 5.3.8. $rank(G : 5A) = 3$.

Proof. Now if G is $(5A, 5A, nX)$-generated, then by Scott's Theorem we must have $d_{5A} + d_{5A} + d_{nX} \geq 2 \times 10$. However, it is clear from Table 5.1 that $2 \times d_{5A} + d_{nX} = 2 \times 4 + d_{nX} < 20$ for each nX a non-identity class of G and therefore G is not $(5A, 5A, nX)$-generated. Thus $rank(G : 5A) \notin 2$. We see that no maximal subgroup of G meets the classes $3C$,

$5A$ and $11A$ of G. We then obtained that $\Delta_G^*(5A, 3C, 11A) \geq \Delta_G(5A, 3C, 11A) = 22 > 0$, proving that G is $(5A, 3C, 11A)$-generated. Applying Lemma 2.2.1, we obatin that the group G is $(5A, 5A, 5A, (11A)^3)$-generated. Since $(11A)^3 = 11A$, the group G will become $(5A, 5A, 5A, 11A)$-generated. Hence $rank(G : 5A) = 3$. □

Proposition 5.3.9. *$rank(G : 6B) = 3$.*

Proof. Now if G is $(6B, 6B, nX)$-generated, then by Scott's Theorem we must have $d_{6B} + d_{6B} + d_{nX} \geq 2 \times 10$. However, it is clear from Table 5.1 that $2 \times d_{6B} + d_{nX} = 2 \times 4 + d_{nX} < 20$ for each nX a non-identity class of G and therefore G is not $(6B, 6B, nX)$-generated. Thus $rank(G : 6B) \neq 2$. We see that no maximal subgroup of G meets the classes $3C$, $6B$ and $11A$ of G. We obtain that $\Delta_G^*(6B, 3C, 11A) \geq \Delta_G(3C, 6B, 11A) = 330 > 0$, proving that G is $(6B, 3C, 11A)$-generated. By applying Lemma 2.2.1 above, then we obtained that the group G is $(6B, 6B, 6B, (11A)^3)$-generated. Since $(11A)^3 = 11A$, the group G will become $(6B, 6B, 6B, 11A)$-generated. Hence $rank(G : 6B) = 3$. □

Proposition 5.3.10. *Let $nX \in T := \{4B, 4C, 5B, 6A, 6C, 6D, 6E, 7A, 8A, 9A, 10A, 11A, 11B, 12A, 12B, 12C, 14A, 15A, 15B, 20A, 21A, 21B\}$ then $rank(G : nX) = 2$.*

Proof. From Table 5.2 we see that M_6 (or M_7) (two non-conjugate copies) is the only maximal subgroup containing elements of order 11. The non-empty intersection of conjugacy classes in M_6 with those in M_7 gives us a subgroup of G which is isomorphic to 11:5. The value of h for each contributing subgroup of G is 1. In Table 5.4, we listed we list the values of $\Delta_G(nX, nX, 11A)$, h, $h\sum_{M_6}(nx, nx, 11a)$, $h\sum_{M_7}(nx, nx, 11a)$, $h\sum_{11:5}(nx, nx, 11a)$ and $\Theta_G(nX, nX, 11A)$. Since $\Delta_G^*(nX, nX, 11A) \geq \Theta_G(nX, nX, 11A) > 0$ in Table 5.4, it follows that G is $(nX, nX, 11A)$-generated where $nX \in T$. This proves that $rank(G : nX) = 2$ for all $nX \in T$. □

The following Table 5.4 gives information on partial structure constants of G computed using GAP and the relevant information required to calculate $\Theta_G(nX, nX, 11A)$. We give some information about $\Delta_G(nX, nX, 11A)$, h, $\sum_{M_6}(nx, nx, 11a)$, $\sum_{M_7}(nx, nx, 11a)$ and $\sum_{11:5}(nx, nx, 11a)$. The last column $\Theta_G(nX, nX, 11A) = \Delta_G(nX, nX, 11A) - h\sum_{M_6}(nx, nx, 11a) - h\sum_{M_7}(nx, nx, 11a) + h\sum_{11:5}(nx, nx, 11a)$ establishes each generation of G by its triples $(nX, nX, 11A)$ because $\Delta^*_G(nX, nX, 11A) \geq \Theta_G(nX, nX, 11A)$, that is $\Delta^*_G(nX, nX, 11A) > 0$ then the group G is $(nX, nX, 11A)$-generated.

Table 5.4: Some information on the classes $nX \in T$

nX	$\Theta_G(nX, nX, 11A)$	h	$h\sum_{M_6}$	$h\sum_{M_7}$	$h\sum_{11:5}$	$\Theta_G(nX, nX, 11A)$
$4B$	1320	1	77	77	-	1166
$4C$	2640	1	-	-	-	2640
$5B$	31680	1	297	297	22	31108
$6A$	55	1	-	-	-	55
$6C$	3960	1	-	-	-	3960
$6D$	8800	1	-	-	-	8800
$6E$	55220	1	154	154	-	54912
$7A$	825	1	-	-	-	825
$8A$	318780	1	429	429	-	317922
$9A$	221760	1	-	-	-	221760
$10A$	11880	1	-	-	-	11880
$11A$	147600	1	35	35	3	147533
$11B$	162000	1	80	80	3	161843
$12A$	8085	1	-	-	-	8085
$12B$	31680	1	-	-	-	31680
$12C$	139260	1	-	-	-	139260
$14A$	23265	1	-	-	-	23265
$15A$	6160	1	-	-	-	6160
$15B$	8976	1	-	-	-	8976
$20A$	44748	1	-	-	-	44748
$21A$	44880	1	-	-	-	44880
$21B$	44880	1	-	-	-	44880

The rank for each conjugacy class of elements for the alternating group A_{11} will be summarized as follows:

- $rank(G : 2A) = rank(G : 3A) = 5$, results are in Propositions 5.3.2 and 5.3.4,
- $rank(G : 2B) = rank(G : 3B) = rank(G : 4A) = rank(G : 5A) = rank(G : 6B) = 3$, results are in Propositions 5.3.3, 5.3.5, 5.3.7, 5.3.8 and 5.3.9,
- $rank(G : nX) = 2$ if $nX \notin \{1A, 2A, 2B, 3A, 3B, 4A, 5A, 6B\}$, results are in Propositions 5.3.6 and 5.3.10.

APPENDIX A

Appendix

A.1 Tables of the symplectic group $Sp(6,2)$

Tables A.1 to A.8 give the partial structure constants of $Sp(6,2)$ computed by Gap [26] that will be used in our calculations.

Table A.1: Structure constants $\Delta_{Sp(6,2)}(2A, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(2A,2A,pX)$	0	0	2	0	3	0	0	0	0
$\Delta_G(2A,2B,pX)$	0	0	1	1	0	0	0	0	0
$\Delta_G(2A,2C,pX)$	30	3	0	3	0	0	0	0	0
$\Delta_G(2A,2D,pX)$	0	12	12	3	0	0	0	0	0
$\Delta_G(2A,3A,pX)$	32	0	0	0	0	0	0	0	0
$\Delta_G(2A,3B,pX)$	0	0	0	0	0	0	0	0	0
$\Delta_G(2A,3C,pX)$	0	0	0	0	0	0	0	0	0
$\Delta_G(2A,5A,pX)$	0	0	0	0	0	0	0	0	0
$\Delta_G(2A,7A,pX)$	0	0	0	0	0	0	0	0	14
$\|C_G(pX)\|$	23040	4608	1536	384	2160	648	108	30	7

Table A.2: Structure constants $\Delta_{Sp(6,2)}(2B, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(2B,2B,pX)$	0	18	8	0	0	0	3	0	0
$\Delta_G(2B,2C,pX)$	15	24	6	3	0	0	0	0	0
$\Delta_G(2B,2D,pX)$	60	0	12	15	0	0	0	0	0

Table A.2:Continued

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(2B,3A,pX)$	0	0	0	0	0	0	0	0	0
$\Delta_G(2B,3B,pX)$	0	0	0	0	0	0	0	0	0
$\Delta_G(2B,3C,pX)$	0	128	0	0	0	0	27	0	7
$\Delta_G(2B,5A,pX)$	0	0	0	0	0	0	0	15	7
$\Delta_G(2B,7A,pX)$	0	0	0	0	0	0	108	30	70
$\|C_G(pX)\|$	23040	4608	1536	384	2160	648	108	30	7

Table A.3: Structure constants $\Delta_{Sp(6,2)}(2C, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(2C,2C,pX)$	0	18	34	6	45	0	9	5	0
$\Delta_G(2C,2D,pX)$	180	36	24	21	0	0	0	0	0
$\Delta_G(2C,3A,pX)$	0	0	32	0	45	0	0	5	0
$\Delta_G(2C,3B,pX)$	0	0	0	0	0	27	0	0	0
$\Delta_G(2C,3C,pX)$	0	0	128	0	0	0	54	20	14
$\Delta_G(2C,5A,pX)$	0	0	256	0	360	0	72	140	14
$\Delta_G(2C,7A,pX)$	0	0	0	0	0	0	216	60	210
$\|C_G(pX)\|$	23040	4608	1536	384	2160	648	108	30	7

Table A.4: Structure constants $\Delta_{Sp(6,2)}(2D, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(2D,2D,pX)$	180	180	84	12	180	108	36	15	7
$\Delta_G(2D,3A,pX)$	0	0	0	32	0	0	0	0	0
$\Delta_G(2D,3B,pX)$	0	0	0	64	0	0	0	0	7
$\Delta_G(2D,3C,pX)$	0	0	0	128	0	0	54	30	28
$\Delta_G(2D,5A,pX)$	0	0	0	192	0	0	108	90	98
$\Delta_G(2D,7A,pX)$	0	0	0	384	0	648	432	420	560
$\|C_G(pX)\|$	23040	4608	1536	384	2160	648	108	30	7

Table A.5: Structure constants $\Delta_{Sp(6,2)}(3A, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(3A,3A,pX)$	0	0	32	0	46	0	2	5	0
$\Delta_G(3A,3B,pX)$	0	0	0	0	0	12	2	0	0
$\Delta_G(3A,3C,pX)$	0	0	0	0	40	12	20	20	7
$\Delta_G(3A,5A,pX)$	0	0	256	0	360	0	72	120	7

Table A.5:Continued

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(3A, 7A, pX)$	0	0	0	0	0	0	108	30	133
$\|C_G(pX)\|$	23040	4608	1536	384	2160	648	108	30	7

Table A.6: Structure constants $\Delta_{Sp(6,2)}(3B, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(3B, 3B, pX)$	0	0	64	0	40	28	20	10	7
$\Delta_G(3B, 3C, pX)$	0	0	0	0	40	120	4	10	7
$\Delta_G(3B, 5A, pX)$	0	0	0	0	0	216	36	30	77
$\Delta_G(3B, 7A, pX)$	0	0	0	384	0	648	108	330	245
$\|C_G(pX)\|$	23040	4608	1536	384	2160	648	108	30	7

Table A.7: Structure constants $\Delta_{Sp(6,2)}(3C, qY, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(3C, 3C, pX)$	0	1152	768	192	400	24	500	150	203
$\Delta_G(3C, 5A, pX)$	0	0	1024	384	1440	216	540	690	441
$\Delta_G(3C, 7A, pX)$	0	4608	3072	1536	2160	648	3132	1890	2289
$\|C_G(pX)\|$	23040	4608	1536	384	2160	648	108	30	7

Table A.8: Structure constants $\Delta_{Sp(6,2)}(5A, qY, rZ)$ and $\Delta_{Sp(6,2)}(7A, 7A, rZ)$

pX	2A	2B	2C	2D	3A	3B	3C	5A	7A
$\Delta_G(5A, 5A, pX)$	0	2304	7168	1152	8640	648	2484	3998	1379
$\Delta_G(5A, 7A, pX)$	0	4608	3072	5376	2160	7128	6804	5910	7483
$\Delta_G(7A, 7A, pX)$	46080	46080	46080	30720	41040	22680	35316	32070	30595
$\|C_G(pX)\|$	23040	4608	1536	384	2160	648	108	30	7

A.2 Tables of the sporadic simple group M_{23}

Tables A.9 to A.12 give the partial structure constants of M_{23} that were used in the calculations.

Table A.9: Structure constants $\Delta_{M_{23}}(2A, qY, rZ)$

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_G(2A, 2A, pX)$	98	30	5	0	0	0	0	0	0
$\Delta_G(2A, 3A, pX)$	448	180	65	35	35	11	11	0	0
$\Delta_G(2A, 5A, pX)$	896	780	605	364	364	253	253	138	138
$\Delta_G(2A, 7A, pX)$	0	450	390	301	462	308	308	184	184
$\Delta_G(2A, 7B, pX)$	0	450	390	462	301	308	308	184	184
$\Delta_G(2A, 11A, pX)$	0	180	345	392	392	341	341	391	391
$\Delta_G(2A, 11B, pX)$	0	180	345	392	392	341	341	391	391
$\Delta_G(2A, 23A, pX)$	0	0	90	112	112	187	187	161	230
$\Delta_G(2A, 23B, pX)$	0	0	90	112	112	187	187	230	161
$\|C_G(pX)\|$	688	180	15	14	14	11	11	23	23

Table A.10: Structure constants $\Delta_{M_{23}}(3A, qY, rZ)$

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_G(3A, 3A, pX)$	2688	1681	855	511	511	275	275	138	138
$\Delta_G(3A, 5A, pX)$	11648	10260	6550	5124	5124	3795	3795	2438	2438
$\Delta_G(3A, 7A, pX)$	6720	6570	5490	4886	4886	4136	4136	3312	3312
$\Delta_G(3A, 7B, pX)$	6720	6570	5490	4886	4886	4136	4136	3312	3312
$\Delta_G(3A, 11A, pX)$	2688	4500	5175	5264	5264	5126	5379	5129	5129
$\Delta_G(3A, 11B, pX)$	2688	4500	5175	5264	5264	5379	5126	5129	5129
$\Delta_G(3A, 23A, pX)$	0	1080	1590	2016	2016	2453	2453	3082	2714
$\Delta_G(3A, 23B, pX)$	0	1080	1590	2016	2016	2453	2453	2714	3082
$\|C_G(pX)\|$	688	180	15	14	14	11	11	23	23

Table A.11: Structure constants $\Delta_{M_{23}}(5A, qY, rZ)$ and $\Delta_{M_{23}}(7X, qY, rZ)$

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_G(5A, 5A, pX)$	108416	78600	61058	54320	54320	45287	45287	37582	37582
$\Delta_G(5A, 7A, pX)$	69888	65880	58200	52584	52584	48576	48576	44160	44160
$\Delta_G(5A, 7B, pX)$	69888	65880	58200	52584	52584	48576	48576	44160	44160
$\Delta_G(5A, 11A, pX)$	61824	62100	61755	61824	61824	62238	61479	61893	61893
$\Delta_G(5A, 11B, pX)$	61824	62100	61755	61824	61824	61479	62238	61893	61893
$\Delta_G(5A, 23A, pX)$	16128	19080	24510	26880	26880	29601	29601	32706	32706
$\Delta_G(5A, 23B, pX)$	16128	19080	24510	26880	26880	29601	29601	32706	32706
$\Delta_G(7A, 7A, pX)$	88704	62820	56340	51948	60412	48400	48400	52992	52992
$\Delta_G(7A, 7B, pX)$	57792	62820	56340	51948	51948	56496	56496	45264	45264
$\Delta_G(7A, 11A, pX)$	75264	67680	66240	71904	61600	64416	64416	67712	67712
$\Delta_G(7A, 11B, pX)$	75264	67680	66240	71904	61600	64416	64416	67712	67712

Table A.11:Continued

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_G(7A, 23A, pX)$	21504	25920	28800	27552	32256	32384	32384	32384	32384
$\Delta_G(7A, 23B, pX)$	21504	25920	28800	27552	32256	32384	32384	32384	32384
$\Delta_G(7B, 7B, pX)$	88704	62820	56340	60412	51948	48400	48400	52992	52992
$\Delta_G(7B, 11A, pX)$	75264	67680	66240	61600	71904	64416	64416	67712	67712
$\Delta_G(7B, 11B, pX)$	75264	67680	66240	61600	71904	64416	64416	67712	67712
$\Delta_G(7B, 23A, pX)$	21504	25920	28800	32256	27552	32384	32384	32384	32384
$\Delta_G(7B, 23B, pX)$	21504	25920	28800	32256	27552	32384	32384	32384	32384
$\|C_G(pX)\|$	688	180	15	14	14	11	11	23	23

Table A.12: Structure constants $\Delta_{M_{23}}(11X, qY, rZ)$ and $\Delta_{M_{23}}(23X, qY, rZ)$

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_G(11A, 11A, pX)$	83328	88020	83835	81984	81984	87485	88520	81029	81029
$\Delta_G(11A, 11B, pX)$	83328	83880	84870	81984	81984	87485	87485	79994	79994
$\Delta_G(11A, 23A, pX)$	45696	40140	40365	41216	41216	38258	38753	42067	42067
$\Delta_G(11A, 23B, pX)$	45696	40140	40365	41216	41216	38258	38753	42067	42067
$\Delta_G(11B, 11B, pX)$	83328	88020	83835	81984	81984	88520	87485	81029	81029
$\Delta_G(11B, 23A, pX)$	45696	40140	40365	41216	41216	38753	38258	42067	42067
$\Delta_G(11B, 23B, pX)$	45696	40140	40365	41216	41216	38753	38258	42067	42067
$\Delta_G(23A, 23A, pX)$	26880	21240	21330	19712	19712	20119	20119	17646	18222
$\Delta_G(23A, 23B, pX)$	18816	24120	21330	19712	19712	20119	20119	17646	17646
$\Delta_G(23B, 23B, pX)$	26880	21240	21330	19712	19712	20119	20119	18222	17646
$\|C_G(pX)\|$	688	180	15	14	14	11	11	23	23

A.3 Tables for the alternating group A_{11}

Tables A.13 to A.16 give the partial structure constants of A_{11} that were used in the calculations.

Table A.13: Structure constants $\Delta_{A_{11}}(2X, qY, rZ)$

pX	2A	2B	3A	3B	3C	5A	5B	7A	11A	11B
$\Delta_G(2A, 2A, pX)$	44	6	84	9	0	5	0	0	0	0
$\Delta_G(2A, 2B, pX)$	105	24	0	0	0	0	0	0	0	0
$\Delta_G(2A, 3A, pX)$	28	0	24	0	0	5	0	0	0	0
$\Delta_G(2A, 3B, pX)$	168	0	0	39	0	30	0	14	0	0
$\Delta_G(2A, 3C, pX)$	0	0	0	0	45	0	0	0	0	0

Table A.13:Continued

pX	2A	2B	3A	3B	3C	5A	5B	7A	11A	11B
$\Delta_G(2A, 5A, pX)$	56	0	168	18	0	95	0	14	0	0
$\Delta_G(2A, 5B, pX)$	0	0	0	0	0	0	90	0	44	44
$\Delta_G(2A, 7A, pX)$	0	0	0	180	0	300	0	175	0	0
$\Delta_G(2A, 11A, pX)$	0	0	0	0	0	0	100	0	110	220
$\Delta_G(2A, 11B, pX)$	0	0	0	0	0	0	100	0	220	110
$\Delta_G(2B, 2B, pX)$	420	84	1260	165	54	225	25	42	0	0
$\Delta_G(2B, 3A, pX)$	0	24	0	0	0	0	0	0	0	0
$\Delta_G(2B, 3B, pX)$	0	176	0	180	0	0	25	28	0	0
$\Delta_G(2B, 3C, pX)$	0	384	0	0	216	0	100	84	110	110
$\Delta_G(2B, 5A, pX)$	0	144	0	0	0	0	0	28	0	0
$\Delta_G(2B, 5B, pX)$	0	1152	0	1080	648	0	825	504	660	660
$\Delta_G(2B, 7A, pX)$	0	576	0	360	162	600	150	644	55	55
$\Delta_G(2B, 11A, pX)$	0	0	0	0	1620	0	1500	420	2145	1320
$\Delta_G(2B, 11B, pX)$	0	0	0	0	1620	0	1500	420	1320	2145
$\|C_G(pX)\|$	20160	1152	60480	1080	162	1800	25	84	11	11

Table A.14: Structure constants $\Delta_{A_{11}}(3X, qY, rZ)$

pX	2A	2B	3A	3B	3C	5A	5B	7A	11A	11B
$\Delta_G(3A, 3A, pX)$	8	0	25	2	0	5	0	0	0	0
$\Delta_G(3A, 3B, pX)$	0	0	112	32	3	30	0	7	0	0
$\Delta_G(3A, 3C, pX)$	0	0	0	20	21	0	0	0	0	0
$\Delta_G(3A, 5A, pX)$	56	0	168	18	0	40	0	7	0	0
$\Delta_G(3A, 5B, pX)$	0	0	0	0	0	0	30	0	11	11
$\Delta_G(3A, 7A, pX)$	0	0	0	90	0	150	0	63	0	0
$\Delta_G(3A, 11A, pX)$	0	0	0	0	0	0	25	0	110	55
$\Delta_G(3A, 11B, pX)$	0	0	0	0	0	0	25	0	55	110
$\Delta_G(3B, 3B, pX)$	728	192	1792	440	42	380	25	168	0	0
$\Delta_G(3B, 3C, pX)$	0	0	1120	280	390	600	100	224	66	66
$\Delta_G(3B, 5A, pX)$	336	0	1008	228	54	540	0	140	0	0
$\Delta_G(3B, 5B, pX)$	0	1152	0	1080	648	0	1080	504	704	704
$\Delta_G(3B, 7A, pX)$	3360	384	5040	2160	432	3000	150	1428	33	33
$\Delta_G(3B, 11A, pX)$	0	0	0	0	972	0	1600	252	3212	2332
$\Delta_G(3B, 11B, pX)$	0	0	0	0	972	0	1600	252	2332	3212
$\Delta_G(3C, 3C, pX)$	5600	1536	7840	2600	1198	2000	900	840	660	660
$\Delta_G(3C, 5A, pX)$	0	0	0	360	180	0	100	168	22	22
$\Delta_G(3C, 5B, pX)$	0	4608	0	4320	5832	7200	4400	5376	4928	4928
$\Delta_G(3C, 7A, pX)$	0	1152	0	2880	1620	3600	1600	3024	990	990
$\Delta_G(3C, 11A, pX)$	0	11520	0	6480	9720	3600	11200	7560	12760	12760
$\Delta_G(3C, 11B, pX)$	0	11520	0	6480	9720	3600	11200	7560	12760	12760
$\|C_G(pX)\|$	20160	1152	60480	1080	162	1800	25	84	11	11

Table A.15: Structure constants $\Delta_{A_{11}}(5X, qY, rZ)$

pX	2A	2B	3A	3B	3C	5A	5B	7A	11A	11B
$\Delta_G(5A, 5A, pX)$	1064	0	1344	324	0	428	2	112	0	0
$\Delta_G(5A, 5B, pX)$	0	0	0	0	648	144	456	336	440	440
$\Delta_G(5A, 7A, pX)$	3360	384	5040	1800	324	2400	100	1092	11	11
$\Delta_G(5A, 11A, pX)$	0	0	0	0	324	0	1000	84	1892	1804
$\Delta_G(5A, 11B, pX)$	0	0	0	0	324	0	1000	84	1804	1892
$\Delta_G(5B, 5B, pX)$	72576	38016	72576	46656	28512	32832	33984	32256	31680	31680
$\Delta_G(5B, 7A, pX)$	0	6912	0	6480	10368	7200	9600	8736	9504	9504
$\Delta_G(5B, 11A, pX)$	80640	69120	60480	69120	72576	72000	72000	72576	76032	69696
$\Delta_G(5B, 11B, pX)$	80640	69120	60480	69120	72576	72000	72000	72576	69696	76032
$\|C_G(pX)\|$	20160	1152	60480	1080	162	1800	25	84	11	11

Table A.16: Structure constants $\Delta_{A_{11}}(7A, qY, rZ)$ and $\Delta_{A_{11}}(11X, qY, rZ)$

pX	2A	2B	3A	3B	3C	5A	5B	7A	11A	11B
$\Delta_G(7A, 7A, pX)$	42000	8832	45360	18360	5832	23400	2600	11996	825	825
$\Delta_G(7A, 11A, pX)$	0	5760	0	3240	14580	1800	21600	6300	29700	29700
$\Delta_G(7A, 11B, pX)$	0	5760	0	3240	14580	1800	21600	6300	29700	29700
$\Delta_G(11A, 11A, pX)$	403200	138240	302400	228960	187920	295200	158400	226800	147600	162000
$\Delta_G(11A, 11B, pX)$	201600	224640	604800	315360	187920	309600	172800	226800	147600	147600
$\Delta_G(11B, 11B, pX)$	403200	138240	302400	228960	187920	295200	158400	226800	162000	147600
$\|C_G(pX)\|$	20160	1152	60480	1080	162	1800	25	84	11	11

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