EXPLORING GRADE 10 LEARNERS' SPATIAL SKILLS OF COMPUTING SIDES AND ANGLES IN FIGURES USING TRIGONOMETRIC RATIOS

by

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DISSERTATION

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DEDICATION

I dedicate this study to my lovely family:

1. My late dad, Molotolo Johannes Selowa
2. My lovely mom, Nkhensani Mongwe
3. My lovely sister, Patience Selowa

DECLARATION

I Selowa Reinhard, declare that the dissertation entitled "**EXPLORING GRADE 10 LEARNERS' SPATIAL SKILLS OF COMPUTING SIDES AND ANGLES IN FIGURES USING TRIGONOMETRIC RATIOS**'' submitted to the University of Limpopo is my own work and altogether the sources that I have utilised are acknowledged by means of a complete reference. I further declare that this dissertation has not been submitted previously for another degree at any institution.

15/05/2023

Selowa R (Miss) Date

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ABSTRACT

In Grade 10, learners are facing challenges to compute sides and angles using trigonometric ratios. This is a result of accessing inappropriate and fragmented trigonometric concepts which complicates their computational procedures. The purpose of this study was to explore Grade 10 learners' spatial skills when computing sides and angles in figures using trigonometric ratios. This was achieved by adhering to the elements of an exploratory case study. The qualitative study was underpinned by the three principles of semantic tenets, conceptual structures, system representation and sense. Grade 10 learners were purposively sampled to explore their spatial skills exhibited whilst computing sides and angles using the three basic ratios. Data were collected in three phases from 22 purposively sampled Grade 10 learners using three mathematical tasks and semi-structured interviews. The two sets of data were analysed thematically. In summary, the textual data from the three mathematical tasks were read critically and the audio data was transcribed. Both sets of data were categorised using axial coding and four themes emerged. The major findings revealed that learners who leave blank spaces have spatial skills that can be expressed verbally. This was shown as learners exhibited appropriate spatial skills during semi-structured interviews. Moreover, learners utilised improper properties of a right-angled triangle during computations of sides and angles using trigonometric ratios. On the other hand, some learners struggled to relate concepts used within the given questions. This resulted from a lack of prerequisite trigonometric concepts to comprehend what the question requires. Based on the findings of the study, the recommendation is that the learning of trigonometry should be based on developing spatial skills that permit the exploitation of relevant properties during the computations of sides and angles using ratios. This should be followed by inclusive formative assessment and verbal assessment to accommodate learners struggling to achieve in written assessment.

Key concepts: Basic Trigonometric Ratios; Spatial skills; Sides: Opposite; Adjacent; Hypotenuse; Spatial figures: Right-angle triangle

KEY CONCEPTS

Angle-is a size within or around a vertex formed by two lines. It can be represented using variables, and symbols such as θ.

Basic Trigonometric Ratios- sine, cosine, and tangent.

Spatial skills: Three subskills that enable learners to understand, analyse and synthesise spatial figures:

- **SV**-Spatial visualisation
- **SR**-spatial relation
- **SO**-Spatial orientation

Spatial figures- a visual representation question using geometric concepts, for example, right-angled triangles.

Right-angle triangle- a spatial figure with an interior angle of 90. It is made up of three sides, namely, opposite, adjacent and hypotenuse.

Side- is a line segment.

Opposite-is a side opposite the right-angle.

Adjacent-is a side located next to the given and right-angle.

Hypotenuse-a longest side opposite the right angle.

Trigonometry- A mathematics topic stated in the mathematics FET CAPS document.

FET-Further education and training. Refer to training and education that is provided from Grades 10 to 12.

CAPS- Curriculum and Assessment Policy Statement-Is a single, comprehensive and concise policy document introduced by the Department of Basic Education for all the subjects listed in the National Curriculum Statement for Grades R-12. It gives teachers guidance on how to teach and assess learners.

GET-General Education and Training.

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CHAPTER ONE: INTRODUCTION

1.1 Introduction

Substantial research on learners' computational abilities showed that learners still have difficulties navigating between tasks involving spatial figures (Khuzwayo, 2019; Sorby & Baartmans, 1996; Ssebaggala, 2019; Utami et al., 2021). Spatial figures provide invaluable support in trigonometric computations (Sánchez et al., 2023), and yet this area of trigonometry has not been well-documented in the literature in terms of spatial skills. Spatial figures are defined as representations derived using geometric ideas (Battista, 1990; Philips et al., 2010). The geometric ideas can be sides which are line segments whereas angles are sizes within or around a vertex formed by two lines represented in terms of variables or symbols used to denote angles or sides. These geometric ideas bring a rich context for learners to notice the interrelatedness between different properties that are essential for computations in trigonometry (Kariadinata et al., 2013). Trigonometry is a content domain that studies the relationships between the properties of sides and angles of spatial figures (Walsh et al., 2017). However, the properties of the spatial figures require an understanding of how meaning is conveyed within the orientation of the figure (Madonsela et al., 2020). This leads to a need for a proper choice of trigonometric ratio to compute sides and angles (Department of Basic Education, DBE, 2013). Hence, for learners to navigate between spatial figures and trigonometric concepts, they require established spatial skills and proper use in the context of trigonometry (Martín-Fernández et al., 2019).

Spatial skills provide learners with an opportunity to mentally visualise, transform, and retrieve the position of figures in a context (Halpern, 2000). Mix and Battista (2018) concur with Halpern that spatial skills assist learners in planning how to navigate from one point to another within a figure. In the context of trigonometry, spatial skills are learners' ability to mentally manipulate properties of figures applied within various orientations to compute sides and angles (Atit et al., 2020). Essentially, if learners can develop spatial skills, solving spatial figure problems will be easier and more interesting. In this study, three types of spatial skills are considered while computing sides and angles using trigonometric ratios. Spatial visualisation (SV) is the process of making mental rotations of figures in various orientations (Carroll, 1993; Tandon et al., 2022). According to Fujita et al. (2020), SV is not making mental rotations only but changing the given figures by drawing additional lines. These additional lines enhance learners' understanding of spatial figures. In this instance, SV is learners' ability to mentally imagine how spatial figures relate to trigonometric concepts such as properties of right-angled triangles, and the theorem of Pythagoras as they appear in a variety of orientations (Kalogirou & Gagatsis, 2012; Jaelani, 2021). Therefore, developing SV is crucial for interpreting spatial figures and spotting patterns.

Interpreting spatial figures deliberately allow learners to handle the second spatial skill, namely, spatial relation (SR). The SR is considered as the ability to manipulate the given figures in various positions using various concepts in diverse contexts (Surtees et al., 2013). Buczkowski (2003) explained SR as the ability to organise visual information based on their similarities and differences. Buczkowski adds that SR is also regarded as a relationship as it deals with the organisation of visual information. In this case, SR is the ability to change mental images, and then choose the appropriate trigonometric ratio to compute sides and angles. As a result, this assists individuals to make sense of the chosen trigonometric ratio.

Making sense of the trigonometric ratio is a key element of operating within the last spatial skill called spatial orientation (SO). The SO is the ability to make a mental transformation for clarifying statements using the SV and SR to make an informed judgement (Kozhevnikov & Hegarty, 2001). This means that both SV and SR are requirements for the justification of statements. In this study, SO relates to the ability to justify chosen trigonometric ratio whilst computing the sides and angles of figures. Emphasis on SO can assist learners to organise their responses logically when given trigonometric problems. Therefore, handling the SV provides learners with an opportunity to operate within the last two skills, the SR and SO during computation (Edler & Kersten, 2021).

1.2 Background and motivation

Studies in various provinces around South Africa such as Kwazulu-Natal, Limpopo, Gauteng and other countries around the world have reported various challenges faced by learners when computing the sides and angles of figures (Molataola, 2017; Ngcobo, 2019; Nurmeidina & Rafidiyah, 2019; Rankweteke, 2020; Walsh et al., 2017). These range from the ability to apply relevant trigonometric ratios (Madonsela et al., 2020), to how learners relate spatial figures and trigonometric concepts (Kozhevnikov & Hegarty, 2001; Fahrudin, 2019). For instance, Rankweteke (2020) conducted a study in Gauteng, South Africa with Grade 11 learners to enhance their conceptual understanding of trigonometry. In that study, findings revealed that most learners have challenges in processing figures against trigonometric questions. Meanwhile, Molataola (2017) conducted a study to improve Grade 10 learners' conceptual understanding of ratios using Pythagoras theorem. In that study, findings revealed that learners struggled to investigate the relationship between sides of a right-angled triangle. However, spatial skills that learners exhibit when computing sides and angles using trigonometric ratios are not explicit in the literature. Besides, Nurmeidina and Rafidiyah (2019) found that learners struggle to understand mathematical statements. Their findings further showed that learners could not identify the necessary information from the statement and the figure to compute sides and angles.

Similarly, Wardhani and Argaswari (2022) pointed out that learners struggle to understand or notice words used in the given statements. Correspondingly, the current study outlined that some learners struggled to compute sides and angles due to failure to notice vital words on the question. This is due to lack of the SV to imagine and recognise the essential words such as given numeric values of sides or angles on the given question and spatial figure. As such, the difficulty prohibited learners to use their SR to choose the correct ratio to compute the required sides or angles. It is known that the topic of trigonometry is initially introduced in Grade 10, however, some concepts within the topic were introduced in the previous grades such as sides, and angles when computing sides using Pythagoras theorem. To ensure that the pre-existing concepts are considered, the mathematical CAPS document emphasised that the learning process of trigonometric ratios can be divided into various stages, including before, during and after (DBE, 2013). In this study, the three stages were implemented within the three lessons to document more details on the computations of sides and angles. The stages were implemented in this chronology: 1) in the introduction, for checking learners' prior knowledge of properties of a right-angled triangle; 2) for introducing the concept of trigonometric ratio; and 3) after, for checking whether the learning of trigonometric ratio has occurred or not. As a result, the three stages enabled me to explore Grade 10 learners' spatial skills to check if learners are capable of using their existing knowledge to discover how the three ratios are used during computations. As such, I recommended teachers to explicitly assist learners in developing spatial skills. Hence, this prompted the study to explore Grade 10 learners' spatial skills when computing sides and angles using trigonometric ratios.

1.3 Problem statement

Spatial skills are increasingly required while computing the sides and angles of figures (Rittle-Johnson et al., 2019). These afford learners an opportunity to explore spatial figures and choose appropriate trigonometric ratios to compute sides and angles (Firdaus, 2017). Parallel to spatial skills, the basic knowledge of trigonometric concepts is required to strengthen learners' ability to compute the sides and angles of various spatial figures (Madonsela et al., 2020). From this perspective, computing the sides and angles of spatial figures allowed learners to move flexibly between the three aspects of computing sides and angles. Firstly, to access relevant concepts from their prior knowledge for computations. Secondly, to use the relevant concepts to analyse the interrelatedness of figures applied within various orientations. Finally, to choose relevant trigonometric ratios to compute sides and angles accurately and consistently towards obtaining the desired solutions (Sánchez et al., 2023; Widada, et al., 2019). As such, failure to access relevant concepts from their prior knowledge forbids learners to move to the second (analysing figures) and last (computing sides and angles accurately and consistently) tenets.

Despite so much research that has been done on factors affecting the learning of trigonometry (Kagenyi, 2016; Kirkland, 2020; Utami et al., 2021), results showed that it is still not clear why learners struggle to access relevant concepts from their prior knowledge when computing sides and angles using trigonometric ratios. This struggle hinders them from analysing the interrelatedness of figures applied within various orientations (Kirkland, 2020). As a result, learners obtain undesired solutions during computations due to the application of irrelevant concepts while analysing spatial figures (Nabie et al., 2018). The results from a study by Nanmumpuni and Retnawati (2021) on the analysis of learners' difficulties in solving trigonometric problems emphasised that the inability to access relevant concepts from prior knowledge poses knowledge gaps related to the second (analysing figures) and last (computing sides together with angles accurately and consistently) tenets (Bishop, cited in Tartre, 1990; Van Garderen, 2003). To address the knowledge gap, the semantic theory is used to consistently explore spatial skills exhibited by learners as they compute the sides and angles of figures (Dimmel et al., 2021; Sujadi, & Subanti, 2019). Hence, the study explored Grade 10 learners' spatial skills in computing sides and angles using trigonometric ratios. Although there are many studies on trigonometry, the literature outlined that it is still uncertain why learners have difficulties in choosing proper trigonometric ratios during computations. Thus, the contribution of this study was to encourage learners in developing the three spatial skills to understand the context of figures during the computations of sides and angles using basic ratios.

1.4 The purpose of the study

The purpose of the study was to explore Grade 10 learners' spatial skills in computing sides and angles of figures using trigonometric ratios. This assists in noticing appropriate strategies to close the knowledge gaps of accessing irrelevant concepts from learners' prior knowledge, analysing figures using irrelevant concepts, and utilising irrelevant trigonometric ratios to compute sides and angles.

1.5 Research questions

The study was guided by two research questions stated below:

1.5.1 How do learners' abilities of computing sides and angles of figures reflect their spatial skills?

1.5.2 Why do learners compute the sides and angles of figures in particular ways?

1.6 Research Methodology

This section outlines the chosen research approach of this study. However, the full details are documented in Chapter 3. In this study, the qualitative approach was used to gain insight into learners' spatial skills exhibited whilst computing sides and angles. Creswell (2009) asserts that the qualitative approach is a procedure followed by researchers to obtain a detailed understanding of a context. Similarly, Jameel et al. (2018) maintained that the qualitative approach is a process of gathering and understanding non-numeric data by observing a social phenomenon as it occurs. The approach was appropriate for this study as it allowed me to apply three tenets of semantic theory to interpret and explain learners' spatial skills shown when computing sides and angles.

1.6.1 Research paradigm

A research paradigm refers to a fundamental set of values, assumptions and beliefs which guide researchers based on their research designs (Creswell, 2009). It is characterised by elements that illustrate the philosophical point of view of the researcher. To be precise, the study is based on the constructivism paradigm, a belief that learners should discover mathematical concepts on their own (Kumatongo & Muzata, 2021). The study chose constructivism as Grade 10 learners were expected to use their spatial skills in discovering trigonometric concepts that would assist during computations of sides and angles. In doing this, three mathematical tasks were implemented at different stages, for checking prior knowledge of side names, developing the trigonometric concepts for computing sides and angles, and checking whether the trigonometric concepts were obtained or not.

1.6.2 Research design

A suitable research design for this study was an exploratory case study (Merriam, 1998). Patnaik and Pandey (2019) defined an exploratory case study as a method of understanding a phenomenon in a single setting to provide a clear description. Similarly, Yazan (2015) identified an exploratory case study as a process of exploring a situation with specific boundaries based on the researcher's interests. According to Zainal (2007), conducting an exploratory case study allows the researcher to closely examine a specific context. The case study is categorised by three principles: 1) a case, Grade 10 learner; 2) boundary, exploring Grade 10 learners' spatial skills when computing the sides and angles of figures; and the type of case study, a single for testing a theory or a multiple case study for developing a rich theory. In this study, the three aspects of an exploratory case study allowed me to identify the difficulties that prohibit learners from using relevant spatial skills during computations of sides and angles using trigonometry ratios.

1.6.3 Sampling

The study used purposive sampling which is a form of selection whereby the researcher relies on their own judgment when choosing participants (Palinkas et al., 2015). Purpose sampling is useful to address concerns that arise during a specific event such as the learning of trigonometry (Ames et al., 2019). In this case, the study purposively sampled a Grade 10 class at a rural public school around Limpopo whereby traditional teaching methods such as chalkboards are still used during the process of teaching and learning. Based on enrolment, the school comprises 487 learners, 233 males, and 240 females maintained by 16 staff members. Hence, the Grade 10 class has 13 females, 9 males (22) and they all participated in this study. The school is dominated by Xitsonga-speaking, and few Sepedi-speaking, learners. The variety assisted in exploring spatial skills shown during the computations of sides and angles using ratios. Moreover, mathematical tasks were purposively selected from online trigonometric questions and previous question papers for computing the sides and angles of figures.

1.6.4 Data Collection

Firstly, three mathematical tasks (**Appendix D**) were used to collect data within the three lessons in responding to the first research question. The lessons were conducted within a period of two weeks. During the first lesson, learners were given a short baseline task for 50 minutes to check their prior knowledge of sides and angles on a right-angled triangle. The task was written under the researchers' supervision. During the second lesson, learners were given a learning activity for 60 minutes to develop the skills and computational concepts such as sine, cosine and tangent. Throughout the lesson, learners were observed as they are working on the learning activity. Also, the researcher was responsible for providing clarity were needed. Thereafter, an assessment task was given to learners for 60 minutes on the last lesson to check whether they can apply the three ratios or not when computing sides and angles. This allowed me to capture learners' spatial skills exhibited through accessing relevant concepts required in the computations of sides and angles. At the end each lesson, all scripts were collected and classified based on their similarities and differences using the semantic tenets. Secondly, semi-structured interviews (**Appendix E**) were conducted for approximately 5 minutes after categorising the scripts for each lesson using the audio recording to probe for reasons why learners compute sides and angles in particular ways. Semi-structured interviews refer to a way of asking a few pre-determined questions while the rest of the questions arise from the interviewee's responses (Ahlin, 2019). All interview questions were based on learners' responses to the given mathematical tasks mentioned above. The semi-structured interviews were conducted with both learners who exhibited inappropriate and appropriate spatial skills. This was a way of ensuring that learners provide their spatial skills in detail such that they can be explored.

1.6.5 Data analysis

Thematic analysis was used to analyse data from mathematical tasks such as baseline tasks, learning activity and semi-structured interviews (Braun & Clarke, 2012). Thematic analysis refers to a technique used to identify, organise and describe a pattern of meaning across a set of data (Clarke, 2012). Thematic analysis was essential as it enabled me to comprehend learners' spatial skills during computations. During the process of analysis, textual data was read in detail, thereby checking learners' spatial skills when computing sides and angles of figures and the audio data was changed into textual data. Thereafter, axial coding was used, and the three appointed coders categorised learners' spatial skills based on their differences and similarities. During the coding process, the three tenets of semantic theory were used to explore the data. Subsequently, four themes emerged from the codes of the three lessons. This is ample evidence that an inductive approach was applied because the themes emerged from the two sets of data.

1.6.6 Quality Criteria

The study ensured four aspects of quality criteria. Firstly, credibility is the process of representing the meaning of the study (Lincoln & Guba, 1985). In this study, credibility was ensured using triangulation, thereby collecting data using a variety of methods. For instance, data was collected using mathematical tasks and semi-structured interviews. Secondly, the study also considered dependability such that similar results can be obtained in other contexts using the same research procedures. As such, an inquiry audit was used to ensure dependability. An inquiry audit is a way of examining research processes and products in the absence of the researcher (Hoepfl, 1997). Thus, appointed coders or teachers examined the research procedures. Thirdly, confirmability emphasises the aspect of objectivity (Patton, 1990), and was ensured through an audit trail, which indicates procedures performed to keep a record of what has been done while conducting the study. In this study, all research steps from the beginning and reporting of findings were documented as emphasised by Korstjens and Moser (2018). Finally, transferability is a process of ensuring that the findings of the study are applied in other contexts (Pandey & Patnaik, 2014). In doing this, the results of the study should be cautiously transferable to a variety of classrooms to address the challenges of computing sides and angles using ratios.

1.7 Ethical considerations

The initial step of carrying out a study is to ask for permission from Turfloop Research Ethics Committee (**TREC**). In doing this, an application was sent, and an ethical clearance certificate was obtained (**Appendix A**). Furthermore, a request letter was sent to the Department of Basic Education Limpopo Province head office (**Appendix B**) and the school manager (**Appendix G**) to be granted an opportunity to collect data. A clearance certificate from the Department of Education (**Appendix C**) and approval from the school manager were obtained. Meanwhile, learners who wished to participate in the study were given consent forms to sign together with their parents (**Appendix F**). This is consistent with the Children's Act 38 of 2005, which states that parents must express their views concerning their children (DSD, 2015). In terms of confidentiality and privacy, the data collected was stored in a locked room to prevent others from accessing personal details of participants. This was a way of implementing the Protection of Personal Information Act of 2013 (POPIA), which emphasises the protection of individuals. During data analysis, learners were given pseudo names to avoid violating their confidentiality, for example, learner **A**, learner **B**, etc.

In terms of respect, dignity and standard of care, participants were granted an opportunity to decide whether to participate in the study or not. Also, a consent form was written in understandable language to avoid confusion (Lamont et al., 2016). During interviews, participants were asked questions politely to freely express their ideas. This was a way of honouring and admiring individuals due to their qualities, and this signifies dignity (Hodson, 2001). Moreover, none of the interviewees were blamed for giving irrelevant responses. The standard of care was ensured through treating all participants equally irrespective of their status (Moffett & Moore, 2011). This relates to implementation of Promotion of Equality and Prevention of Unfair Discrimination Act 4 of 2000, which emphasises equal treatment and prohibits unfair treatment (DSD, 2015). Hence, these aspects are essential and should be applied in a proper manner.

1.8 Significance of the study

The theoretical significance of this study was directed to the use of contexts to teach content. In this instance, the context is spatial figures while the content is computations of sides and angles using basic trigonometric ratios. This emphasises the use of spatial figures in teaching learners how to compute sides and angles. Therefore, the study contributes to the use of spatial skills to operate within three tenets of semantic theory during the process of computations of sides and angles in trigonometry. The three tenets provide a clear image of how to navigate between tasks that involve spatial figures.

1.9 Outline of the study

The University of Limpopo Postgraduate Manual (Postgraduate manual, 2022) outlines that a master's dissertation should consist of six chapters. As such the study is divided into six chapters. The first chapter gives an overview of the study. It comprises the background (issues of the problem) and purpose of the study, research questions, introduction to the research methodology, ethical considerations, and significance of the study. The second chapter presented a semantic theoretical framework as well as the literature on the spatial skills in the computations of sides and angles, trigonometry concepts for computing sides and angles together with the learning of trigonometry. In chapter three, a literature review was used to justify the chosen research methodology and design, data collection and analysis methods, and ethical consideration issues. The fourth chapter presented results from the three mathematical tasks and semi-structured interviews. The fifth chapter interpreted the research findings of the study, which were related to the differences and simplified findings reported in a literature review, the semantic theory and the design adopted for this study. The final chapter described the reflection of the methods used for collecting and analysing data, interpretation of the findings, answers to research questions and recommendations. The chapter further presented contributions together with the limitations of the study and conclusion.

1.10 Chapter Summary

Chapter one briefly explained the overview of the dissertation. To ensure that the overview is well understood, I started by giving the background and motivation of the study. The background entails recent research, key constructs from the title, my teaching experience on the computation of sides and angles of spatial figures and what prompted me to conduct the study. Secondly, I provided a research problem statement. Thirdly, I described the purpose of the study. Fourthly, research questions were documented. Fifthly, an overview of the research methodology was provided. Sixthly, the chapter provided ethical considerations. Seventhly, the chapter outlined the significance of the research. Lastly, the chapter described the structure of the whole dissertation.

CHAPTER TWO

THEORETICAL FRAMEWORK AND LITERATURE REVIEW

2.1 Introduction

This chapter presents a review of the literature on studies related to the exploration of spatial skills when computing sides and angles using trigonometric ratios. The review is based on the learning of trigonometry and spatial skills required during the computations of sides and angles using trigonometric ratios such as sine, cosine and tangent. Concisely, the subheadings of this chapter are arranged as follows: 1) semantic theory; 2) spatial skills in the learning of trigonometry; 3) relevant trigonometric concepts in the computations of sides and angles; and 4) the learning of trigonometry. Within the pillars, related studies were discussed and criticised. Also, the discussions were coupled with challenges that hinder learners from computing sides and angles accurately.

2.2. Semantic theory

The study used Frege‘s semantic theory (Frege, 1962, cited in Martin-Fernández et al., 2019) to explore spatial skills exhibited during computations of sides and angles. The Frege semantic theory illustrates that an understanding of mathematics involves providing ideas and procedures with structured meanings. This means that individuals should express their understanding of ideas using multiple representations such as figures, symbols, tables, words and notations. Martin-Fernández et al. (2016) purport that since learners express their understanding differently, semantic theory is an essential tool for interpreting the relevancy of their knowledge. Despite this, the theory has been used by Capraro et al. (2010) to identify how learners develop mathematical misunderstandings while solving problems. In contrast, Martin-Fernández et al. (2019) used the theory to analyse the meaning and understanding of mathematical concepts. Moreover, the theory has been used by Castro-Rodríguez et al. (2022) in a descriptive qualitative study to check how primary school learners align numeric, graphical and verbal representation when solving Partitioning tasks. In this study, the theory was used to analyse how Grade 10 learners utilise their spatial skills to access relevant concepts from their prior knowledge when computing sides and angles using ratios.

The semantic theory was built on the three tenets as depicted in a study by Martin-Fernández et al. (2019). The tenets are the conceptual structures, system of representation and sense. They are coherently explained below.

2.2.1 Conceptual Structures

A conceptual structure is a prerequisite in the computations of sides and angles. According to Martín-Fernández (2021), a conceptual structure involves ideas, notions and procedures for solving mathematical problems. As explained by Bell et al. (1983), Hiebert and Lefevre (1986) together with Arnold et al. (2021), a conceptual structure is prearranged into two dimensions in terms of conceptual and procedural. Specifically, a set of mathematical concepts lies within the conceptual dimension whilst processes related to the set of concepts expressed in terms of steps form part of the procedural. Along the lines defined by the authors above, here the conceptual structure comprises trigonometric concepts such as properties of spatial figures, procedures, the theorem of Pythagoras, and ratios that provide an adequate understanding of trigonometry. These schemas are essential for applying spatial skills to express an understanding of spatial figures. Consequently, understanding the schemas permits individuals an opportunity to illustrate steps for computing sides and angles logically. Moreover, it illuminates a habit of writing more than two procedures because of doubting thoughts (Pfende et al., 2022). In contrast, failure to manage trigonometric schemas leads to difficulty computing sides and angles correctly (Fatmanissa et al., 2020). Therefore, an appropriate conceptual structure is a requirement for computing sides and angles.

A recent study by Castro-Rodríguez et al. (2022) postulates that a conceptual structure assists teachers in determining how learners understand mathematical concepts. Gärdenfors (2009) also concurs that conceptual structures should be considered as they consist of important ideas for representing meanings based on spatial constructions. As such, this has a significant impact on the accomplishment of the objective for the computations of sides and angles as stated in the curriculum document. In the data analysis, this tenet was used to check whether the trigonometric schemas are relevant for the computations of sides and angles using ratios. As a result, having relevant trigonometric schemas is an illustration of understanding what the question requires (Yiğit Koyunkaya, 2016).

2.2.2 System of Representation

A conceptual structure is a requirement to navigate between the systems of representations. The system of representation relates to the use of components such as figures, symbols, signs and rules to support an understanding of mathematical ideas (Martin-Fernández, 2016; Mulligan & Mitchelmore, 2009; Post & Prediger, 2022). Hasanah et al. (2022) also agree that representations can be symbols, words, figures or rules that enable learners to write and describe mathematical ideas using pre-existing knowledge, which strongly assists learners to describe ideas logically. In this study, a system of representation relates to a way of symbolising the sides and angles of figures using relevant conceptual structures. This led to the proper choice of relevant trigonometric ratios during computations. Choosing relevant ratios enhances individuals' understanding of how spatial figures should be visualised (Rankweteke, 2020). Hence, visualising spatial figures enables learners to notice their understanding of various properties of spatial figures.

Studies related to the importance of representation in solving problems were widely carried out by numerous researchers (Mahama & Kyeremeh, 2023; Novira et al., 2019 cited in Hasanah et al., 2022; Suningsih & Istiani, 2021 cited in Hasanah et al., 2022). Findings by Novira et al. (2019) emphasised that representation is a vital component for supporting learners to represent their own thoughts fluently. The findings were supported by Suningsih and Istiani (2021), who stated that when solving problems, learners try different representations as a manifestation of their thoughts and strategies.

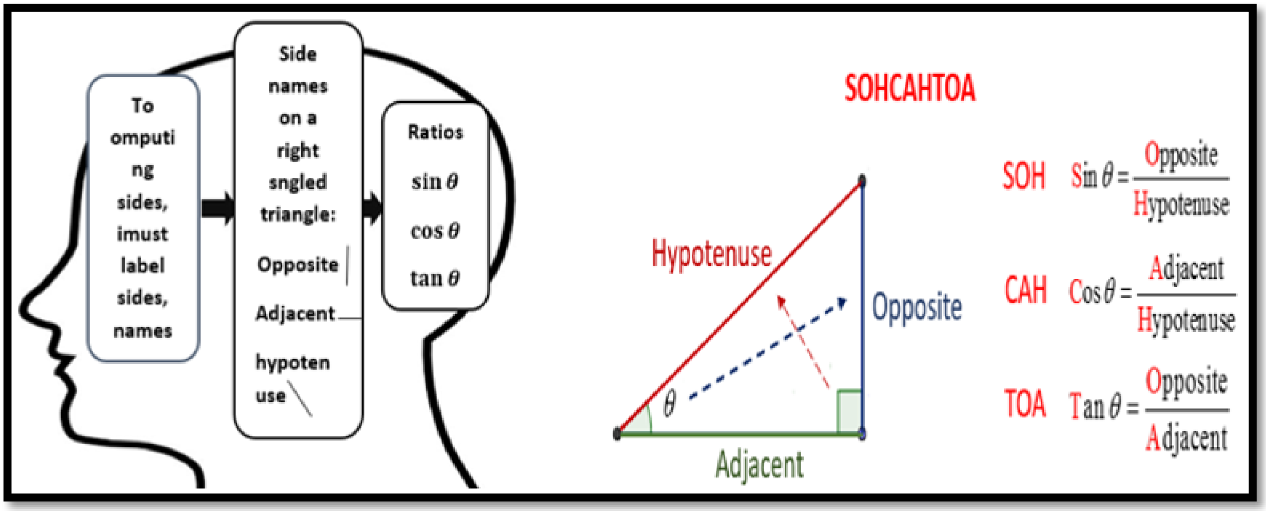
As a result, using a representation that matches the problem becomes easier for learners to obtain the desired solution. Meanwhile, inappropriate representation leads to failure to solve trigonometric problems.

Moreover, Mahama and Kyeremeh (2023) carried out a study on the impact of multiple representations in solving fractions. In the study, data were collected from 96 learners using pre and post-tests and analysed descriptively. Essentially, the authors found that multiple representation is an effective approach that needs to be incorporated while learning fractions. Hence, fractions also contribute to the computations of sides and angles while using ratios, meaning their findings also contribute to the current study.

The importance of system representation was also illustrated in a Translational model depicted by Lesh et al. (2003). The model was developed to emphasise that mathematical concepts should be expressed in multiple representations. The three vital components to be considered while computing sides and angles are explained below. Firstly, Pictorial representation demonstrates spatial concepts using hand-sketched or computer-generated figures, which is used for checking learners' interpretive skills and SV. Secondly, symbolic representations are written variables for illustrating procedures of computing sides and angles and are vital for analysing learners’ procedures on how SR was exhibited. Lastly but not least, verbal representation is a way of expressing mathematical concepts orally. Within the verbal representation, learners clarify their written responses which relate to SO. Concisely, in the data analysis, this tenet was used to check how learners used their spatial skills in choosing relevant concepts such as side names to symbolise sides and angles of spatial figures during computations (Kamber & Takaci, 2018). This justifies the mastery of ideas required to explore figures during computations.

Malhotra (2022) is one of the authors who conducted a qualitative study to emphasise the role of Lesh’s model of representation in the learning of mathematics. In the study, data was collected by observing a lesson planned on Lesh’s mode of representation in teaching Grade 4 fractions and division. Dissimilarly, in the current study, the role of representation is based on the ability to compute sides and angles of spatial figures using ratios. The findings of that study indicate that the Lesh representations encourage learners to construct and apply mathematical concepts. Therefore, it is the responsibility of the teacher to facilitate and empower learners during the process of learning. In this study, Lesh's representation is effective in assisting learners to move between representations during computations. For instance, from pictorial which are spatial figures to symbolic representation such as symbolysing the given sides or angles on the ratio. Hence, representation cannot be left out in trigonometry as it is one of the vital aspects utilised in computations.

Moh'd et al. (2022) have recognised the importance of representation and carried out a quasi-experimental study titled "The effect of representation strategies on secondary students' mathematics achievement". The study was carried out with learners from a public school in the urban district of Unguja Island. In terms of data collection, pre and post-test were used to notice the effectiveness of representation. The collected data was analysed through descriptive statistics and t-tests, which differ from the analysis method of the current study. The study was guided by two research questions: 1) is there a significant difference in the score of the experimental and control group before and after intervention? and 2) is there a significant difference in the mathematics achievement scores of the experiment group before and after the treatment? The author decided to establish a control and experimental group. However, the interventions indicated that the experimental group was more actively engaged due to the application of multiple representations than the control group. This was an accomplishment of the 21st-century directive that learners are expected to acquire mathematical knowledge as well as skills using a representation that assists in interpreting problems. In this instance, the knowledge is referred to as trigonometric concepts while the skills are spatial abilities. Most importantly, the successful acquisition of knowledge and skills is through active learning. Hence, it can be concluded that the application of multiple representations is a fundamental aspect in the learning of trigonometry (Mahama & Kyeremeh, 2023). For successful support, the patience and commitment of teachers are core elements for an effective 21st-century directive.



Source for 2.1.b: online, https://www.onlinemathlearning.com/trigonometric-ratio.html

Figure 2.1. a) Internal (what the learner is thinking) Figure 2.1.b) External (written down)

Figure 2. 1 Internal and external representation of trigonometric ratios

Figure 2.1, indicates that representation is a vital component for the computations of sides and angles. This view is in accordance with results from a study by Samsudin and Retnawati (2018), which states that mathematical representation assists individuals in solving various mathematical problems. Findings from Mainali (2021) contend that learners solve various mathematical problems through translating mathematical ideas using two representations. The internal representation is created mentally and is used to establish mathematical meaning (Komala & Suryadi, 2018), and the external representations which are conventional representations such as symbols, equations, algebraic expressions, tables and graphs (Azmidar & Husan, 2022; Goldin, 2001). However, the external representation is considered as a system of representation. In this study, the internal representation is linked with the SV, whereby learners mentally strategise steps for computing sides and angles. In figure 2.1.a, the scenario illustrates how learners mentally categorised side names to choose the relevant ratio. On the other hand, external representation in figure 2.1.b relates to SR, whereby learners choose and write down relevant ratios for computing sides and angles (Surtees et al., 2013). Therefore, using representations to compute sides and angles correctly illustrates having appropriate trigonometric conceptual structures and a system of representation (Martin-Fernández et al., 2019).

Similarly, a prior study by Mainali (2021) on representation in the learning and teaching of mathematics outlined the essential roles of representation to: 1) support an understanding of mathematical ideas and interrelatedness; 2) apply knowledge in realistic situations; and 3) make learning more interesting. The roles are further supported by various scholars who viewed representation as a vital element of learning trigonometry (Lesh et al., 2003). Despite the roles emphasised by various scholars, representing trigonometric problems remains an issue. The statement is consistent with that of DeReu (2019), who argues that learners experience difficulties in expressing and managing their knowledge of representation. This could be a consequence of being forced to follow educators' preferred procedures without an opportunity to reflect on their work to link given figures with underlaid ideas (Kang & Liu, 2018). To fill this literature gap, there should be early exposure to representations.

2.2.3 Sense

Conceptual structures and systems of representation inspire learners to categorise and make sense of phenomena. The phenomena are shown in different trigonometric problems which serve as contexts. A sense is a meaningful impression intending to stipulate an agreement or attention (Sheehan, 2016). However, the impression necessitates the mind to interpret contexts using a set of concepts. Most importantly, learners with sense can critically think to find appropriate ways of computing sides and angles. Moreover, Heuvel-Panhuizen (2014) argues that sense is considered while applying mathematical ideas in various contexts. In this study, the sense is applied during computations of sides and angles to confirm solutions. Shapiro et al. (1997) also contend that sense is the ability to use a variety of concepts to navigate between representations. In this instance, sense relates to the use of SV to mediate between trigonometric concepts involved within the ratios (Lamon, 1995). Pfende et al. (2022) purport that teachers should encourage learners to make sense of their strategies and concepts applied when solving problems. In the data analysis, this tenet was used to explore how learners expressed their SV by moving flexibly between varieties of representations using the properties of figures. Expressing SV is an indication of knowing how to link an understanding of ratios with pictorial representation (Martin-Fernández et al., 2021). Linking an understanding of ratios with pictorial representation enabled individuals to change their mental images. This leads to a proper choice of ratio for computing sides together with angles, and relates to SR (Cartwright, 2006). Hence, the tendency of using SO to justify the chosen ratio becomes easier since SV and SR are applied. Therefore, the trigonometric conceptual structures and a system of representation should be relevant to move flexibly in the sense (Martin-Fernandez et al., 2019). In concussion, the three tenets of the theory are used to explore Grade 10 learners' spatial skills exhibited when computing sides and angles.

2.3 Literature review

A series of recent studies in south Africa, Limpopo province and other countries have indicated that most learners have an incomplete understanding of trigonometric ratios (Faturohman & Amelia, 2020; Molataola, 2017; Nanmumpuni & Retnawati, 2021; Suri et al., 2021), which leads to difficulty computing sides and angles. This is ample evidence that learners cannot differentiate between angles and sides of a right-angled triangle. As a result, they grapple to identify an opposite, adjacent or hypotenuse side on a right-angled triangle. In this study, the computation of angles and sides is accomplished through three pillars: 1) the spatial skills in the learning of trigonometry; 2) trigonometric concepts; and 3) the learning of trigonometry. The three pillars are intertwined, and they provide invaluable support to each other. Concisely, spatial skills in the learning of trigonometry enable individuals to comprehend the size and location of figures (Hegarty & Waller, 2005). However, a collection of trigonometric concepts acquired during the process of learning is required. Arguably, applying irrelevant trigonometric concepts prohibits individuals to move malleably between the three pillars. Grito (2018) highlighted that the way in which learners perceive figures predicts their spatial skills. Therefore, following the cyclic process below is crucial for determining learners' spatial skills and encouraging opportunities to establish trigonometric connections. Hence, a diagram is drawn to illustrate how the subheadings are intertwined.

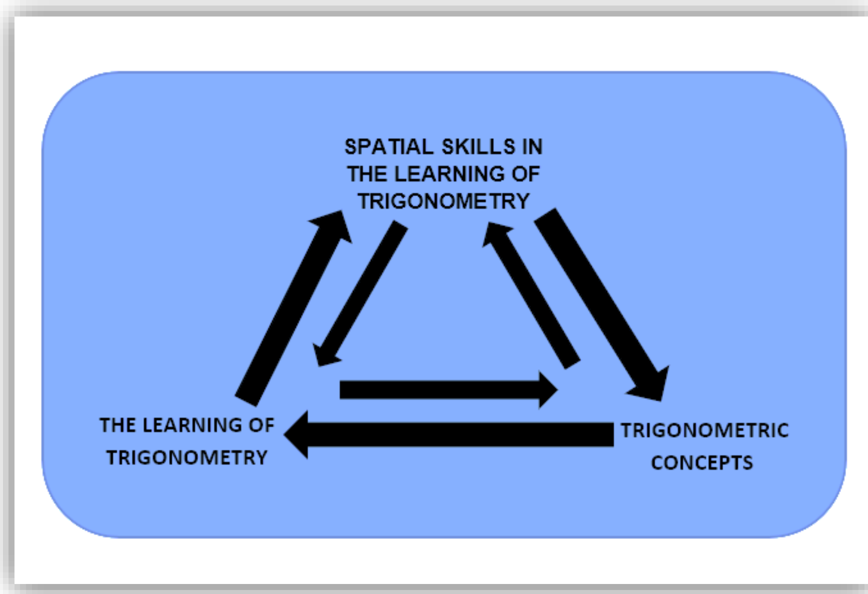


Figure 2.2 Three pillars of mastering computation of sides and angles

2.3.1 Spatial Skills in the Learning of Trigonometry

Fostering appropriate spatial skills to learners holds a great promise for computing sides and angles. In the learning of trigonometry, spatial skills enable learners to explore figures and manipulate such forms of representations mentally (Gridos et al., 2018). More importantly, sufficient knowledge of different spatial figures is key to computations. Sufficient knowledge permits learners an opportunity to mentally transform figures accurately (Lohman, 1996). I believe that this is a form of intelligence for demonstrating mental imagination and rotations of figures by differentiating how they relate in real and imagined contexts. According to Rocha et al. (2022), appropriate spatial skills are effective in improving learners' trigonometric knowledge. The idea is accomplished once learners view trigonometry as an interesting idea that is part of their daily lives. Rocha et al. further revealed that spatial skills allow learners to organise their thinking. Consequently, organising their thinking contributes to ways of expressing their ideas such as side names to choose the appropriate ratio for computations. Generally, previous studies emphasised that the three spatial skills are essential in laying a mental foundation for learning spatial figures (Clements et al., 2018; Gagnier et al., 2022). In this study, the three spatial skills enhance learners' ability in navigating between figures to obtain desired solutions during computations of sides and angles. These assist learners in communicating, reflecting, and transforming problems in trigonometry (DBE, 2013). Hence, I think understanding the elements of spatial skills is imperative for addressing literature gaps such as confusion during computations of sides and angles using ratios.

Rocha et al. (2022) carried out a study to examine how teachers affect learners' spatial skills. The study was carried out with 62 primary and secondary teachers across the USA. The teachers were recruited through social media platforms. They were given an online mental rotations test, and spatial skills assessment to select the matching figure with the given statement. Moreover, a teaching activity questionnaire was used to understand teachers' spatial pedagogical practices. The results were analysed using the R version by Corp Team (2019). Their findings revealed that teachers with high spatial skills have proper spatial pedagogical practices. The findings are consistent with that by Atit and Rocha (2020), who state that teachers with high or appropriate spatial skills make great use of spatial pedagogical practices. This shows that their proper conceptual structures of figures allow them to operate within the system representation during the process of teaching and learning. As a result, making sense of various figures while teaching becomes easier since the conceptual structure and system representation are relevant. The authors made a good point, however, that having spatial skills does not mean the teacher can handle spatial pedagogical practices, meaning a teacher can have the skills but struggle to deliver spatial content to learners. Therefore, it cannot be concluded that having spatial skills predicts proper spatial pedagogical practices. Consequently, failure to deliver content contributes to learners' difficulties to acquire knowledge at different levels. For instance, failure to deliver spatial content at the primary school level forbids learners from developing an understanding of new concepts at high school level. Hence, additional pedagogical support is required to enhance their classroom practices.

Additional pedagogical support should be linked with the three subskills depicted by Lohman (1996). The spatial visualisation (SV), whereby learners are encouraged to comprehend figures through imagination and mental manipulations on a space (Açikgül et al., 2023; Mix et al., 2017). Comprehending figures is a significant step of illustrating the articulated properties. Badmus and Jita (2022) also argue that the articulated properties also contribute to interpreting the position of spatial figures. In the context of trigonometry, Lohman indicates that learners use the SV to comprehend the formation of figures through shifting between ideas involved in trigonometry. Lowrie et al. (2020) highlighted that in the mathematics classroom, the SV predicts learners' trigonometric knowledge and ensures that individuals handle the schemas as they appear within the orientation of a figure. Despite the prominent roles shown above, learners still struggle to handle the trigonometric schemas within figures such as side names on a right-angled triangle regarding reference angles (Nader, 2021). Machisi (2023) adds that sometimes learners tend to display mathematical anxiety, fear or worry when confronted with a mathematical problem as their existing schemas are not sufficient to respond to the given questions. Hence, introducing an intervention programme in early childhood to draw figures in various positions as suggested by Lowrie et al. (2018) might enhance learners' SV to recognise schemas applied to various figures.

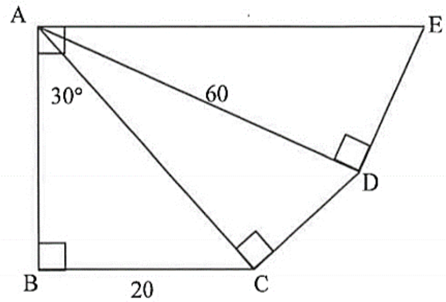
The importance of appropriate SV within (SR) is currently under debate (Rittle-Johnson, et al., 2019). The SR enables learners to perceive and communicate different positions of figures using schemas (Surtees et al., 2013). Ideally, SR is an essential aspect that assists learners to specify how spatial figures are in space. In the current study, SR indicates learners' ability to change mental images and choose the appropriate trigonometric ratio to compute sides and angles (Pittalis & Christou, 2010). However, changing the mental image is part of SV. It seems obvious that the properties of SV also contribute to the SR when choosing the specific ratio. The trigonometric curriculum outlined that SR assists learners to understand the dimensions and positions of figures in a variety of contexts (Corcoran et al., 2012). Corcoran et al. further state that SR provides individuals with essential tools to describe the location of figures in different contexts. Although most learners have SR to unpack properties of a figure in different dimensions, some face challenges to symbolise side names of figures. As a result, they compute the sides and angles of figures incorrectly. Thus, mastery of SR advances learners' ways of accurately analysing and synthesising spatial figures.

Research continues to bring to light how appropriate SV and SR contribute to spatial orientation (SO) (Pollitt et al., 2020). The SO is responsible for making a mental transformation to justify statements using the SV and SR (Surtees et al., 2013). In this instance, SO relates to learners' ability to justify chosen trigonometric ratio when computing sides and angles. The SO is crucial for improving individuals' reasoning and creativity by applying to the SV and the SR. In contrast, Hurrell (2021) revealed that learners have procedural knowledge, or a series of steps for solving mathematical problems across different domains. They use these procedures without understanding why they work. Consequently, they struggle to make justifications for their procedures. Practically, in mathematics classrooms, teachers are compelled to complete an annual teaching plan (ATP) in each period. As such, learners are not given a chance to understand how the variety of steps infers. I personally think this could be one of the reasons for the failure to make a justification of procedures. Hurrell suggests that teachers should use teaching techniques that encourage creativity and reasoning, not memorisation. Thus, the study emphasises the application of spatial skills during the computations of sides and angles using ratios.

The importance of appropriate spatial skills has been described by a great number of authors in the literature (Dhlamini et al., 2019; Lohman, 1988; Lowrie et al., 2018; Yang et al., 2020) However, they were not specifically described in relation to trigonometry but geometry. The importance of spatial skills was explained based on the chosen research designs such as correlational, experimental and survey. It is known that the design of the study is determined by the research approach. As a result, their findings are slightly varying but contribute to the learning of trigonometry.

QUESTION 4

4.1 In the diagram below, ABC, ACD and ADE are right-angled tirangles.
$B\hat{A}E = 90°$ and $B\hat{A}C = 30°$. BC = 20 units and AD = 60 units.



Calculate the:

4.1.1	Length of AC	(2)
4.1.2	Size of $C\hat{A}D$	(2)
4.1.3	Length of DE	(3)

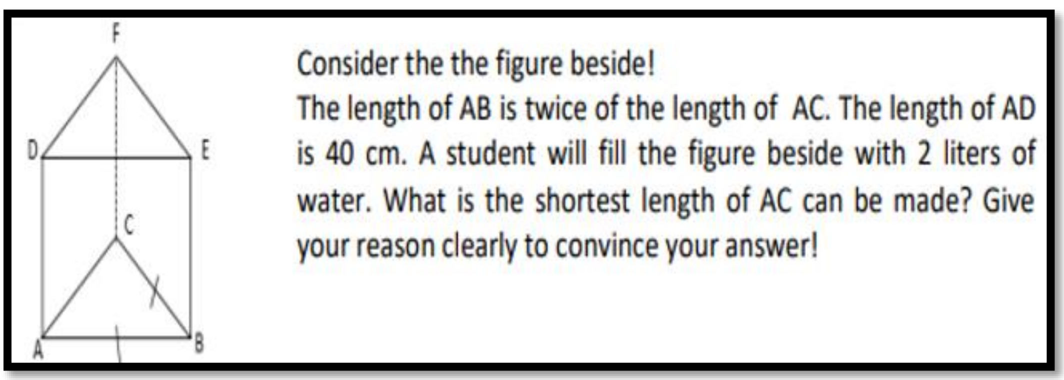
Source (DBE, 2015, P6)

Figure 2.3. Grade 10 question

An experimental study by Clements et al. (2018) regards learners as problem solvers when they can explain spatial skills using figures. The authors examined three studies to explore the role of spatial skills in different contexts: 1) the effects of geometric figures in mathematics; 2) the impact of early foundational knowledge; and 3) the impact of teachers' language in early childhood. Learners were instructed to design pictures using different figures. Findings from the pre- and post-tests on the three projects highlighted that knowledge of geometric figures is effective in mathematics. As a result, designing pictures was interesting and easier for learners with relevant properties of different geometric figures only. Arguably, learners with irrelevant knowledge of figures struggled to design pictures. This relates to the application of inappropriate conceptual structure that affects the last two tenets, a system of representation and sense.

In the current study, geometric figures are used as contexts to explain ratios and how spatial skills are used. In figure 2.3, there are four triangles (∆ABC, ∆ACD, ∆ADE, and ∆BAE). Below figure 2.3, are questions for computing sides and angles. In this instance, the SV relates to the ability to use relevant schemas to comprehend the four figures as they appear within the orientation of ABCDE (Lowrie et al., 2020). The schema should include AB, AC and AD which are opposite sides of ∆ABC, ∆ACD, ∆ADE and ∆BAE. Furthermore, the schema should include the hypotenuse sides for all figures: AC, AD, AE, and BE (Gyan et al., 2021). Thus, outlining these schemas correctly shows success in having relevant SV and effective knowledge of geometric figures.

Outlining schemas enables learners to apply SR. The SR relates to learners' ability to utilise trigonometric concepts to contextualise the SV into trigonometric ratios (Corcoran et al., 2012). For instance, perceive the four figures and indicate the trigonometric ratios for computing sides of the given figures. The trigonometric ratio includes sine, cosine and tangent (DBE, 2013). Lastly, involving the SO transforms both the SV and SR in justifying the written procedures (Fernandez-Baizan et al., 2021). Ideally, if the learner chooses sine, cosine or tangent, SV and SR must be applied to justify the chosen trigonometric ratios.



Maknun et al. (2018)

Figure 2.4 Question from Maknun

Similarly, an exploratory study by Maknun et al. (2018) on mathematical argumentation in trigonometry revealed that learners should make arguments regarding their written procedures. Making arguments provides teachers with sufficient details on learners’ knowledge and difficulties in trigonometry. Mathematical argumentations are reasons provided by individuals to support their solutions (Sriraman & Umland, 2014b;

Marufi et al., 2022). In the current study, mathematical argumentations are specified as SO, which are reasons for supporting the chosen ratio. Their study was carried out with a group of Grade 10 learners in a private school. The learners were already exposed to trigonometric concepts and were at the equivalent level of skills. In contrast, the current study was carried out with learners from rural areas and trigonometry was a new topic to them. In their study, comprehensive data were collected using open-ended tests made of three questions to determine if the given answer was well structured or not. One of the questions is shown in figure 2.4. Findings from the comprehensive data argued that most learners obtained the desired solution but could not justify it. From figure 2.4, seventeen learners managed to find the length of AC and justified it, whereas nineteen struggled to answer and justify their answers. Based on teaching experience, learners encounter challenges to provide the SO due to insufficient knowledge of ratios. This is considered difficult to recall appropriate conceptual structures. The reason is in line with that by Sriraman and Umland (2014b), who emphasised that learners learn numerous concepts without connecting them, which prohibits them from developing the SO. Hence, learning numerous concepts without connecting them remains an important issue to be addressed.

Lowrie et al. (2019) also conducted an experimental study to analyse the influence of SV on SO. An overall of 327 learners from different classrooms participated in the study. A three-week intervention programme was implemented, and SV tests were designed using the Learning framework proposed by Lowrie et al. (2018), which tests the knowledge of spatial figures. Findings from their study signified that the programme increased learners' performance, specifically on the SV and the SR. Critically, their study was somehow quantitative, since teachers believed that learners understand spatial figures by obtaining certain grades (performance). By contrast, in the current study, the criteria for examining the effective SV on SO resides in giving clear explanations of written answers. Findings from Rocha et a. (2022) argue that the SV, SR, and SO should not be used for testing learners' performance but for the understanding of concepts such as computations of sides and angles. Although evidence from Lowrie et al. emphasised that intervention programmes enhance both SV and SO, teachers should not rely on grades but on precise descriptions of the written answers.

2.3.2 Trigonometric concepts for computing sides and angles of spatial figures

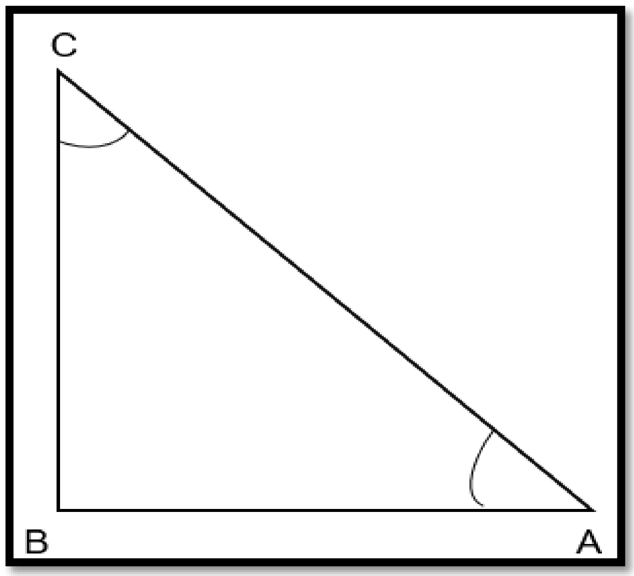


Figure 2.5 A triangle for demonstrating the first trigonometric concepts

In this study, learners encountered challenges to compute sides and angles using trigonometric ratios. As such, it is necessary to clarify the concept ‘trigonometry’ and its aspects in computations. According to Weber (2005), trigonometry is an area of mathematics that links mathematical concepts such as geometry, algebra and graphical reasoning. This means that the three concepts serve as a precursor for manipulating and comprehending symbols involved in trigonometric problems. Trigonometry covers the relationship between sides and angles; the functions of sine, cosine and tangent (Walsh et al., 2017). Williams (2019) also agrees with Walsh et al. that trigonometry focuses on studying the relationship between angles and sides of spatial figures. Williams further stated that these spatial figures are right-angled triangles and can be drawn in a variety of contexts. Similarly, Weber (2008) supports that trigonometry is a topic that presents the interrelatedness between angles and sides of spatial figures that should be symbolised. I concur with the explanations of the four authors since the study emphasised the link between the sides and angles during computations and should be symbolised. Ideally, symbolising the interrelatedness obliges learners to have sufficient knowledge of the three categories of trigonometry namely: trigonometric ratios, general solutions and reduction formulae (Spangenberg, 2021). Regardless of the importance made above on trigonometry, the knowledge prepares learners for advanced mathematics such as calculus, civil engineering, geophysics and architecture. Thus, I strongly believe that it is crucial to understand the trigonometric basics to apply them in the field mentioned above.

The essence of computation of sides and angles essentially departs from knowledge of trigonometric ratios, which are expressed in terms of the sides and angles of right-angled triangles (DBE, 2013). Knowledge of the three sides, namely, hypotenuse, opposite and adjacent is compulsory. The trigonometric ratio is a core element of pairing the two sides with the specified angles (Maknun et al., 2018; Sánchez et al., 2023). The pairing of sides with the specified angles requires proper: 1) SV to mentally analyse how the side names and size of angles are formed; 2) SR for recognising the link between the sides, an angle of the figure and choosing a specific ratio; and 3) SO for justifying the chosen trigonometric ratio. For instance, in figure 2.5, learners should visualise and comprehend all side names, based on $\hat{A}$ and $\hat{C}$. In relation to ΔABC, the **hypotenuse** is consistently opposite the right angle, the **opposite** side is the length facing $\hat{A}$ and the **adjacent** side is the length beside $\hat{A}$. Pollitt et al. (2020) conclude that a lack of prerequisite knowledge such as the one stated above prohibits learners from effectively and essentially judging processes for computing sides and angles.

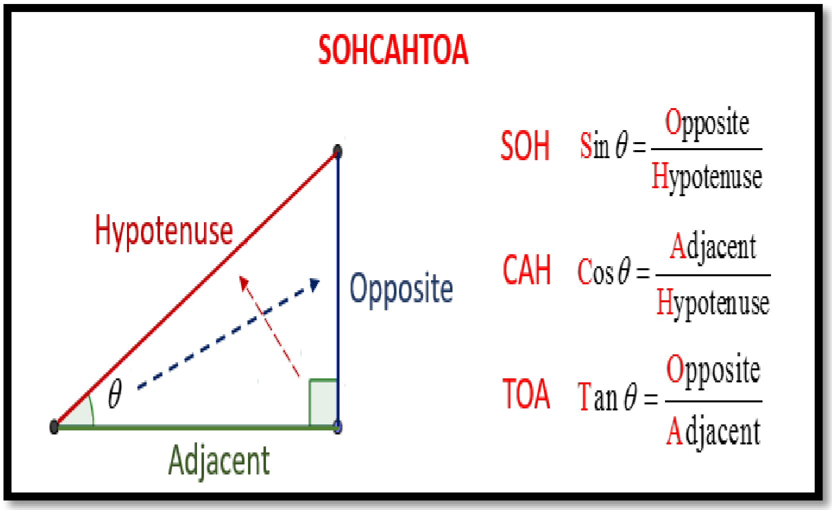


Figure 2.6 Mnemonics of trigonometric ratios

Developing the knowledge of trigonometric ratios advances learners' spatial skills. In a broader context, Faturohman and Amelia (2020) carried out a quantitative study on the analysis of learners' difficulties in solving trigonometric problems. A survey approach was used to collect data. Learners were given $\cos A = \frac{4}{5}$, to represent it pictorially such that writing other ratios becomes easier. Their results highlighted that trigonometric ratios are effective in determining unknown distances and sizes of angles for different spatial figures. However, a proper SV should be considered to comprehend the dimension and location of figures (Cao and Ouyang, 2019; Nagy-Kondor, 2014). The results have been supported by Fahrudin and Pramudya (2019), who emphasised how the right-angled triangle and a circle unit are frequently used to explain the three trigonometric ratios. Firstly, the sine, which illustrates the interrelatedness between the opposite and hypotenuse sides based on an angle (Yigit Koyunkaya, 2016; Walsh et al., 2017). Secondly, the cosine, which equates to the ratio of adjacent and hypotenuse based on the acute angle. Finally, tangent for relating the ratio of the opposite and adjacent sides based on the angle (Rohimah & Prabawanto, 2019). These are basic trigonometric ratios initially introduced in Grade 10. Pictorially, figure 2.6 supports the similarities and differences discussed above between the three trigonometric ratios. Thus, complete knowledge of the similarities and differences mentioned above eliminates confusion on when to use sine, cosine and tangent. This stimulates the use of mnemonics with the ability to justify their choices.

Faturohman and Amelia (2020) further indicated that learners had difficulties distinguishing sides when given trigonometric ratios. The difficulty emanates from irrelevant conceptual structures of ratios such as side names of a right-angled triangle and reference angles. As a result, learners could not represent the given ratio in the form of a diagram. In this study, the struggle to represent $\cos A = \frac{4}{5}$ is interpreted as a failure to operate within a system representation since the learner struggled to notice that 4 is adjacent whereas 5 is a hypotenuse side. According to Lesh et al. (2003), system representation entails pictorial representation, symbolic representation and verbal representation. To my knowledge of Lesh's tenets, within the pictorial representation, learners grappled to indicate $\cos A = \frac{4}{5}$ on a right-angled triangle. In the context of symbolic representation, they did not notice that the opposite side should be computed using the theorem of Pythagoras. This is ample evidence that the properties of the theorem of Pythagoras were not developed in prior grades. As a result, justifying their answers would have been an issue even if they were required to do so. The difficulty relates to the inability to handle verbal representations. Based on my teaching experience, the difficulty might be a consequence of not paying attention during the process of learning. Additionally, it might be a result of being given procedures without illustrating how they work. Hence, failure to handle the three systems of representations causes a struggle in making sense of spatial figures. The question that naturally arises is: how can teachers assist learners to make sense of spatial skills in trigonometry? To address this question, further research must be conducted.

1. Find the General Solution for the following:

a. $2\cos x = 1.5$

Figure 2.7 General solution questions

General solutions are part of the three categories of trigonometry. The general solutions are equations for determining angles that lie within the specified interval by adding or subtracting multiples of the appropriate period (Rohimah & Prabawanto, 2019). The statement by Rohimah and Prabawanto also emphasises the issue of determining angles, which is like the purpose of the current study. However, general solutions go on to add or subtract multiples of the specified period. This is a limitation of the current study. As stated in the curriculum, general solutions are introduced after the concept of ratios (DBE, 2013). This is ample evidence that ratio schemas are frequently applied during the process of solving general solutions. Mix et al. (2017) defined general solutions as an ordinary differential equation that involves arbitrary constants and should be simplified either using CAST or the algebraic method. Although CAST is a procedural dimension of finding signs of ratios, a conceptual dimension such as knowledge of ratios in different quadrants is a requirement to obtain desired solutions. Bell et al. (1983), together with Hiebert and Lefevre (1986), emphasised that the two dimensions form part of a conceptual structure. Therefore, the two dimensions cannot be treated as separate concepts as they provide invaluable support to each other. Mastering the conceptual structures such as understanding the link between ratios and quadrants assists learners to represent the sign of ratios symbolically. Representing the signs of the ratio forms part of system representation. According to Martín-Fernández (2021), handling the conceptual structure and system representation fluently enables learners to make sense of the given problems to become easier. Although the importance of the three tenets is inscribed using diverse words, they all embrace each other during computations.

Operating within the conceptual structure, system representation and sense also necessitates proper spatial skill. Essentially, Nagy-Kondor (2014) and Chamizo (2019) assert that learners have efficient SV if they can deal with the trigonometric schemas involved in the given equations. Adhikari and Subedi (2021) also emphasised that learners should be able to think about the trigonometric concept used in the question. I agree with the four authors above, however, the current study emphasises the application of spatial skills to the given question through accessing trigonometric concepts to obtain desired solutions using ratios. In figure 2.7, the equation is a product of 2 which is an arbitrary constant and a cosine ratio with an angle represented as a variable equated to the value of 1.5. These schemas establish learners' SR to comprehend the equation and then devise a plan for determining the reference angle. The argument agrees with the finding from a study by Atit et al. (2022), who emphasised that spatial skills enable learners to organise what has been visualised on the equation. Once the reference angle is determined using an inverse method, a relevant SO is illustrated by adding 360 degrees to the answer obtained if the equated value is positive for sine and cosine. Dissimilarity, if the equated value is positive for the tangent, SO should be illustrated by adding 180 degrees to the answer obtained. From the knowledge I have obtained on the importance of spatial skills, I would encourage teachers to assist learners in utilising the three spatial skills in different contexts to obtain the desired solution.

$$\text{Simplify } \frac{\tan(180° - x)\cos(360° - x) + \cos(90° + x)}{\cos(90° - x)\sin(90° + x)}$$

Figure 2.8 Reduction formulae question

The knowledge of reduction formulae is an essential tool for simplifying trigonometric expressions. Reduction formulae are trigonometric expressions that necessitate the knowledge of the CAST method to determine whether the ratio is positive or negative within the specified interval (DBE, 2013). Similarly, Kalanov (2022) defined reduction formulae as expressions that enable individuals to write trigonometric ratios in terms of acute angles. Based on my understanding, the two definitions are well articulated as how ratios and knowledge of the Cartesian plane contribute to reduction formulae. A study by Rankweteke (2020) on the learning of reduction formulate argues that teachers tend to introduce the idea of reduction formulae without explaining its functions in relation to real life. On the other hand, some teachers introduce the idea without checking learners' prior knowledge. The difficulties of introducing trigonometric ideas originate from a failure to practically implement the aspects of the chosen teaching technique (Spangenberg, 2021). The argument is consistent with one by Ramaligela et al. (2019), who revealed that teachers introduce new concepts without establishing learners' prior knowledge. Consequently, learners encounter challenges to handle the language and symbols when simplifying expressions. Thus, the findings illustrate that teachers should modify their ways of teaching trigonometry.

Ideally, modifying their ways of teaching should enable learners to utilise their spatial skills in various expressions. In figure 2.8, proper SV is required to visualise the expression of the quotient. The numerator entails tangent in the second quadrant, cosine within the fourth quadrant, plus cosine in the first quadrant. The quotient of the denominator has the cosine ratio within the first and the sine within the second quadrant. The identified schemas are part of conceptual structures and assist learners to use SR to represent the sign of the given ratios which is part of system representation. Writing the signs of the given ratios allowed learners to use SO in making sense of the signs of the ratio while simplifying the given expressions. Although the three categories of trigonometry provide invaluable support to each other, the current study focuses on trigonometric ratios only. The idea of utilising trigonometric ratios helped me to explore learners' spatial skills in handling procedures while computing sides and angles.

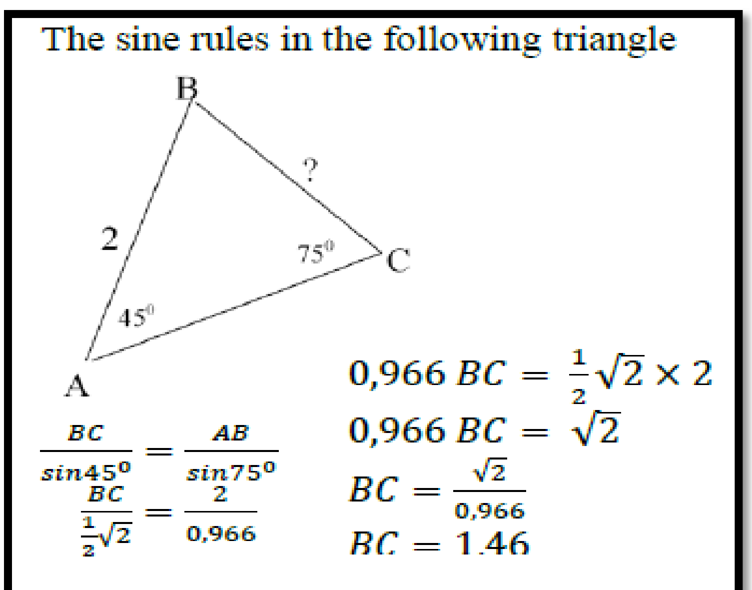


Figure 2.9 Learners' written response from Suri et al. (2021)

Suri et al. (2021) conducted a descriptive qualitative study with the aim of analysing Grade 10 learners' mathematical problem-posing using the what-if not strategy during computations. Although Suri et al. and the current study focus on computation, different skills were used to check whether learners are accessing relevant or irrelevant trigonometric concepts. For instance, Suri et al. used mathematical problem-posing while the current study used spatial skills. According to these authors, mathematical problem-posing develops skills for understanding a problem, connecting trigonometric concepts such as angles and sides, and enhancing high-order thinking. Three mathematical problems posing were used to direct learners on how to solve problems. Firstly, pre-solution is a method of formulating questions based on the specified situation. Secondly, the within-solution posing, formulating questions from problems being solved. Lastly, post-solution posing, modifying the conditions of the solved problem to generate new challenging questions. The three problems posing relate to the four stages of solving a problem proposed by Polya (1985), namely: 1) understanding the problem; 2) devising a plan to solve the problem; 3) carrying out the plan; and 4) and looking back. Although the current study does not focus on Polya stages, understanding the problem is linked with the use of SV to access appropriate conceptual structures, which assist in devising a plan to solve the problem. Carrying out the plan necessitates learners to have knowledge of system representation, which is a key element for applying SR to handle BODMAS rules and symbols in solving problems. While handling BODMAS rules and symbols, the process of sense is taking place. Hence, having relevant semantic tenets eradicates difficulties to justify written procedures.

Literature review (Creswell, 2009) shows that qualitative research should consider data collection techniques such as interviews, observations, focused groups, etc. Similarly, in Suri et al.'s (2021) study, tests, interviews and observations were used to collect data whereas the current study used mathematical tasks and semi-structured interviews only. Although the current and Suri et al. studies intended to compute the sides and angles of figures, different approaches were used to obtain the desired solution. In the current study, learners were restricted to use ratios, whereas in Suri et al.'s (2021) study, learners used the sine rule. Their findings highlighted that in the first session, learners could not understand how to use the what-if strategy, and the researchers encouraged learners to ask themselves a few questions based on the given problem. Consequently, their mathematical posing was enhanced in the session. For instance, in figure 2.9, the learners had appropriate conceptual dimensions on the sine rule such as understanding of fractions, operations, constants and variables. Moreover, the procedural dimension was also illustrated fluently by applying the knowledge of cross multiplication while computing side BC. While cross-multiplying, the BODMAS rule was well implemented. According to Lesh et al. (2003), when solving mathematical problems, pictorial, symbolic and verbal representation should be considered. Similarly, in figure 2.9, the learner managed to use appropriate SV to visualise that 45 is opposite to the required side, whereas 75 is opposite side AB which is 2. This relates to operating within the pictorial representation. All the schemas helped the learners to use SR in choosing the sine rule to compute side BC, and this relates to handling symbolic representation accurately. Lastly, the sense was considered as the learner used SO to express sine rule concepts orally, which relates to verbal representation. From my point of view, I think the success of expressing the sine rule orally clarifies that learners have prerequisite knowledge of trigonometry.

The idea of having prerequisite knowledge is supported by DBE (2013) in the curriculum document. For every topic, prerequisite knowledge is stated. Although the prerequisite knowledge is shown, some teachers do not consider it. This is the reason learners have fragmented trigonometric concepts, disjoined trigonometric concepts such as sides and angles which prevent learners from obtaining desired solutions. Correspondingly, descriptive research by Maknun et al. (2018) posited that prerequisite knowledge is also essential to teachers for competency in conducting classroom practices. This means that it is not strict for learners. Although the current study focuses on learners, most of the difficulties documented in the literature are caused by teachers. Nabie et al. (2018) are some of the descriptive studies illustrating that most of the learners' difficulties in trigonometry are caused by teachers. Maknun et al. explored 119 pre-service teachers' understanding and difficulties in trigonometry. The teachers were given $tanx =$

$\frac{4}{5}$ to illustrate it pictorially and to compute the unknown side. APOS theory was used as a technique for analysing teachers' understanding and difficulties. Their findings highlighted that 61.7 percent of teachers operated with the four APOS levels while illustrating the given ratio on a right-angled triangle. In contrast, 36.2 percent of teachers struggled to recognise the side names of the given ratio. As a result, the teachers could not compute the hypotenuse side. These difficulties relate to the background of the current study.

Using the first tenet of the semantic theory, it can be concluded that, in that study teachers' conceptual structure on trigonometry was limited. For instance, on the given ratio, $tanx = \frac{4}{5}$, teachers could not recognise that 4 is an opposite side whereas 5 is an adjacent side. Ideally, this could be a result of not willing to mentally organise the given sides. The knowledge of differentiating side names is considered as a deficient conceptual structure. In terms of system representation, drawing a right-angled triangle on the Cartesian plane remained an issue. As such, the irrelevant conceptual structures prohibited learners from computing the hypotenuse side. Hence, the teachers did not proceed to the sense since the conceptual structure and system representation were not appropriate. Failure to proceed to sense affects learners' understanding during the process of learning. The argument has also been reported in the literature by Rocha et al. (2022), who stated that teachers influence learners' understanding of trigonometry. Although studies have emphasised that teachers should develop better teaching techniques, the problem is still insufficiently explored on how they negatively affect learners.

- Giri with a height of 1.5 m will measure the height of the flagpole. From Giri's spot, the flagpole top was seen with an angle elevation of 45° from the point of view of Giri. The horizontal distance from Giri to the flagpole is 18 m.
  1) Make the illustration of the story
  2) How long is the height of the flagpole?

- The students' answers analysis:
  1) One students have difficulty understanding mathematical statements, this can be seen from the illustration that is still wrong (Figure. 4).

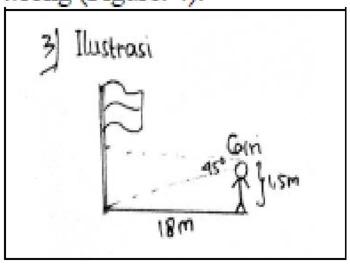


Figure 2.10 Scenario from Nanmumpuni and Retnawati (2021)

Nanmumpuni and Retnawati (2021) also conducted descriptive qualitative research titled "Analysis of Senior High School Students' Difficulty in Resolving trigonometry conceptual problems" in the province of Yogyakarta. The purpose of the study was to identify learners' difficulties in solving trigonometric problems. The desired data were collected using three tests together with in-depth interviews to find details on how they solve trigonometric problems. Two sets of data were analysed. Findings contended that learners had difficulties understanding the given question. In figure 2.10, the learners struggled to understand the concept of "angle of elevation". The struggle is considered as the inability to operate within the conceptual structures due to application of inappropriate SV. As a result, the learner could not represent the problem pictorially, which prohibited him or her to explain how illustrations were made in representing the problem. This is a failure to understand the given context. Moreover, the learner did not compute the height because accessing relevant concepts which assist in choosing the correct ratio was an issue. Nabie et al. (2018) argue that this might be a result of being taught foundational concepts using procedural instructions without understanding how they work. In contrast, Kovač et al. (2023) discovered that not providing meaningful feedback also affects learners' skills of understanding what the question requires. The authors add that teachers who incorporate the process of teaching and learning with immediate feedback inspire learners to engage when given activities. Essentially, this triggered them to change their attitudes towards mathematics.

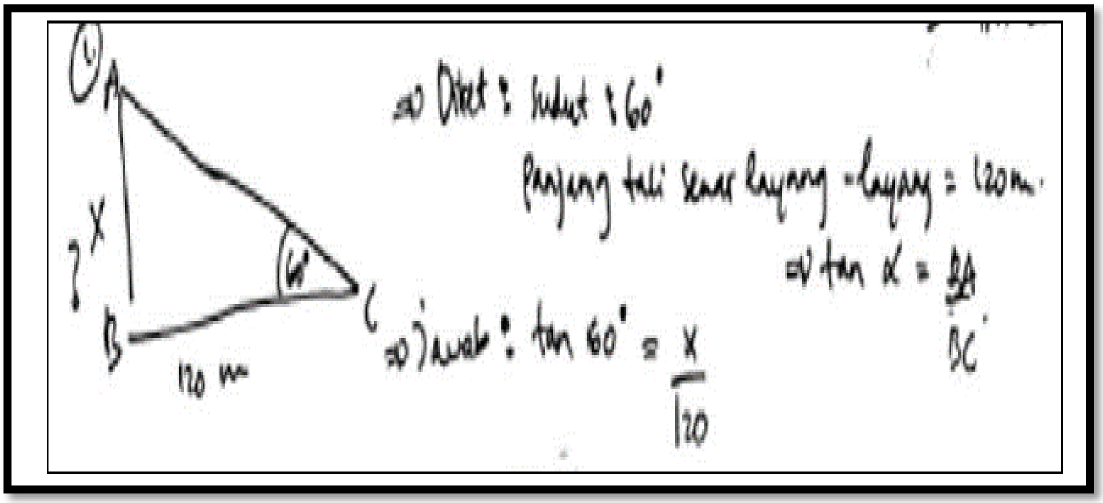


Figure 2.11 Scenario from Wardhani and Argaswari (2022)

Wardhani and Argaswari (2022) also carried out a descriptive study to analyse difficulties that hinder learners from solving word trigonometric problems. The study asked one research question: “What are difficulties made by learners when solving word trigonometric problems?” In responding to the question, data were collected through an online test. The textual data was categorised using the Newman error hierarchical model. Six learners with lower marks were sampled for interviews to document the difficulties encountered when computing sides. Their findings revealed that learners lack understanding of words applied to the given statement. Consequently, they represented the angle incorrectly. As a result, learners used tangent ratio instead of sine. I think the use of tangent ratios indicates that their conceptual structures were not well developed. Subsequently, they applied an incorrect ratio due to the failure to use appropriate conceptual structures to represent the problem pictorially and symbolically. The findings are consistent with those by Arlinwihowo et al. (2021, as cited in Wardhani & Argaswari) as well as Rosidin et al. (2019, as cited in Wardhani and Argaswari), who maintained that more than forty percent of learners still struggle to understand procedures for solving word problems. Therefore, the difficulties should be used as key elements for improving the learning of trigonometry.

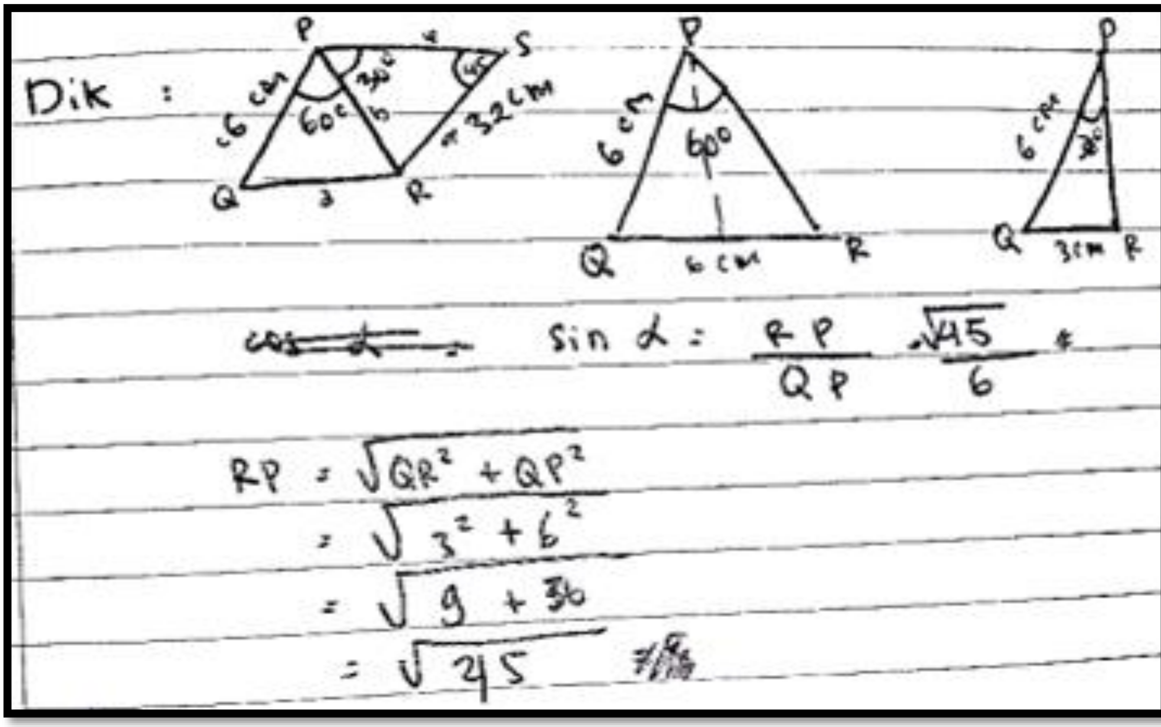


Figure 2.12 Scenario from Ekaludini and Darhim (2020)

A series of studies have recognised the importance of mathematical communication in trigonometry (Rustam & Ramlan, 2017; Ekaludini & Darhim, 2020; Utami et al., 2021). Rustam and Ramlan defined mathematical communication as the ability to organise mathematical ideas orally, in writing, in the form of tables, diagrams, symbols, or graphs. Similarly, NCTM (2000) defined it as the ability to organise, consolidate and analyse mathematical ideas clearly and coherently to teachers, peers, and others using proper mathematical language. In relation to this study, mathematical communication is learners’ ability to use the appropriate SV to make a mental collection of concepts on the given spatial figure then use the SR to organise and express side names and angles symbolically whilst computing sides and angles. Essentially, learners should develop mathematical communication as it sharpens their way of thinking and applying the SO. This will enhance their ability to notice the interrelatedness between concepts used in a question.

Ekaludini and Darhim (2020) is one of the qualitative descriptive studies that investigated the importance of mathematical communication when computing sides and angles. The researcher grouped participants into twelve. Each group was given a test to illustrate how they express their solution. The findings of their study were categorised into three aspects. Firstly, learners had difficulties in computing sides and angles presented in the form of figures. In figure 2.12, the learner could not compute side PR on ΔPQR due to failure to notice that the triangle is equilateral. Hence, the length of QR and PQ are equal. The difficulty relates to the lack of both SV and SR, which assists in outlining the required schemas and choosing the appropriate procedures for obtaining the length of PQ. Secondly, learners could not express mathematical ideas. The learner used the theorem of Pythagoras to compute the length of PQ instead of indicating that ΔPQR is equilateral. This relates to failure to operate within the pictorial representation using conceptual structures of an equilateral triangle. Moreover, the difficulty affected symbolic representation. Lastly, the learners explained what the question requires but lacked the mathematical skills to compute sides and angles. Relating to the current study, the learners seem to lack appropriate SV to visualise side names, SR to choose a correct strategy and SO to make justification procedures during interviews. As a result, their written solutions were isolated. This is due to the failure to communicate interrelatedness between side names of spatial figures and symbols.

According to Utami et al. (2021), learners can notice the interrelatedness between side names of spatial figures and symbols by considering the three aspects of mastering mathematical ideas, namely, drawing, writing and mathematical expressions. As such, these three aspects are vital in assisting learners to express their thinking. The three aspects were clearly perceived after conducting a qualitative descriptive study with 28 learners from class XI IPA. In terms of gathering data, a mathematical communication skill test was given to the purposive sample learners. The findings of that study were categorised based on learners' abilities to respond to the given test involving the three aspects. Firstly, in the writing aspects, there were 38% of learners in the deprived category, 50% in the moderate category, 11% in the good category and 4% in the outstanding category. On the indicator for drawing, there were 7% of learners in the deprived category, 57% in the lower category, and 36% in the moderate category. Lastly, on the mathematical expression, there were 14% on the deprived category, 50% in the lower category, 18% in the moderate category, 11% in the good category and 7% in the outstanding category.

The percentages above-mentioned show that the three aspects have not reached classical completeness, meaning few learners have mastered the three aspects of mathematical communication skills. Sujadi and Subanti (2019) are some of the scholars who utilised these three aspects in examining learners' mathematical connection ability in solving trigonometric problems. Although their study was based on mathematical connection, I strongly believe that the three aspects used in their study contribute to the current study. However, the three aspects are classified in terms of semantic theory. In this study, learners are given spatial figures which are drawings using trigonometric ratios to substitute the given numeric values to obtain the required answer which forms part of writing and mathematical expression. Sujadi and Subanti stated that although the aspects assist in expressing any mathematical idea, there are certain factors that hinder some learners from moving flexibly during computations. Firstly, failure to understand how symbols are utilised. Secondly, failure to determine concepts that can be used to compute sides and angles for various contexts. Therefore, the three aspects should be developed continuously to improve learners' skills in tackling questions. However, it is advisable to firstly examine their current mathematical communication and factors that influence it to achieve lesson objectives.

2.3.3 Learning of trigonometry

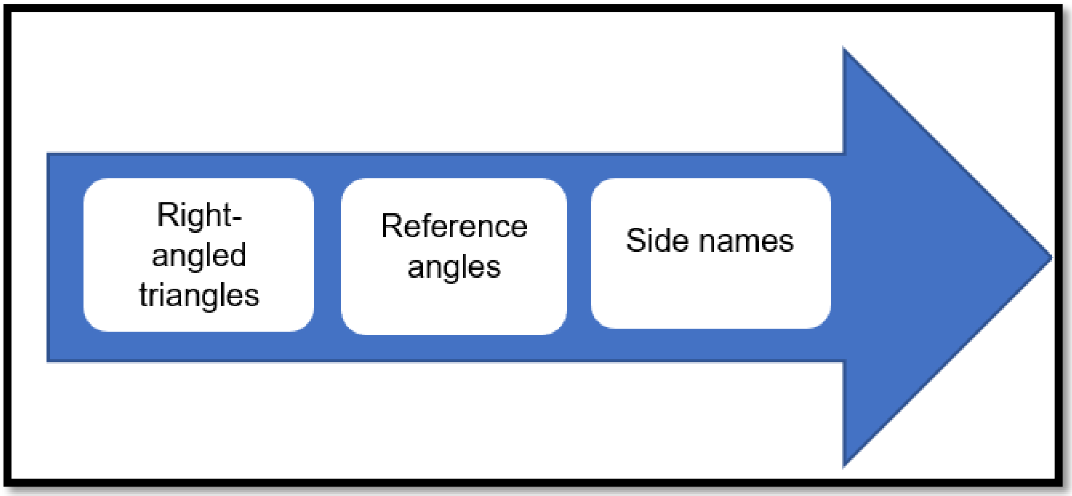


Figure 2. 13 The process of understanding trigonometric ratios

The learning of trigonometry should be enhanced to move flexibly between spatial skills during computations of sides and angles. Before explaining the learning of trigonometry, it is vital to clarify the general meaning of learning. Mayer (2019, cited in Adebiyi, 2019) explained learning as a permanent alteration or modification that occurs in individuals due to experience. Mayer goes on to say that the definition involves three aspects: 1) the duration of the alteration or modification; 2) alteration or modification which is related to the learners' knowledge; and 3) experience in the environment as a cause of the alteration or modification. The three aspects are keys for examining whether learning has occurred or not as shown in figure 2.13. According to Piaget (2016), learning is a process of constructing knowledge by incorporating new knowledge and existing schemas. However, success is based on the relevance of learners' existing schemas. This idea is consistent with that by Fiorella and Mayer (2015), who present learning as a process of making sense of ideas through selecting and arranging existing knowledge. Arguably, Vygotsky (1934) encourages social interactions, experience, and an essential Vygotskian principle, scaffolding which is a technique of providing learners with hints to solve problems. Ideally, all these definitions are based on the authors' philosophical views and approaches.

From the definitions above, the learning of trigonometry refers to a process of, 1) acquiring new knowledge, 2) understanding, and 3) skills through experience or studying. The current study considers trigonometric ratios as new knowledge since the concepts are introduced in Grade 10 using properties of a right-angle triangle to understand the ratios, and spatial skills to compute sides and angles. This can be accomplished through applying the three stages of learning as suggested by DBE (2013): before to check prior knowledge, during to develop the new concepts which are ratios, and after to check whether learners can apply the ratios or not. However, during the process of learning on the, during stage, constructivists believe that knowledge of trigonometric ratios cannot be transmitted but constructed by learners (Bada & Olusegum, 2015). However, not all individuals can construct trigonometric knowledge on their own. For instance, some learners lack adequate prior knowledge to conduct their own learning (Simonsmeier et al., 2022). The difficulty hinders learners from fully engaging in the learning process of trigonometry. All difficulties that hinder learners from fully engaging during the process of learning trigonometry should be minimised through the application of effective and efficient teaching strategies and methods that are learner centered (Ogunjimi & Gbadeyanka, 2023; Sakata, 2022).

Vital facts about meaningful learning were outlined: 1) advancing learners' ability to recall mathematical concepts better; and 2) boosting their ability to connect previously learned and new mathematical knowledge (Andjamba, 2021; Mystakidis, 2021). The facts illustrate that meaningful learning does not enable learners to recall pre-existing and new knowledge only, but also to apply it to various contexts. Meaningful learning is a precondition for recalling aspects of computing sides and angles using spatial skills. However, the learning processes and suitable classroom settings should be considered for successful meaningful learning (Burden, 2020). This view is consistent with that by Bah et al. (2019), who emphasised that the mastery of mathematical concepts is not determined by cognitive abilities only, but rather by the proper classroom settings. For instance, having sufficient material for learning trigonometry practically such as tables, rulers and protectors. Once learning of ratios has occurred, a mnemonic strategy for recalling SOH-CAH-TOA can be used with understanding (Ni et al., 2019). However, in rural areas, a proper classroom setting is still an issue that limits the development of spatial skills to critically analyse figures during computations.

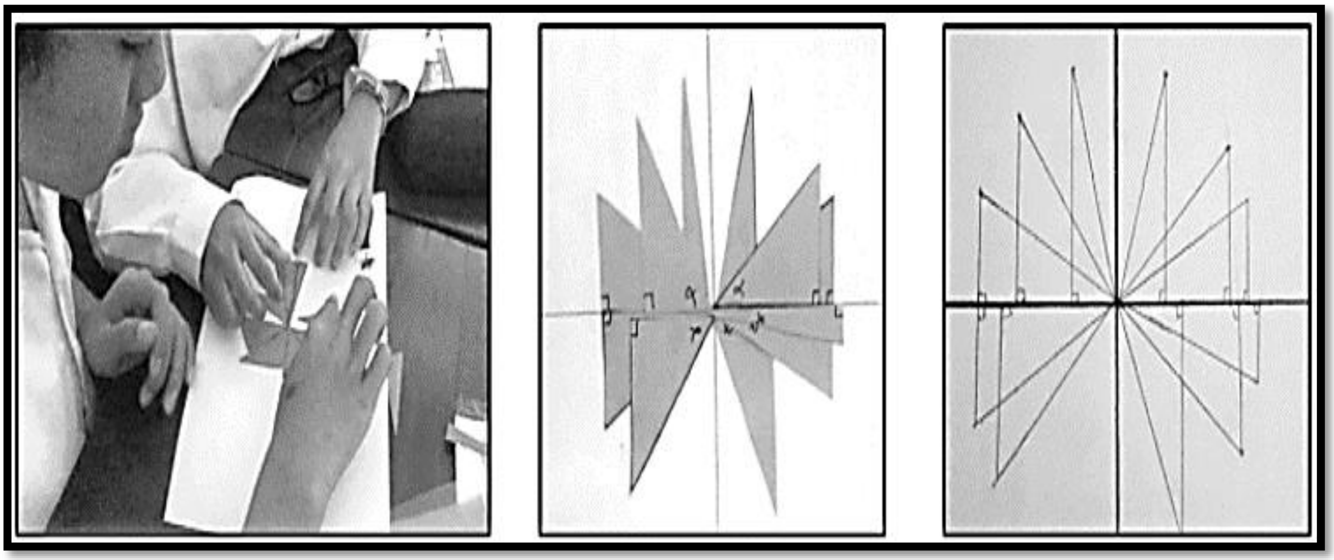


Figure 2.14 Scenario from Maknun et al. (2019)

The literature review outlined that there are few studies on trigonometry founded on a phenomenographic design. Maknun et al. (2019) are some of the prior phenomenographic studies that showed that teachers can easily deliver the definitions of ratios using the context of a right-angled triangle or circle unit. Their study asked two research questions: 1) "What lesson design instructions can teachers implement in linking trigonometric ratios with unit circles; and 2) what difficulties do learners face in the learning process based on the lesson design instruction implemented". The lessons were implemented in a class of 33 learners in Indonesia. During the learning process, learners were encouraged to work in groups to draw different triangles with varying lengths as shown in figure 2.14. Thereafter, cut the triangles and arrange them on the Cartesian plane such that their vertices meet at the center. All lessons were recorded using a device for further reference. Within the initial implementation of the lesson, findings outlined that some learners found it difficult to arrange the triangles on the Cartesian plane. This implies that the knowledge of locating the right-angled triangle which forms part of the conceptual dimension was limited. However, after revision, learners managed to define ratios fluently in various quadrants. The study recommended teachers to develop hands-on activities for learners to discover ratios on their own. For a successful lesson, the teacher should ensure that all learners have the required material, and their knowledge of coordinates is modified.

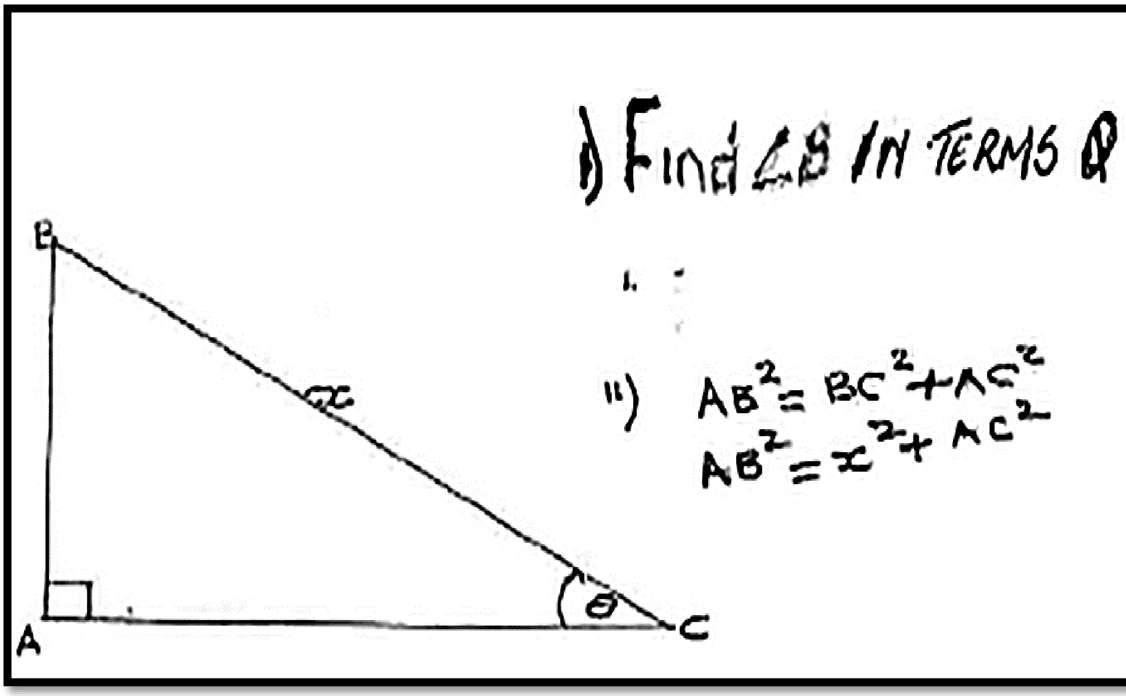


Figure 2.15 Scenario from Ngcobo (2019)

Prior research by Ngcobo (2019) argues that having the required material and knowledge of coordinates is not enough. Thus, understanding the concept of the hypotenuse, adjacent, opposite, reference angles and their relationship is essential. This was a qualitative study carried out with Grade 12 learners in Kwazulu-Natal. The study intended to check whether learning had occurred through solving unknown sides and angles of triangles. Three research questions were asked: 1) “what is the nature of Grade 12 learners’ mental constructions of solutions on triangles? 2) to what extent do learners’ mental constructions of triangles align with genetic decomposition? 3) why do these mental constructions arise?” Although Ngcobo and the current study collected data through mathematical tasks and interviews, different theories were used to analyse the textual data.

In terms of theory, Ngcobo (2019) was founded on APOS whereas the current study adopted the semantic theory. Findings from that study revealed that five learners accessed irrelevant trigonometric concepts to compute the required angle. In figure 2.15, the question required angle B, but the learner used the theorem of Pythagoras to compute the length of AB. This shows that action conception was not interiorised. Consequently, operating within the process, object and schema remained an issue. Interpreting the findings using spatial skills and semantic theory, the learner could not use SV to differentiate between an angle and a side. This means that the learner manipulated incorrect conceptual structures. As a result, having conceptual structures would have allowed the learner to use appropriate SR in choosing a ratio to compute angle B. Eventually, the sense was not interiorised since the concept of Pythagoras was used in the wrong context. Therefore, conceptual structures are key elements for computing sides and angles of spatial figures.

According to Polman et al. (2021), learning trigonometry should be meaningful to learners. The ample evidence was outlined after five teachers with varying beliefs on meaningful learning were interviewed. Their findings revealed that learners' understanding of trigonometric ideas is influenced by teachers' pedagogical practices, contextualised as a method of assisting individuals to acquire both knowledge and spatial skills for computing sides and angles. A recent study by Andjamba (2021) maintains that meaningful learning should motivate learners to make sense of, remember, and apply the acquired knowledge when given activities and in real-life situations. This strengthens learners' skills to operate precisely from the initial level of semantic theory which is conceptual structure towards the sense (Martin-Fernández et al., 2019). Despite this, the literature review contends that learners tend to forget key constructs required in computations of sides and angles since they seem to be meaningless to them (Kamber & Takaci, 2018). The sincere way to address this difficulty is grounded in the constructivism paradigm, which portrays that individuals construct knowledge through the spirit of experimentation (Piaget, 2016).

Ideally, constructing knowledge through experimentation also relies on mathematical connections. Learners with appropriate mathematical connections have skills for visualising the process of computing sides and angles. In contrast, learners without proper mathematical connections struggle to imagine the connection between sides and angles. Mathematical connection refers to the ability to link existing ideas in other fields with new mathematical ideas while solving problems (Pambudi et al., 2020). In the context of trigonometry, mathematical connections refer to learners' ability to link algebraic, geometric and fraction ideas learned previously with trigonometric ideas. Moreover, in this study, mathematical connection plays a vital role in assisting learners to manage their SV, mental manipulations of spatial figures, SR, computing sides and angles using appropriate ratios and SO, and clarify concepts involved in their procedures. However, Rohimah and Probawanto (2019) pointed out that most learners still suffer from a weakness in connecting ideas applied to a problem due to the absence of adequate prior knowledge. For instance, struggling to link the concept of grouping like-terms from algebra in computations of sides and angles (Tastepe & Yanik, 2023). The weakness was outlined during interviews as learners explained their knowledge of trigonometry in terms of procedures yet found it challenging to provide definitions in their own words, thus resort to memorisation.

Based on my teaching experience, the process of assisting learners to use mathematical language seems to be an adequate technique for improving the learning of trigonometry. This view is supported by the Department of Education in South Africa, which introduced a mathematical language policy for both teachers and learners in the learning process (DBE, 2013). According to DBE, mathematical language is a unique and distinct system for communicating mathematical concepts. It contains its own conventions, vocabulary and rules for combining them into a full set of thoughts. Gürefe (2018) explained mathematical language as communication between learners using mathematical vocabulary and symbols. In this context, mathematical language is a strategy of using trigonometric vocabulary to illustrate procedures for computing sides and angles symbolically and verbally. Essentially, efficient mathematical language enables learners to use spatial skills to comprehend and perceive relevant elements for computations. Hence, having a proper mathematical language is a requirement to illustrate the computation of sides and angles clearly.

Wahyuni et al. (2020) carried out a study titled “The Effect of Mathematical Language on Learning of Mathematics”. In the study, the authors utilised a method of examining existing literature concerning learners' mathematical language. Thereafter, a conclusion was made using results obtained from the existing literature. This is an indication that the set of data was strictly on the existing studies. After examining the literature, the authors delineated that comprehending mathematical language enables learners to acquire mathematical ideas. In relation to this study, the SV is a prerequisite for recognising vocabulary that can be used to respond to computations of sides and angles. Consistently, the appropriate SV activates the learners' SR to organise the conceptual structures such as relevant ratios to compute sides or angles. The process of organising leads to a skill to represent answers to the given question, and this forms part of using mathematical language such as using symbols. Based on semantic theory, the use of symbols forms part of system representation. From my point of view, I would say mathematical language is used during the process of operating within the symbolic representation which forms part of the system representation. Moreover, the ability to use mathematical language fluently during computations also assists in justifying the written procedures using the SO, and this is categorised as a skill to consider the sense. Therefore, learners must be given tasks to assess their spatial skills in using mathematical language.

It is vital to give learners tasks at different stages to monitor their learning progress. When learners are given those tasks, they present answers differently. Huang and Xiao (2019) are some of the authors who emphasised that learners present answers differently. For instance, these two authors highlighted that sometimes learners leave blank spaces as they are not allocated enough time to think independently about the given task. Similarly, Jakwerth and Stancavage (2003) as well as Hamukwaya (2022) concur that learners leave blank spaces due to a lack of time or inability to figure out answers. Despite this, learners often spend a lot of time attempting to make sense of the given questions. On the other side, Asbupel et al. (2021) add that some learners leave blank spaces as they do not understand the concepts used in the question or not willing to make mental manipulations. This is an indication that learners leave blank spaces for various reasons.

Medaille and Usinger (2020) carried out a qualitative phenomenological study to examine the experience of quiet learners in classrooms. The study was carried out at the University of Nevada with 10 students. Moreover, in responding to the research question posed, “How do self-identified quiet undergraduate students understand how the postsecondary classroom environment affects their learning?”, reflections and interviews were used based on collaborative learning. The two sets of data were analysed through NVivo 11 TM software. The authors found that some learners prefer an expression of engaging through listening, writing and observing. In this context, these forms of learning are categorised as system representations. From my point of view, I believe that learners learn differently. Hence, all learners should be accommodated in a classroom.

2.3.4 Learners’ challenges in trigonometry around South Africa

Studies in South Africa have outlined general difficulties encountered by learners in trigonometry. Khuzwayo (2019) is one of the studies carried out to explore the learning of mathematical identities in Grade 11. Although the study intended to explore the learning of trigonometric identities, learners difficulties on trigonometric ratios were indicated as it contribute to trigonometric identities. Findings in that study indicated that learners struggle to understand that trigonometry is a topic that links fractions and algebra such as handling the like and unlike terms. Specifically, when required to use trigonometric ratios during computations, learners could not recognise the necessary sides. As such, learners ended up memorising the ratios without knowing how to apply them. This is consistent with an idea of Hurrell (2021) emphasising that learners tend to memorise mathematical ideas.

Moreover, Khuzwayo (2019) highlighted that KZN learners found it difficult to respond to questions in 2017. This was a result of lacking the complicated skills necessary in trigonometry. Precisely, lack of necessary skills in trigonometry resulted into the following challenges: 1) failure to comprehend the words on the question; 2) lacking skills to attack the question; 3) being confused; 4) lacking knowledge to solve the problem; 5) struggling to use trigonometric language such as symbols; and 6) ignoring rules for computing sides. Ideally, some of the challenges might be due to lack of adequate training to teach the topic. Arguably, Long and Wendt (2019) postulate that this cannot be considered as schools in South Africa are supplied with qualified mathematical teachers. Yet, the teachers might be qualified but struggle to deliver the content to learners. Therefore, it is vital to ensure that the qualified teachers are capable of delivering or assisting learners to acquire trigonometric concepts.

Molataola (2017) also carried out a quantitative study titled ‘‘Improving Grade 10 learners conceptual understanding of trigonometric ratio using Pythagoras theorem’’. The study was carried out with 30 Grade 10 learners between the age of 14 and 18 years at Capricorn district around Limpopo to investigate the effects of cooperative learning in understanding of trigonometric ratios. The study asked two research questions: 1) how teaching using cooperative learning does enhances learners understanding of trigonometric ratios? 2) Has grade 10 learners’ achievement in trigonometry improved after the study? Pre and post-tests were used to collect data. The two set of data were analysed using descriptive statistics and this is in contrast with the current study setting as it was underpinned on qualitative approach to explore spatial skills learners’ exhibited during computations. In that study, findings outlined that learners struggled to investigate the link between properties of a right angled triangle. As a result, they could not compute sides or angles. Therefore, all challenges must be addressed to enhance learners’ skills and knowledge of computing sides and angles. This can also be accomplished if the content provided to student-teachers at universities can be aligned with that are offered in public schools.

2.4 Chapter Summary

The chapter started by giving an overview of the literature reviewed. Secondly, a theoretical framework called the semantic theory that guided and structured my study was discussed. Each tenet was precisely explained and supported by the existing literature. Thirdly, the chapter discussed three pillars, namely, spatial skills in the learning of trigonometry, trigonometric concepts in trigonometry and the earning of trigonometry which were formulated using the key constructs research title. Each pillar entails conceptual literature which is supported by findings that support my research gap.

CHAPTER THREE: METHODOLOGY

3.1 Introduction

This chapter describes an array of research procedures that were followed in conducting the study. The research procedures include the research paradigm, approach, research design, sampling, and how mathematical tasks and semi-structured interviews were used to collect data in Grade 10 on how the three basic trigonometric ratios are used during the computations of sides and angles. Moreover, the chapter describes how thematic analysis was used to analyse the two sets of data. Each aspect of the research was given a satisfactory justification for why it was chosen. The chapter further explained components for improving the rigor of the study and concludes with details on ethical considerations.

3.2 Research paradigm

A research paradigm plays a crucial role in indicating the researcher's beliefs or philosophical viewpoints. A research paradigm refers to a fundamental set of values, assumptions and beliefs which guide researchers based on their research designs (Creswell, 2018). Similarly, Perera (2018) concurs that a paradigm can be a set of common agreements shared by various scientists on how problems that arise during the process of teaching and learning are comprehended and addressed. According to Park et al. (2020), a research paradigm assists researchers to have an idea of how to conduct their research. For instance, the kind of methods to collect and analyse data as well as the theory to interpret the data. I concur with the authors mentioned above that researchers should have a philosophical point of view that guides their classroom practices and processes of conducting research.

There are various sets of paradigms. Some were clearly explained by Kumatongo and Muzata (2021), namely, constructivism, positivism, interpretivism, etc. Firstly, constructivism explains that learning occurs when learners discover knowledge through the process of experimentation or doing while the teacher plays the role of a facilitator. Secondly, positivism refers to a philosophical view that believes that learners gain knowledge through the process of observation which occurs by employing traditional teaching methods. Lastly, interpretivism assumes that mathematical knowledge is based on interpreting what had been seen or said. Above all, the current study used constructivism as a worldview. In Grade 10, trigonometry is a new topic. Hence, I wanted learners to develop skills and trigonometric concepts that would assist during the computation of sides and angles using ratios. To accomplish this idea, three tasks were designed such that: 1) the first one checks learners' prior knowledge; 2) the second one, learners develop trigonometric concepts and spatial skills on how to use trigonometric ratios during the computation of sides and angles; and 3) finally, for checking whether learners have gained trigonometric concepts to use ratio during the computation of sides and angles. Thus, how the process of discovering ratios was applied.

3.3 Qualitative research approach

The study used a qualitative approach. According to Creswell (2009), the qualitative approach is a procedure followed by researchers to obtain a detailed understanding of a context. Jameel et al. (2018) defined the qualitative approach as a process of gathering and understanding non-numeric data by observing a social phenomenon as it occurs. Jackson et al. (2007) concurs with the authors above by mentioning that the qualitative research approach is an interpretive approach to understanding human beings' experiences. In this instance, human experiences are spatial skills exhibited when computing sides and angles using ratios. The approach was effective as it allowed the researcher to gain an insight into learners' spatial skills exhibited whilst computing sides and angles. According to Brace (2018) and Cohen et al. (2017), the qualitative research approach assists in presenting, analysing and interpreting images and words based on the raw data that was collected. In line with Brace's statement, the qualitative research approach allowed the researcher to apply three tenets of semantic theory to elaborate on learners' behaviour towards the computations of ideas and angles. Specifically, learners' behaviours are spatial skills exhibited on the given mathematical tasks and interviews. Therefore, relying on the chosen participants to offer detailed responses to questions on how they understand the given tasks is an essential aspect of the qualitative research approach (Mohajan, 2018). As a result, participants are crucial components that provide answers to research questions. The greatest strength of the qualitative research approach is that it generates a rich description of participants and focuses on why a phenomenon has occurred.

3.4 Research design

Creswell (2009) defined research design as a basic plan that guides the processes of collecting, analysing, interpreting and reporting data in research. Research design is an overall strategy that sets procedures for the data required, and the method to collect and analyse the required data. According to Kerlinger (1986), a research design is a structure, or a plan of investigation adopted by researchers to obtain answers to research questions. The literature review from Kerlinger emphasises that the research approach determines the design. Correspondingly, a suitable research design for this study was an exploratory case study. Patnaik and Pandey (2019) defined an exploratory case as a method of understanding a phenomenon in a single setting to provide a clear description. Similarly, Yazan (2015) identified an exploratory case study as a process of exploring a situation with specific boundaries based on the researcher's interests. This implies that the researcher is interested in studying a certain behaviour of a chosen group. According to Zainal (2007), conducting an exploratory case study allows the researcher to closely examine a specific context. Based on the three descriptions, an exploratory case study is characterised by three important aspects. Firstly, a case can be an individual, group, organisation, or an event (Gustafsson, 2017). In this instance, the case is a class of Grade 10 learners. Secondly, a specific boundary which is the typical phenomenon of interest (Neuman, 2014). The boundary for this study is exploring Grade 10 learners' spatial skills when computing the sides and angles of figures. Lastly, the type of case study, single for testing a theory or multiple case study for developing a rich theory. In this study, an understanding of learners' spatial skills was gained by conducting a single case study of a class. Merriam asserts that the three aspects assist researchers to use a variety of data-collecting methods to document enough data about learners' behaviour. In this study, the three aspects of an exploratory case study enabled the researcher to recognise difficulties that hinder learners from exhibiting relevant spatial skills while computing sides and angles of figures using trigonometric ratios.

3.4.1 Sampling

According to Merriam (1998), an exploratory case study should be guided by purposive sampling, not population since it is suitable for collecting qualitative data. Thus, the study used purposive sampling which is a form of selection whereby the researcher relies on their own judgment when choosing participants (Palinkas et al., 2015). Crossman (2020) maintained that purposive sampling selects participants based on the interest of a study. It is useful for addressing concerns that arise during a specific event (Ames et al., 2019). In doing this, the study purposively selected a Grade 10 class at a rural public school around Limpopo. The school was under-resourced, and procedure teaching methods such as chalkboards are still used during the process of teaching and learning. Also, the school has 487 learners comprising 247 males and 240 females, supported by 16 staff members. To be specific, the Grade 10 class has 22 learners and they all participated in this study. In terms of language, the school is dominated by Xitsonga-speaking, and a few Lobedu-speaking, people. This was a great way of exploring a variety of spatial skills exhibited by learners when computing sides and angles as they access relevant concepts learned previously in GET. In terms of mathematical tasks, the researcher purposively selected three trigonometric questions from previous question papers, which require learners to compute the sides and angles of figures.

3.4.2 Data collection

Creswell (2018) highlighted that data-collecting methods should be in line with the research approach. Since the study is qualitative, mathematical tasks (**Appendix D**) and semi-structured interviews (**Appendix E**) were used to collect data. The details on how the two methods were used are discussed:

3.4.2.1 *Implementation of the three mathematical tasks*

Three stages were implemented to collect data: 1) before, on the first lesson; 2) during, on the second lesson; and 3) after, on the third lesson. In each lesson, a mathematical task was used. However, before the implementation process, learners were informed that at the end of the three lessons they must be capable of computing sides and angles using ratio. In the first lesson, learners were given a baseline task, mathematical task 1 to inspect their prior knowledge of side names required during computations. The task was written under the researchers' supervision for 50 minutes such that each learner could exhibit his or her own spatial skills. This was also a way of ensuring learners do not copy from each other. The baseline task contained three questions. The first question required learners to differentiate between sides and angles. Meaning it was necessary for the learners to mentally visualise the position of an angle and sides. This is a process of using the SV. Once the positions were noticed, the SR was required to access appropriate words to express the differences. Hence, an SO was required for learners to justify their written responses during interviews. The second and third questions required learners to use their SV to notice the given sides and the one to be computed. Thereafter, the noticed sides enabled learners to use their SR to choose the appropriate equation to compute the unknown side. In doing so, knowledge of Pythagoras' theorem was mandatory. This was a strategy of emphasising that the concept of side names of the right-angled triangle is well comprehended. After 50 minutes all scripts were collected and marked using a marking guideline.

On the second lesson, learners were given feedback on the baseline test so that they can know what they were expected to do. Thereafter, a learning activity, mathematical task 2 was given to learners for 60 minutes to develop skills and knowledge of ratios and how they work during computations. This indicates that learners were expected to use their spatial skills to acquire the concept of basic ratios on their own. Most importantly, they were allowed to use any source, a textbook, study guide, or internet to obtain different ideas. During this process, I acted as a facilitator of learning to observe as learners are working and clarify their difficulties. The learning activity contained seven

(7) questions. The first question required learners to use their SV to outline side names in reference to the given angle. The second question obliged learners to use the sides mentioned on the first question to define the three basic ratios. The third question expected learners to read the given statement using the SV and the SR to represent the given sides on the spatial figure on question one such that the unknown side can be computed. The fourth question necessitated learners to use the two given sides and the computed side to define the three ratios. On the fifth question, a right-angled triangle was given and learners had to observe it using the SV and use the SR to access properties of a right-angled triangle and use the SO to explain the link between the size of the angles and the length of sides. On the sixth question, learners had to use the SV to notice that the required side is hypotenuse. Therefore, the SR was necessary to notice that hypotenuse can be computed using cosine ratio whereas a tangent was an appropriate ratio for computing the adjacent side. These questions were sequential such that learners can notice that a proper acquisition of side names influence the choice of ratios.

In the last lesson, learners were given an assessment task, mathematical task 3 to apply knowledge of three basic ratios using spatial skills to compute sides and angles when given numerous spatial figures as one figure. The task had six questions which required learners to use appropriate spatial skills to recognise sides and angles for various figures in reference to angles. Learners were monitored such that each can exhibit his or her spatial skills. Briefly, the three tasks were used to respond to the first research question. This allowed me to capture learners' spatial skills exhibited through accessing relevant concepts required in computations of sides and angles. Although some of the questions required learners to use Pythagoras' theorem during the computation of sides, the ability to use the three basic ratios during computations remains an important aspect of this study. This was a way of showing learners that knowledge of Pythagoras' theorem does contribute to the ability to use the three basic ratios. Thereafter, all scripts were collected, marked and grouped based on their similarities and differences in spatial skills using the semantic categories. Concisely, the three mathematical tasks enabled learners to notice that acquisition and comprehension of ratios necessitate pre-existing knowledge of sides and angles from the previous Grade.

3.4.2.2 Implementation of semi-structured interviews

After classifying learners' written responses in each lesson, semi-structured interviews (**Appendix E**) were conducted for approximately 5 minutes to probe the reasons why learners computed sides and angles in particular ways in each lesson. Semi-structured interviews refer to a way of asking a few pre-determined questions while the rest of the questions arise from the interviewee's response. This indicates that the interview questions were based on individual responses to the given mathematical task. The interviews were conducted such that the sampled learners included those with inappropriate and appropriate spatial skills. Learners with inappropriate spatial skills are those with the inability to mentally imagine how figures relate to trigonometric concepts, which prohibits them to perceive given figures and justify written procedures (Kozhevnikov & Hegarty, 2001). In contrast, learners with appropriate spatial skills can access relevant knowledge during the computations of sides and angles. This allowed learners to provide detailed insight and further perspectives on spatial skills exhibited when working on mathematical tasks (Patton, 1990; Merriam, 1998). During the interviews, the researcher was given a pseudo name called Faith whereas alphabets were used to represent learners. For example, learner A. The two methods are relevant to the current study as they required learners to represent trigonometric concepts in multiple ways. Hence, the two methods allowed the researcher to analyse the exhibited of trigonometric concepts using the three tenets of semantic theory.

3.4.3 Data analysis

Thematic analysis was used to analyse data from mathematical tasks and semi-structured interviews (Braun & Clarke, 2012). Thematic analysis refers to a technique used to identify, organise and describe a pattern of meaning across a set of data (Clarke, 2012). This definition is in line with that by Kiger and Varpio (2020), who emphasised that thematic analysis enables researchers to understand behaviour or experiences across sets of data. In this study, the thematic analysis allowed me to explore data and check how learners exhibited spatial skills during computations. Moreover, the approach allowed me to check if the two sets of data answer the two research questions.

The two sets of data were analysed in two phases. Firstly, the textual data was read in detail, checking how learners exhibited their spatial skills when computing the sides and angles of figures. Also, the audio data of the sampled learners with various spatial skills were changed into textual data. This was a strategy of managing and exploring the two sets of data using the three tenets of semantic theory, conceptual structures, system representation, and sense.

Secondly, Gibbs (2012) argued that to ensure that the data responds to research questions, a coding strategy must be used. Similarly, axial coding was used to categorise learners’ spatial skills based on their differences and similarities. The coding process started after the audio data was transcribed. In doing this, the two sets of data on the three lessons were explored and categorised by me and the three appointed coders. However, the appointed coders were trained before exploring the data to inform them about the categories of semantic theory and how they can be used in this study. After categorising the data, the coders noticed that some learners: 1) left blank spaces for different reasons; 2) recalled irrelevant prior conceptual structures of the right-angled triangle; 3) struggled to compute the required sides; 4) recalled relevant prior conceptual structures such as properties of a right-angled triangle; 5) and handled the BODMAS rule and symbols accurately while computing the required sides and angles. Thereafter, codes were generated for each lesson using the categories mentioned above. Hence, three tables are shown in chapter four to emphasise that the responses from the three tasks together with their transcripts were grouped. Most importantly, it is crucial to know that the cells in the three tables without codes illustrate that there are no learners who exhibited certain spatial skills in those questions.

Lastly, four themes emerged from the codes of the three lessons and the semantic tenets were used to interpret learners' spatial skills. This shows that an inductive approach was followed as themes emerged from the two sets of data not from the theory.

3.4.4 Quality criteria

The study ensured four aspects of quality criteria, credibility, dependability, confirmability, and transferability.

3.4.4.1 Credibility

Credibility is considered an appropriate criterion that is consistent in qualitative research and refers to a process of representing the real meaning of a study (Lincoln & Guba, 1985). This is a criterion for ensuring that a study measures what is intended (Pandey & Patnaik, 2014). In this study, credibility was achieved using the triangulation method, which is a technique of using two or more methods to collect data (Korstjens & Moser, 2017). Noble and Heale (2019) also explained triangulation as a method of utilising various methods to respond to research questions. As such, data were collected using various methods to answer the how and why research questions. Firstly, data were collected using a mathematical task to answer the how question. Lastly, data were collected by conducting semi-structured interviews to respond to why question. The two methods allowed me to comprehend learners' challenges and their causes. Moreover, the methods allowed me to confirm that learners exhibit different spatial skills when given mathematical tasks and during semi-structured interviews.

3.4.4.2 Dependability

Dependability indicates that similar results must be obtained if the study is done repeatedly in the same context using the same methods and participants (Shenton, 2004). Moon et al. (2016) argue that dependability is the consistency of findings when an inquiry is replicated with similar participants and context. However, the research procedures should be well documented to allow others to follow the process. In this study, dependability is attained through an inquiry audit which is a way of examining the process and products of the research in the absence of the researcher (Hoepfl, 1997). However, a detailed explanation of how the data was analysed is a priority to conduct an inquiry audit. For instance, in this study, data were analysed by the appointed coders into phases: 1) read and explored the textual and transcribed data in detail; and 2) categorised the data based on their similarities and differences. The different categories were coded, and four themes emerged from the codes due to the application of the inductive approach. Thereafter, procedures followed while collecting data in the absence of the researcher were examined by the appointed mathematics coders or teachers (Tracy, 2010).

3.4.4.3 Confirmability

Confirmability concerns the aspect of objectivity (Patton, 1990). As such, an audit trail was used to ensure real objectivity. Patton further explains an audit trail as a procedure performed to keep a record of what has been done during investigations. In doing this, the researcher documented and provided a complete set of research steps taken from the beginning of the research to the development and reporting of the findings (Korstjens & Moser, 2017). Thereafter, the documented details were given to the appointed coders to explore to comprehend aspects of this study.

3.4.4.4 *Transferability*

Transferability ensures that the findings of the study can be applied in other contexts (Pandey & Patnaik, 2014). Shenton (2004) concurs that the findings of a study should be applied in various settings. Thus, to ensure transferability of this study, the results may be cautiously transferable to other classrooms to address spatial skills in trigonometry. However, I was cautious when interpreting the data of the purposively sampled participants to avoid generalisation. For better management, pseudo-names were used to represent participants. This enabled me to explore their spatial skills. Therefore, while using data from this study in other contexts, there might be different opinions and findings due to the strategic sampling chosen.

3.5 Ethical considerations

3.5.1 Permission to carry out a study

It is essential to ask for permission from the Turfloop Research Ethics Committee (**TREC**) to conduct a study. In doing this, I have applied for ethical clearance from TREC to get permission to carry out the study. This was done as soon as the proposed study was accepted by the faculty. Once an ethical clearance certificate was obtained, I requested permission to collect data by sending a request letter to the Department of Basic Education Limpopo Province head office (**Appendix B**). Also, a request letter was given to the school manager (**Appendix E**) for permission to collect data in Grade 10. The request letters contained important aspects of research such as the purpose of the study and methods of collecting data for a brief about the proposed study.

3.5.2 Informed consent and voluntary participation

Diener and Cranddall (1978) defined informed consent as procedures that enable individuals to choose whether to participate in a study after being informed about facts that would influence their choices. Neuman (2014) argues that the description involves five vital elements. Firstly, competence refers to the ability to make relevant choices. This was ensured by allowing participants to decide whether to participate in the study or not. Secondly, voluntary participation relates to a choice taken by participants to freely engage in a study (Marczyk et al., 2010). Specifically, the researcher should not be able to force learners to participate in a study. Therefore, all learners have the right to withdraw at any time without being criticised.

Thirdly, full information whereby a consent form contains relevant information about the study. In this instance, the consent form specified the purpose, procedure of the study, and benefits of participating in the study. These aspects were clearly explained to all participants at the initial stage of the study. Fourthly, comprehension is when participants understand the nature of the study (Neuman, 2014). To ensure that participants understand the nature of the study, I explained all aspects of the study such as the purpose, and methods of collecting and analysing data. This was a greater opportunity for learners to understand the reasons for conducting the study.

Lastly, assent is an expression of the agreement for young individuals to participate in a study (Al-Sheyab et al., 2019). The definition mentioned above argues that it is necessary to consider parental approval for young individuals to participate in a study. This is consistent with the Children's Act 38 of 2005, which states that parents must be given an opportunity to express their views concerning their child (DSD, 2015). In this study, Grade 10 learners are still minors, therefore, they cannot choose independently to participate in a study. As such, parents were expected to give their learners permission to participate in the study. In this case, I fairly indicated a slot for parents' signatures **(Appendix C)**. All young individuals with an interest in participating were given the consent form so that their parents could sign it. Hence, signing the form was a way of permitting their children to participate in this study.

3.5.3 *Confidentiality and Privacy*

Confidentiality refers to the principle of not disclosing individuals' personal information in an identifiable way (Neuman, 2014). In this criterion, the researcher ensured that information that can affect the dignity of participants is protected. For instance, the data collected was stored in a locked room to prevent others from accessing personal details of participants. This was a way of implementing the Protection of Personal Information Act of 2013 (POPIA), which emphasises that individuals' information must be protected (DSD, 2015). Moreover, during data analysis, pseudo names were used to represent the participants instead of indicating their names to avoid violating their confidentiality, for example, learner **A**, learner **B**, etc. Also, the pseudo name for the researcher was Faith.

Marczyk et al. (2010) argues that privacy, which is concerned with access to participants' information, should be entwined with confidentiality. In doing this, during the consent process, the researcher indicated people who will have access to their information. For instance, I was the only one with access to their details. Although the supervisor was there to monitor the progress, personal information was never given to him. Thus, the information was anonymous to others excluding the researcher.

3.5.4 Respect, dignity, and standard of care.

According to Bell and Bryman (2007), the researcher should firstly prioritise respect which concerns beliefs, values, safety and well-being of participants. This manifested through the consent form by 1) allowing individuals to decide whether to participate in the study or not; and 2) ensuring that the consent form is in an understandable language, free from errors to avoid misconceptions (Lamont et al., 2016).

During interviews, participants were not asked sensitive questions so that they could freely engage. This was a way of honouring and admiring individuals due to their qualities; and this signifies dignity (Hodson, 2001). Moreover, none of the interviewees was blamed for giving irrelevant responses. When an interviewee was not understood, the researcher displayed a positive attitude and used a probing technique such that they do not feel disgraced. To ensure the standard of care, all participants were treated equally irrespective of their status, and this is a way of maintaining their dignity (Moffett & Moore, 2011). For instance, all participants with irrelevant spatial skills were supported and respected like those with relevant spatial skills (Lamont et al., 2016). This was a way of applying the Promotion of Equality and Prevention of Unfair Discrimination Act 4 of 2000, which emphasises that individuals must be treated equally without being discriminated (DSD, 2015). Since the three aspects are intertwined, they were not treated as separate concepts for a successful study.

3.5.5 Benefits and harm

The literature highlighted that participants need to gain in a study (Marczyk et al., 2010). In this instance, the research questions of the study enabled participants to notice their knowledge gaps in trigonometry, and to modify their knowledge and ways of thinking. All these aspects enhanced how learners compute the sides and angles of figures. However, to minimise injuries, I asked questions politely and avoided using hurtful words when interviewees encountered challenges in responding to questions.

3.6 Chapter Summary

This chapter examined the research approach, research paradigm, research design and methods of collecting and analysing data employed in this study. In addition, the chapter presented quality criteria and ethical considerations.

CHAPTER FOUR: PRESENTATION OF RESULTS

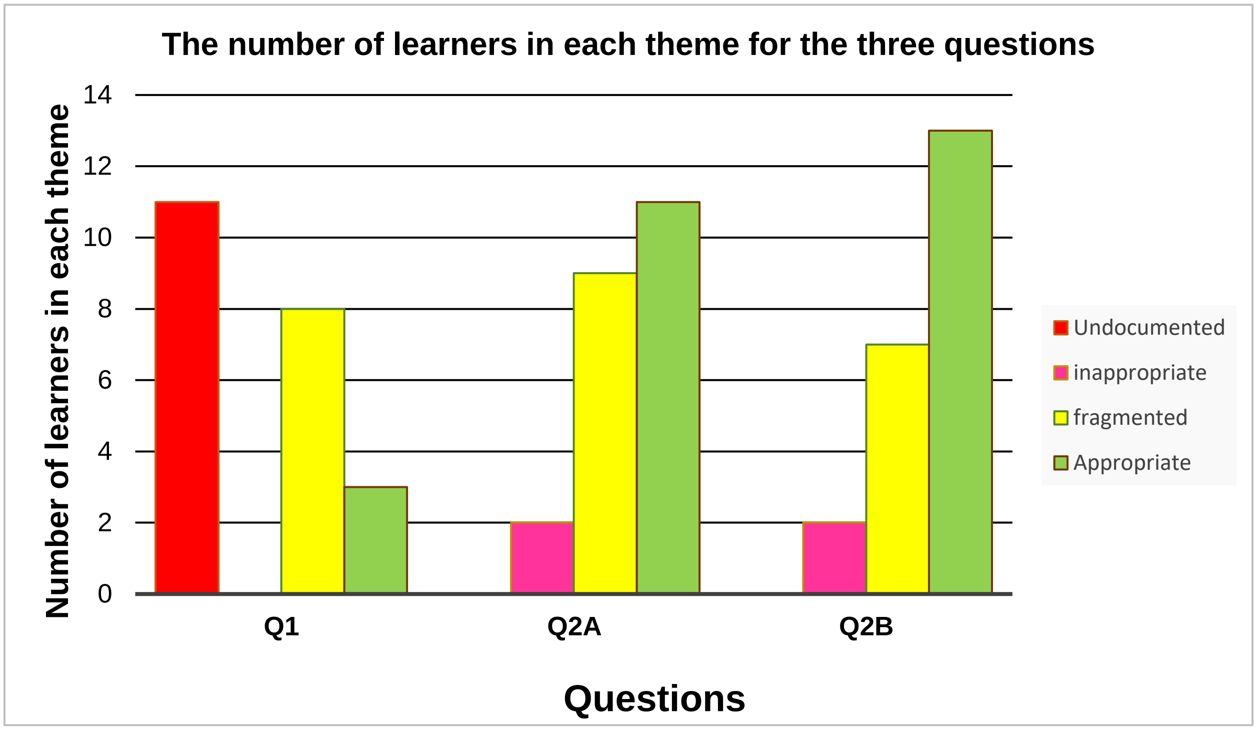
4.1 Introduction

The chapter presents the results of Grade 10 learners' spatial skills whilst computing sides and angles of spatial figures using trigonometric ratios. In this study, the three lessons were conducted to document more details on the computations of sides and angles as stated in Chapter 3. For instance, in the first lesson for checking learners' prior knowledge of properties of a right-angled triangle; 2) in the second lesson for introducing the concept of trigonometric ratio; and 3) in the third lesson, for checking whether the learning of trigonometry has occurred or not. Semi-structured interviews were conducted to understand why learners compute sides and angles in particular ways. The three mathematical tasks in each lesson as well the semi-structured interviews aided me to use the semantic theory in the next chapter to interpret how the learners utilised their spatial skills to access relevant concepts when computing sides and angles.

In addition, the results obtained through the mathematical tasks and semi-structured interviews were coded using the axial strategy and further analysed thematically. Concisely, four themes emerged from the codes, namely: 1) undocumented trigonometric conceptual structures, leaving blank spaces due to: i) difficulties to think in a short period of time, ii) not knowing what to write, and iii) understanding the question but does not know what to write; 2) inappropriate procedures for computing sides and angles through accessing irrelevant concepts; 3) having fragmented trigonometric concepts; and 4) appropriate procedures for computing sides and angles through accessing relevant trigonometric concepts. All results are presented within the four themes below.

4.2 Presentation on computations of sides using prior conceptual structures

Learners present their spatial skills differently when computing sides. Erroneously, some tend to use appropriate skills to access irrelevant concepts which lead to incorrect solutions whereas others access relevant ones and obtain desired solutions. Both the relevant and irrelevant skills are exhibited in the trigonometric tasks and semi-structured interview data. Below is a bar graph and table presenting codes from the first mathematical task and semi-structured interviews, followed by results within the four themes.



Graph 4. 1 A bar graph illustrating spatial skills illustrated within the four themes

The graph above shows the number of learners within the four themes for the three questions 1, 2a, and 2b (**Appendix D**). In detail, the first question required learners to differentiate between sides and angles. This was a question that required learners to think and express their ideas in the form of word representation. However, 11 learners left blank spaces. Despite this, most learners left blank spaces in question 1, transcripts enlighten that learners leave blank spaces for various reasons. For instance, indolent, ignorant, articulation, etc. In this regard, the trend as shown in the graph above was lightly high in the second question compared to others. On the other hand, question number 2 required learners to compute the hypotenuse and adjacent sides and there are no blank spaces in the two questions. The mathematical task and the transcripts designate that learners are interested in solving questions that are based on symbolic representation than reasoning. For instance, question number one required learners to think and provide the response in the form of words whereas the last two question required learners to used Pythagoras theorem to compute sides. Although blank spaces were not left in question 2, 2 learners showed inappropriate procedures for both 2a and 2b as they used inappropriate equations to compute the required sides. This indicates that the learners could not recall suitable equations to compute the required sides. Deducing from the bar graph, 9 learners showed fragmented conceptual structures in question 2a whereas there are 7 learners in question 2b. This implies that many learners struggled to access relevant properties of the Pythagoras theorem to compute the required sides. Although the number of fragmented conceptual structures was high initially, it decreased after conducting semi-structured interviews since some fragmented conceptual structures are due to mistakes. Lastly, there were 3 learners with spatial skills to illustrate appropriate procedures on question 1, 11 learners on question 2a and 13 learners on question 2b.

Table 4. 1: Codes from learners' responses in question number 1, 2a, and 2b

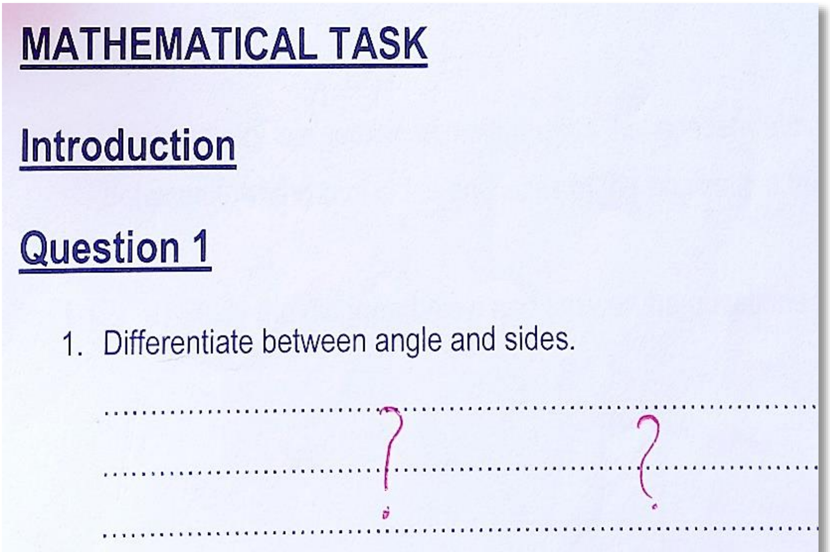
Questions	***Codes from the mathematical task and semi-structured interviews***			
1. Differentiating between angles and sides	BS1: Blank spaces for various reasons, incomprehensible, ignorant (lacking relevant concepts), neglection (not paying attention), clueless (not having an idea of what to do), indolent (not willing to think).	IS1: Incorrectly linked angles and sides due to confusion, and memorisation, having unsuitable words.	FS1: Inability to locate sides and angles due to fragmentation, doubtful, unverified and inarticulation (Inability to express ideas in the language of teaching), ignorant	AS1: Appropriately linking sides and angles through association and retrievals.
2a) Computing the length of AB	BS2a: Blank spaces for various reasons, failure to compute the hypotenuse side, recalling Pythagoras equation.	IS2a: Incorrectly linking the variables on the Pythagoras equation (inaccurate procedures), doubtfulness, ignorance, inarticulation, unverified response, and confusion.	FS2a: Inability to apply **BODMAS** rules and the Pythagoras equation to compute the length of AB. Undecided or doubtful thoughts	AS2a: Appropriately ` recalled Pythagoras equation and handled the **BODMAS** rule efficiently.
2b) Computing the length of LM.	BS2b: Blank spaces for various reasons, failure to recall Pythagoras equation, compute the adjacent side.	IS2b: Inappropriately recalled the link between the variables on the Pythagoras equation to compute AB and doubtfulness	FS2b: Inability to handle **BODMAS** rules, inability to differentiate side names, and to compute the length of AB.	AS2b: Appropriately recalled side names on the Pythagoras equation, handling the **BODMAS** rule efficiently.

4.2.1 Undocumented trigonometric conceptual structures

The trigonometric mathematical task 1 for the first lesson was designed in such a way that it tests learners' prior knowledge of the side names and angles on the right-angled triangle. This worked as a strategy for identifying learners' spatial skills in explaining the concept of an angle and a side as well as computing the hypotenuse, AB and opposite, LM using the Pythagoras theorem. In responding to the first question, some learners left blank spaces. However, these spaces do not reveal what learners are thinking in terms of the computations of sides. To understand what they are thinking, I have conducted semi-structured interviews. Learners' transcripts from interviews on why they left blank spaces in the three questions are shown below.

4.2.1.1 Differentiating between an angle and a side

Figure 4.1.a) Learner A written response



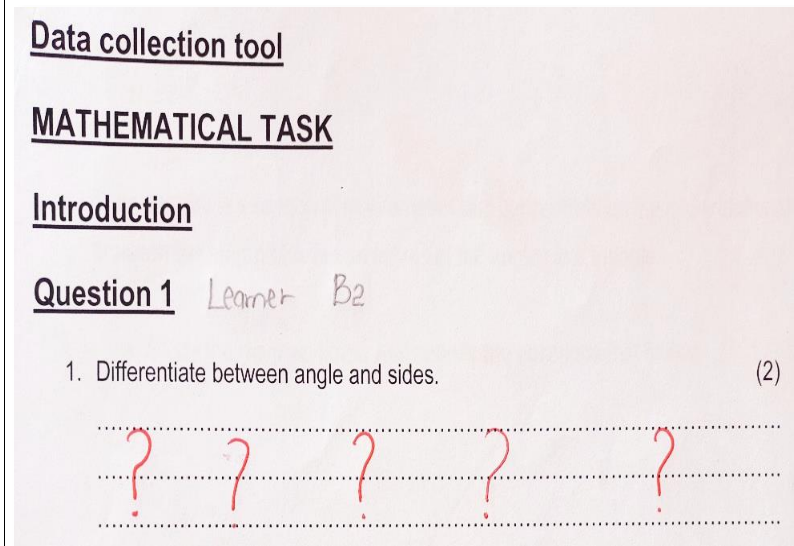
MATHEMATICAL TASK

Introduction

Question 1

1. Differentiate between angle and sides.

Figure 4.1b) Learner B2 written response



Data collection tool

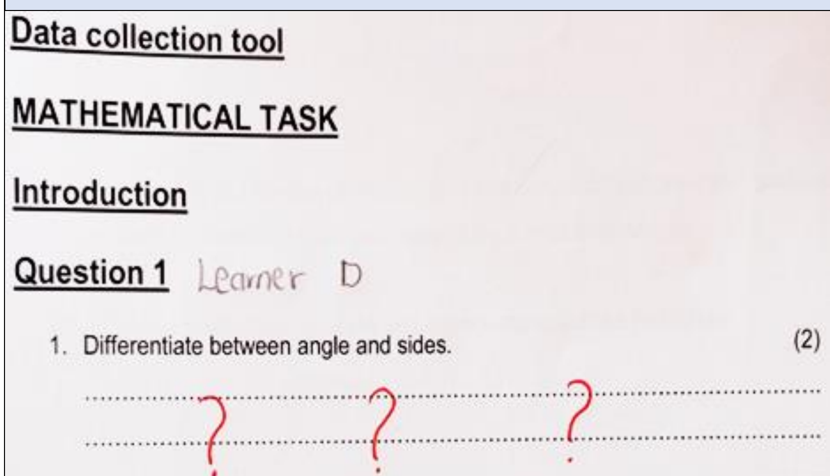
MATHEMATICAL TASK

Introduction

Question 1

1. Differentiate between angle and sides. (2)

Figure 4.1c) Learner D written response



Data collection tool

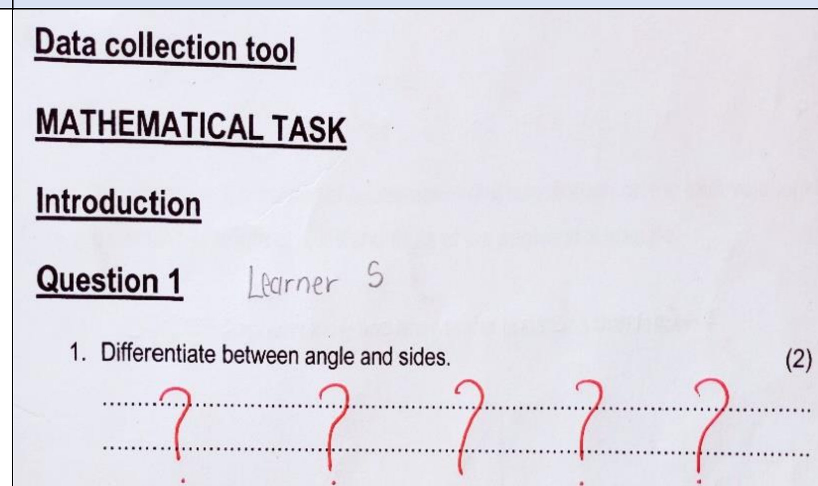
MATHEMATICAL TASK

Introduction

Question 1

1. Differentiate between angle and sides. (2)

Figure 4.1d) Learner S written response



Data collection tool

MATHEMATICAL TASK

Introduction

Question 1

1. Differentiate between angle and sides. (2)

Figure 4. 1 Learner A, B2, D, and S question number 1 written response

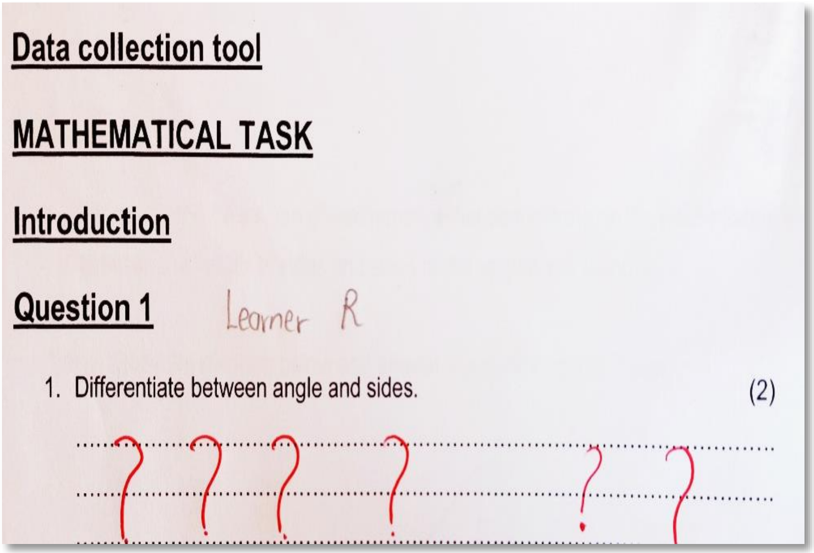
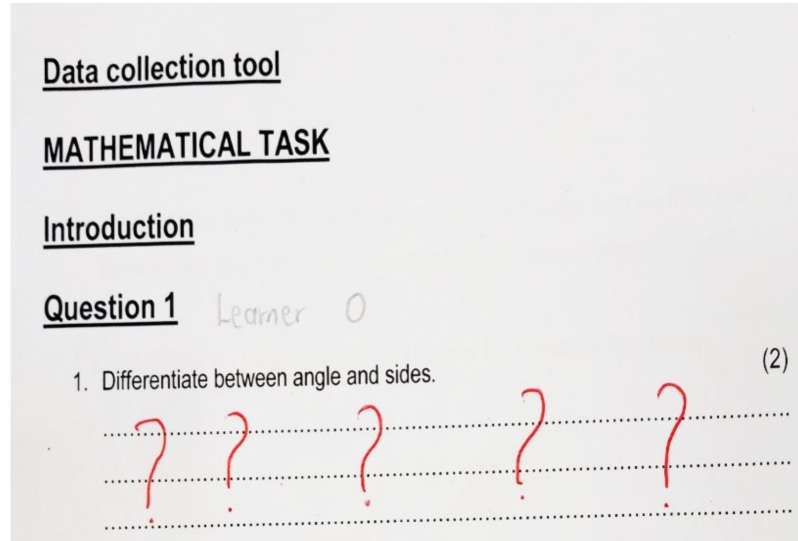
<table>
<tr><th>4.2a) Learner R written response</th><th>4.2b) Learner R written response</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

Figure 4. 2 Learner R and O question number 1 written responses

Faith	: Why did you leave a blank space in question number 1?
Learner A	: I did not understand the question.
Faith	: Can you give me an example of an angle or a side?
Learner A	: Angle is where two points meet to make up 90 degrees.
Faith	: Actually, you can give examples of angles, but you cannot give precise explanations of an angle or sides.
Learner A	: yes madam.

The transcript above elaborates on explanations that transpired during the interview. The question was 'differentiate between an angle and a side'. This question required learners to, 1) recall and notice that an angle is formed when two lines intersect; and 2) a side is a line formed from a point to the left or right. Additionally, learners were required to manipulate and explain what an angle or side is using the keywords mentioned above. However, learner A explained an angle as a location where two points meet to specifically form 90, and this was coded **BS1** category incomprehensible. Firstly, the learner indicated that she did not understand the question. At this stage, the learner could not recall and notice the relationships of sides and angles. However, after requesting the learner to provide an example of either an angle or a side, the learner provided an incomprehensible response, which attests to the provision of a general definition of an angle as examples of angles were uttered. In this instance, the learner could not provide a specific definition of an angle, which is characterised by two lines intersecting each other. Furthermore, the learner agreed with Faith that providing examples of angles is more achievable than giving precise explanations. This might be a result of lacking all the basic skills that could have allowed her to manipulate and provide reasons that the given figure is a right-angled triangle.

Faith : You did not write anything on number 1. Why?

Learner B2 : I could not differentiate between an angle and a side.

Faith : Can you give an example of a side or an angle?

Learner B2 : All sides in shape.

Faith : Example of an angle.

Learner B2 : Angle can be a 90° corner in a right-angled triangle.

Faith : Another example, please.

Learner B2 : 60 in an equilateral triangle. I can give you any examples of angles, but I cannot explain what an angle is.

Faith : I understand.

The transcript from the interview explains that the learner indicated that she could not differentiate between sides and angles. The response confirmed that the learner lacks skills of visualising and noticing the link between the sides in a right-angled triangle such as how sides and angles are formed and interpreting the side names against the angles. This was coded as **BS1** category ignorance. Evidence for ignorance was noticed after the learner was requested to provide an example of an angle or a side. In responding to the question, the learner provided a generic response instead of specifying that side names could be the hypotenuse, opposite or adjacent. Although the learner could not specify examples of sides, examples of angles were clearly outlined. Furthermore, the learner confirmed that she can provide numerous examples of angles but could not explain how they are formed. The insufficient constituent mentioned above contributed to a failure to explain how the sides and angles differ.

Faith : The first question required you to differentiate between sides and angles, but you did not write anything. Why?

Learner D : I do not know the differences between an angle and a side.

Faith : If I can give you a right-angled triangle will you be able to illustrate the side names?

Learner D : Yes

Faith : Okay. Let us use the figure in question number 2, to confirm what you are saying. What is the name of side AB?

Learner D : Hypotenuse.

Faith : what about AC and BC?

Learner D : AC is an adjacent whereas BC is an opposite.

Faith : Noted. Between AC and BC, which one represent r?

Learner D : AB

Faith : Meaning you understand the differences between hypotenuse, adjacent and opposite?

Learner D : yes

Faith : You also understand the differences between and?

Learner D : Yes.

Faith : Noted with thanks.

The transcript above explains that learner D also found it challenging to differentiate between sides and angles. This was noticed as the learner indicated that she was interested in providing examples of side names, namely, the hypotenuse, adjacent and opposite instead of differentiating sides and angles. This could be a result of failure to notice that a side is a line formed from a point to the left or right. Also, the learner could not notice that an angle is formed when two lines intersect at a point. This is also coded **BS1** category ignorance, which indicates that the learner seemed to lack the essential keywords mentioned above to differentiate between sides and angles. Lack of all these vital words, lines intersecting forms angle led to failure to provide relevant differences between the sides and angles. Moreover, the learner indicated that he could represent side names using variables, and this was not relevant to the question.

Faith : You did not write anything in question number 1, why?

Learner R : I did not know what to write.

Faith : Did you understand the question?

Learner R : Yes, I did, but I did not know what to write.

Learner R clearly indicated that he did not know what to write. This was coded as being clueless. Being clueless entails that the learner mentally apprehended the question but could not figure out concepts such as rays or lines and points that assist in organising the response. This led to a failure to explain what the question required.

Faith : You did not write anything in question number 1, why?

Learner O : I did not know the difference between an angle and a side.

Faith : Can you give me an explanation of an angle or a side?

Learner O : An angle?

Faith : Yes

Learner O : Eish, I do not know how to answer this one.

Faith : Just give me an example of an angle, you can even use your own language.

Learner O : This one is tricky.

Faith : Example of an angle or a side?

Learner O : Okay, when two sides combine, they form an angle.

Faith : Why didn't you write what you are saying now?

Learner O : Ahh, I never thought of this.

Faith : The idea just came now?

Learner O : Yes

From the transcript above, learner O initially indicated that he could not differentiate between sides and angles. However, the learner agreed to provide the explanations. Later, the learner mentioned that he cannot differentiate between sides and angles. This is categorised as BS1 category indolent. The category indicates that the learner does have skills of observing properties of angles and sides but is not interested in arranging and presenting the required response. However, after requesting the learner to give an example of an angle or a side repeatedly, the learner clearly explained what an angle is. This implies that the learner visualised the position of angles. Consequently, it became easier to explain their position using the required keywords as shown on the learner's explanation. The explanation comprises an essential key 'combines' that evidently illustrates how an angle is formed. Although the essential word was utilised on the explanation, the learner did not clarify what a side is, and this might be a result of not willing to mentally organise other keywords quickly. The point of not willing to mentally organise words was presented after the learner was asked why the verbalised explanation was not written and indicated that the idea just came while conversating. Hence asking one question frequently worked as a key for checking learners' skills in responding to questions.

4.2.2 Inappropriate procedures for computing sides

In this section, learners were to compute the length of the hypotenuse labelled AB and adjacent, LM using the Pythagoras equation. The equation necessities the skill of interpreting sides' names in reference to θ. However, some learners wrote incorrect equations and could not differentiate between side names. Consequently, the inability to differentiate side names led to the failure to obtain the desired solutions. The inappropriate procedures on the two questions are shown below.

4.2.2.1 *Computing two sides*

Figure 4.3.a) Learner D2 written response

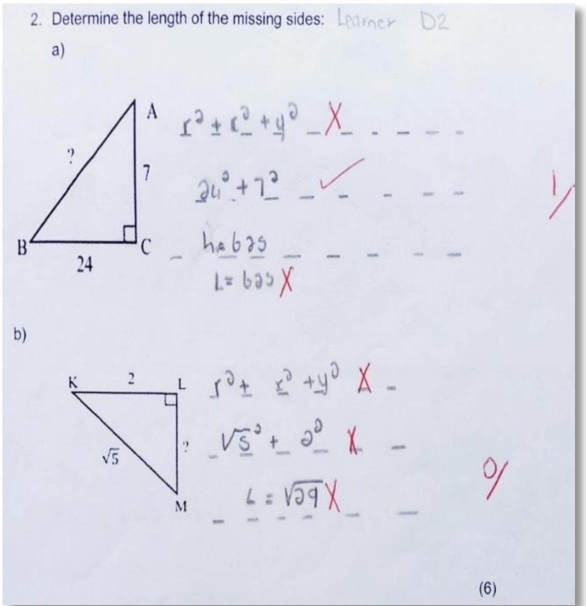


Figure 4.3.b) Learner K response

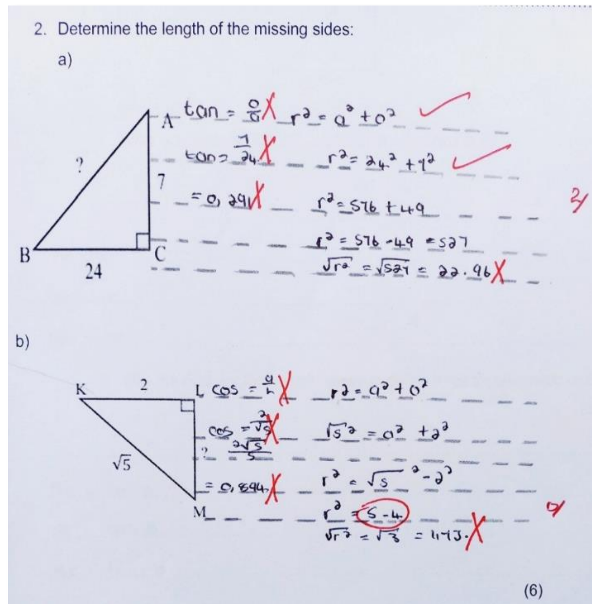


Figure 4. 3 Learner D2 and K written responses

During the process of computing the length of AB, learner D2 wrote the equation incorrectly but substituted 24 as the value of x and 7 as the value of y. Although the values were substituted correctly, the learner wrote $h = 625$ which is another variable equated to the values of $24^2 + 7^2$. This shows that the learner knew what the question required but had limited Pythagoras concepts such as hypotenuse, opposite and adjacent. In responding to the second question, the learner wrote $r^2 + x^2 + y^2$ again. Since the three variables were not separated by an equal sign, it was difficult to know whether 2^2 represents the value of r^2, x^2 *or* y^2.

Faith : Why did you write $r^2 + x^2 + y^2$?

Learner D2 : *A ni nga swi twisisangi* (I did not understand).

Faith : Remember you learned about side names using the Pythagoras theorem in grade 9.

Learner D2 : Yes madam, but I forgot. I was not sure about the equation.

Faith : I see.

The transcript above explains that the learner knows Pythagoras' theorem but the sequence of writing the equation was forgotten. The learner was requested to explain why the equation was written incorrectly, and she indicated that she did not understand. This was coded as **IS2a** the category of the inappropriate linkage of variables, which is a result of deficient knowledge of side names based on the reference angles on the right-angle triangle. Consequently, the deficient knowledge mentioned above could not allow the learner to write the equation correctly. Hence, a conclusive statement was provided by the learner indicating that she was not sure about the equation.

Similarly, learner K is one of the participants who illustrated inappropriate procedures during computations of sides. In responding to questions 2a and 2b, the learner wrote two approaches. This was coded **IS2a** and **IS2b** category of doubt. Firstly, the learner wrote trigonometric ratios without indicating the reference angles, $tan = \frac{7}{24}$, $cos \frac{2}{\sqrt{5}}$ etc. Secondly, the learner substituted values on the Pythagoras equation. This indicates that the learner had doubts on whether the first or second approach is correct or not. Thereafter, the learner was interviewed, and the transcript is shown below.

Faith	: In question number 2a and 2b, you were required to compute unknown sides. Why did you write two responses?
Learner K	: I was not sure. But in grade 9 we learned about the side names of a right-angled and the Pythagoras equation. Remember you talked about trigonometric ratios, so I was confused whether I should use ratios or Pythagoras equation.
Faith	: Oh, I understand.

The transcript above explains that learner K wrote two approaches while computing the hypotenuse and adjacent sides as she was not sure. This was coded as **IS2a** and **IS2b** in the category of doubt, which means that the learner does have knowledge related to computations of the side but does not have the basic skills of choosing the concepts relevant to the given questions. Furthermore, the learner stated that the confusion started after Faith talked about ratios while explaining the purpose of the study. As a result, the learner did not know whether she should use ratios or the Pythagoras equation.

4.2.3 Fragmented trigonometric concept from prior knowledge

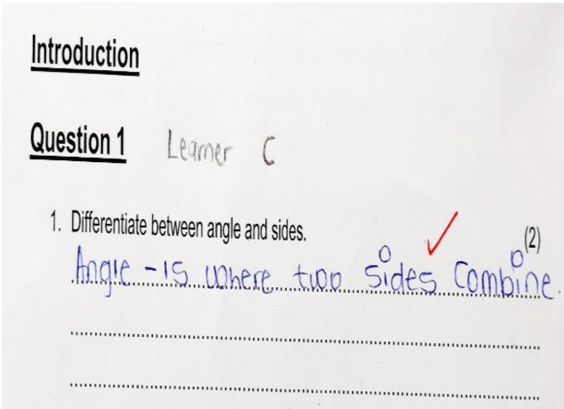
Computing sides requires proper spatial skills to recognise words and symbols used within the given statements and spatial figures. These key elements allow learners to choose relevant equations and substitute values accordingly. In contrast, a lack of skills to know when to utilise the key elements led to the inability to use appropriate words to explain the differences between angles and sides. In addition, insufficient skills also contribute to confusion displayed when substituting numeric values representing variables. This is referred to as having fragmented trigonometric concepts. Specifically, learners' fragmented trigonometric concepts from question number 1, 2a and 2b are presented below.

4.2.3.1 Differentiating sides and angles

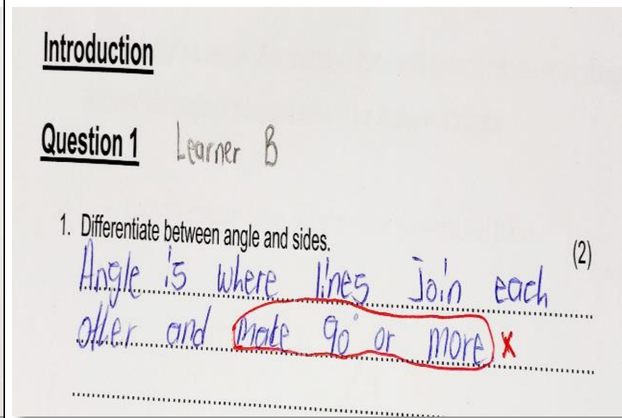
In figure 4.4.a, learner C managed to explain the concept of angle only. The properties of angles were carefully organised logically through visualising and noticing their location in the right-angled triangle. According to the learner, an angle is formed when two points combine. Although the learner tried to explain what an angle is, a differing statement for sides was not given. The transcript from an interview is shown below:

Faith : I see you have tried to clarify the concept of angle.
Learner C : Yes
Faith : But you did not write anything on the side.
Learner C : Yes madam. I did not know what to write.

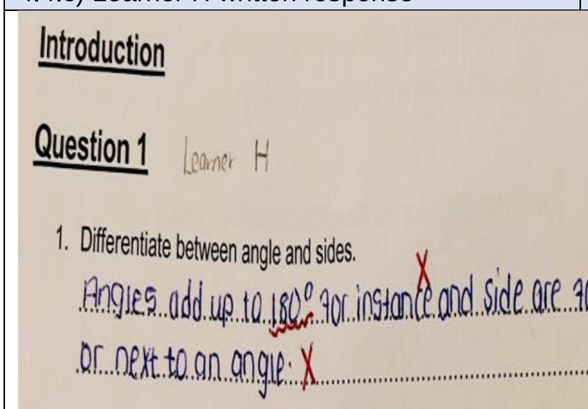
4.4.a) Learner C written response



4.4.b) Learner B written response



4.4.c) Learner H written response



4.4.d) Learner E written response

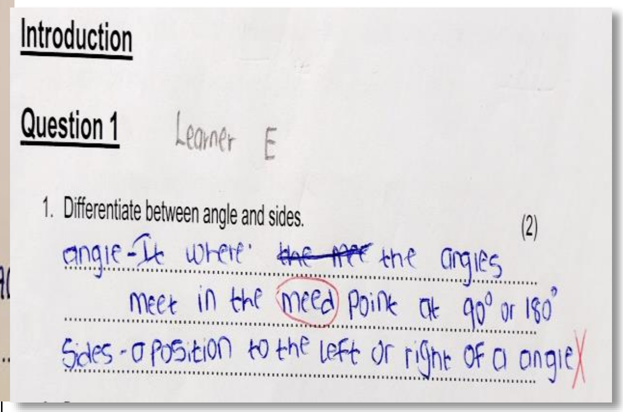


Figure 4. 4 Learner C, B, H and E question no. 1 written responses

The transcript above indicates that learner C could not explain what a side is. This is referred to as ignorance, which is part of **FS1**. The ignorance category states that the learner could not access the existing concept such as lines from a point to the left or right to explain what a side is. As a result, the insufficient skills to aspect a right-angled triangle prevented the learner from providing an explanation of the sides.

Contrasting with learner C's responses, learner B explained an angle as a point where lines join to form an angle which is 90 or more. The words used in explaining the angle were not acceptable as they specified an angle. This stipulates that the learner recalled incorrect concepts which did not assist in noticing the relationship between angles and sides in a right-angled triangle. This was coded FS1 category ignorance.

Subsequently, semi-structured interviews were conducted to document details on why those words were used. The transcript from the interview is shown below.

Faith	: Is it a must for angles to make 90° or more?
Learner B	: No
Faith	: Do you think it is good to say that angles should make 90° or more?
Learner B	: I am not sure.
Faith	: Oh ok.

From the transcript above, learner B clearly indicated that she is not sure whether she should explain the concept of an angle while specifying 90 degrees or not. This was coded as **FS1** category doubtful. The doubtful category indicates that the learner is not sure how to respond to the given question. The state of being not sure emanates from insufficient basic skills of how a right-angled triangle was formed.

In responding to question number 1, learner H also explained what an angle is. The learner explained an angle using inappropriate words, "an angle adds up to 180 °". This was coded fragmentation which forms part of **FS1**. The learner could not differentiate between an angle and a side; instead, she wrote an example of an angle (180°). However, the responses outlined that the learner seems to know what an angle is but struggles to give an exact description or explanation. The transcript from the interview is illustrated below.

Faith	: The first question required you to differentiate between an angle and a side. According to your understanding, what is the side?
Learner H	: I do not know the exact definition, but I can be able to identify different side names when given a right-angled triangle.

Faith : Alright, let's use ΔACB. What is the name of side AB, AC, and BC?

Learner H : In reference to $\hat{B}$, AC is the opposite side.

The longest side AB is called hypotenuse and it is opposite the 90° angle.

BC is based on 90° and the longest side. It is called the hypotenuse.

Faith : Mmhh, I have noticed that you enjoy identifying the side names in reference to different angles.

Learner H : Yes madam. The difficult part is to give brief explanations of angles and sides.

Faith : Okay.

The transcript above explains that the learner cannot provide a precise explanation but examples of side names in reference to different angles. This was coded as **FS1** the category of ignorance, insufficient knowledge of side names. The insufficient knowledge could not allow the learner to explain what a side or angle is.

Figure **4.4.d** illustrates that learner E could not recall the correct words to differentiate angles and sides. As such, the learner ended up specifying angles, 90° or 180°. This is considered **FS1** category ignorance. Also, the learner misspelt the word midpoint. Although some words were misspelt, the explanation showed that the learner had little knowledge of an angle and sides. The learner was interviewed to document more details on the written responses. The transcript from the interviews is illustrated below:

Faith : What do you mean when you say an angle is where points meet?

Learner E : ………………. (silence)

Faith : you can use your own language to explain what you wanted to say.

Learner E : Angle yiva kona loko ku hlangana ti layeni ti mbhiri (angle is where two sides meet)

Faith : Oh, I understand, meaning you cannot express your ideas in English?

Learner E : Yes mam, like I do understand what the question requires but am struggling to express what I am thinking in English.

Faith : Soo can you explain the characteristics of a side?

Learner E : Yes, side ima hlelo ya ti shape (vertical or horizontal lines).

The transcript above explains that learner E tried to explain what an angle is. To check the learner's computation skills, a further elaboration of the given explanation was requested. However, the learner kept quiet for a while, and a directive to use her mother tongue was given. Most importantly, the learner clearly differentiated between angles and sides using her mother tongue. This was coded **FS1** category inarticulation. An inarticulation indicates that the learner does know the differences between angles and sides but could not express the ideas in the language of teaching and learning.

4.2.3.2 Computing two sides

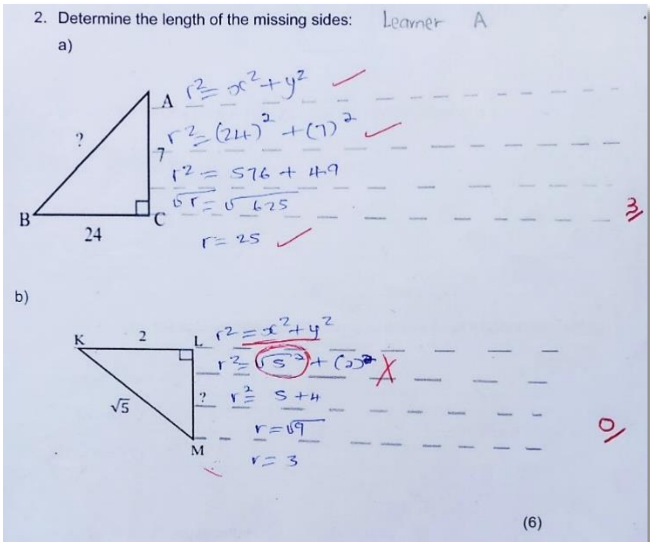
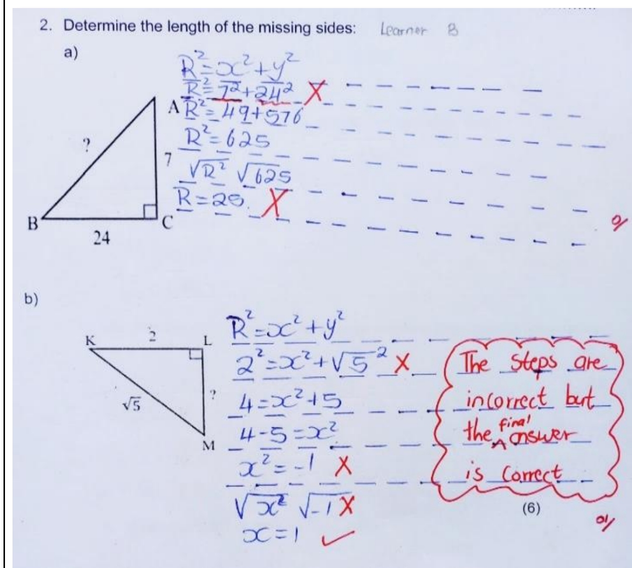
4.5.a) Learner A written response	4.5.b) Learner B written response
	

Figure 4. 5 Learner A and B question no. 2a and 2b written response.

Figure 4.5.a illustrates how learner A responded to the given question. The learner was given question 2b to check whether the concept of side names is fragmented or not. In responding to the question, the learner noticed that two sides are given. Therefore, the Pythagoras equation can be used to compute the unknown side which is the opposite. The learner wrote the equation correctly, $r^2 = x^2 + y^2$. In the given figure, the value of r is $\sqrt{5}$ and x is 2. According to the learner, $\sqrt{5}$ represents x while the value of 2 represents the y variable and the unknown side to be computed is r. This is referred to as fragmentation. The incorrect numeric values mentioned above were substituted, and the learner obtained an incorrect answer which is 3. The transcript from the interview is shown below:

Faith	: Which one represents x between KL and LM?
Learner A	: KL
Faith	: But you wrote $\sqrt{5}$ as the value of x, is it correct?
Learner A	: No
Faith	: Please clarify me, what is the name of side KM?
Learner A	: Hypotenuse.
Faith	: LM?
Learner A	: opposite
Faith	: KL
Learner A	: Adjacent
Faith	: Between KL and LM which one represent the value of y?
Learner A	: LM

The transcript from interviews shows that the learner has appropriate skills of naming side. This was noticed as the learner was capable of interpreting side names using symbols. However, the incorrect manipulations shown in the answer sheet occurred due to failure to verify the written responses. This was coded **FS2b** category unverified responses.

In addition, during the computations of the opposite side, learner B computed the adjacent side using Pythagoras theorem. The learner managed to substitute the value of the hypotenuse and the adjacent, yet indicated the final answer incorrectly, $a = 1$. In question 2b, learner B also wrote that $r^2 = x^2 + y^2$. The value of x was 2 and was labelled as KL. The value of r was $\sqrt{5}$, which was labelled as KM and the unknown value to be computed was y, which was labelled as LM. In contrast, according to the learner, the value of r is 2, whereas the value of y is $\sqrt{5}$. As a result, the learner substituted the incorrect values as $2^2 = x^2 + (\sqrt{5})^2$. The incorrect values resulted in obtaining $4 = x^2 + 5$. The learner noticed that there are like terms. Hence, it was proper to group them as $4 - 5 = x^2$. Although the learner grouped the like terms, the incorrect manipulations resulted in making incorrect simplifications. The transcript from the interviews is shown below:

Faith	: Let's check figure 2b clearly. What is the value of r?
Learner B	: 2
Faith	: What about the value of x?
Learner B	: The value of x is not given.
Faith	: Does this mean the unknown side is x?
Learner B	: Yes.

In responding to question 2a, learner B also wrote the correct equation but could not substitute values correctly. According to the learner, the value of 7 represents the radius while 24 represents the y-axis. The schemas indicate that the learner seemed to lack a clear imagination of the radius, x-axis and y-axis. Although the learner substituted numeric values on the wrong variables, the correct answer was obtained. This was coded **FS2a** category confusion.

4.2.4 Appropriate procedures for differentiating and computing sides

In this subheading, learners were requested to differentiate, compute the hypotenuse and adjacent sides. Positively, some learners used applicable skills to access the required concepts when differentiating between sides and angles. In addition, proper skills were applied to operate fluently while using the Pythagoras equation, and all operations were handled correctly. Details on the three questions are presented below:

4.2.4.1 Differentiating sides and angles

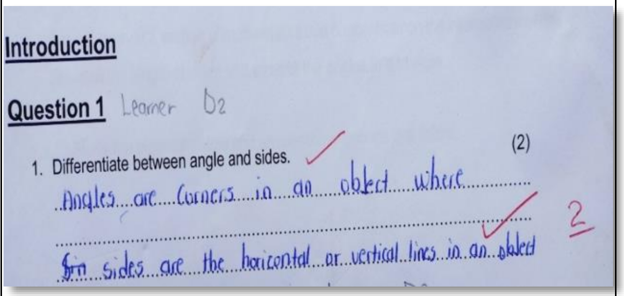
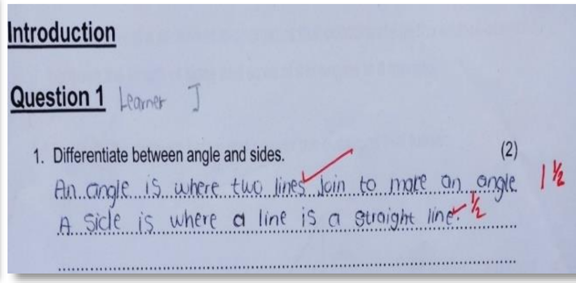
4.6.a) Learner D2 written response	4.6.b) Learner J written response
	

Figure 4. 6 Learner D2, J question 1 written response

Figure 4.6 illustrates that learners D2, and J managed to recall and notice the basic properties of angles and sides such as vertex, vertical and horizontal lines. This was coded **AS1** category appropriately linking sides and angles through association and retrievals. The category emphasises that the learner can select concepts relevant to the question from their existing knowledge. Essentially, the selected concepts enabled them to differentiate between sides and angles. The transcript from the interview is shown below to support the written responses.

Faith	: Can you give me an example of an angle?
Learner D2	: I do not know.
Faith	: Example of a side?
Learner D2	: Are lines.

Although learner D2 operated within the category appropriately linking sides and angles through association and retrievals on the answer book, the transcript above explains that the learner could not provide examples of angles whereas an explanation was given. This was coded AS1 category inarticulation. Under this category, the learner has the basic skills to write ideas but cannot express them orally. For instance, Faith requested the learner to give examples of sides and the learner said are lines and this is not an example.

Faith	: Can you differentiate between an angle and a side?
Learner J	: An angle is where two-line joins while a side is just a straight line.
Faith	: Can you give an example of an angle?
Learner J	: 90°.

The transcript above explains that learner J provided differences between angles. After differentiating, Faith requested the learner to give examples of angles and correct answers were given. This is categorised appropriately linking sides and angles through association and retrievals.

4.2.4.2 Determining the length of the missing sides for question a and b

4.7. Learner B2 written response

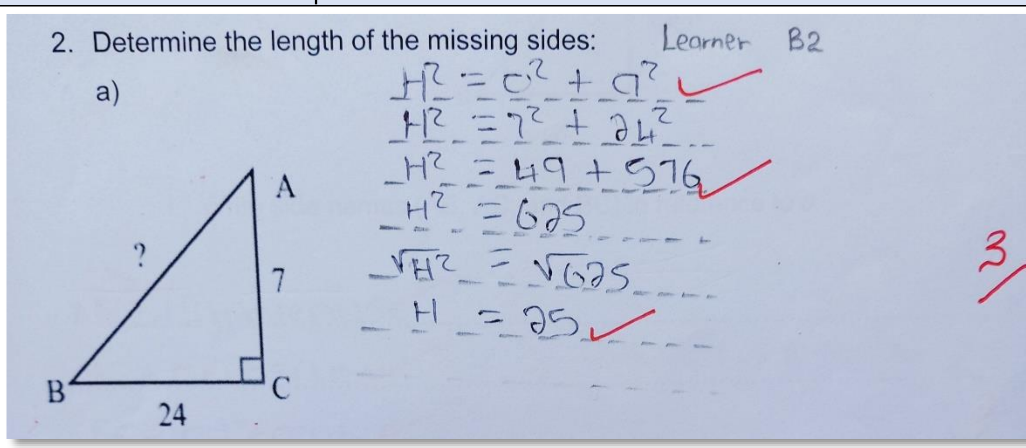


Figure 4. 7 Learner B2 question 2a written response

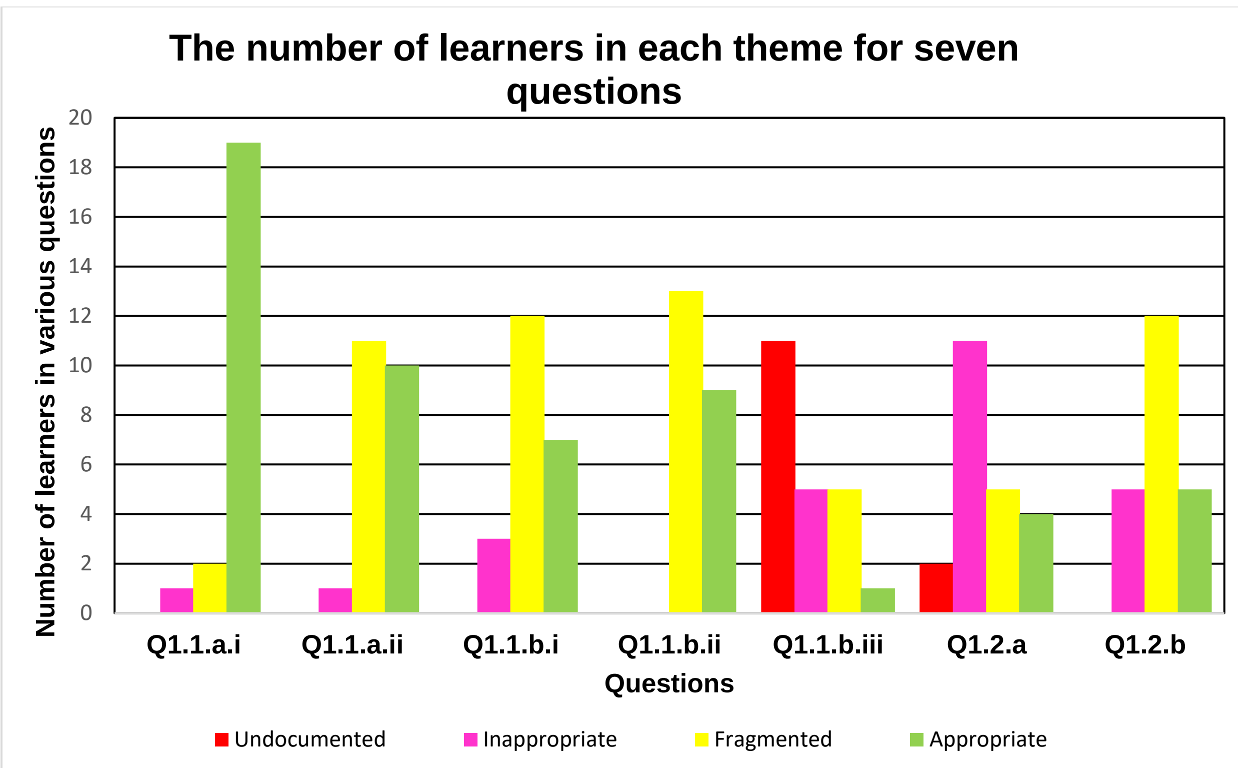
Figure **4.7** illustrates that learner B2 recalled relevant concepts from GET, the theorem of Pythagoras and obtained the correct value. The learner noticed that the value of the opposite is 7 whereas the adjacent side is 24. This implies that the unknown side to be computed is the hypotenuse. Most importantly, while solving the problem, the learner managed to handle all operations and procedures of utilising the theorem of Pythagoras' equation. This was coded **AS2a** category handling the BODMAS rule efficiently, which is a skill of knowing how to use operations sequentially so during computations.

Faith	: What is the name of side AB?
Learner B2	: Hypotenuse.
Faith	: AC?
Learner B2	: Opposite
Faith	: Why did you choose the formula, $h^2 = a^2 + o^2$?
Learner B2	: It helped to notice that the required side is hypotenuse. *Unlike the formula for* $r^2 = x^2 + y^2$
Faith	: Meaning cannot use the formula for $r^2 = x^2 + y^2$
Learner B2	: Yes
Faith	: What if you were instructed to use $r^2 = x^2 + y^2$ to compute the missing side?
Learner B2	: Madam I did not learn this equation in lower grades but now I see that it can also be used to compute the unknown sides.
Faith	: Why?
Learner B2	: It seems like I can use any variable to represent the equation. Soo using different variables will not make a difference.
Faith	: I understand.

The transcript above explains that the learner can name sides when using symbols. Furthermore, the learner was requested to elaborate why $h^2 = a^2 + o^2$ was used. Positively, the learner indicated that she wanted to compute the hypotenuse side. Despite that correct explanation was given, initially the learner could not relate the $h^2 = a^2 + o^2$ with $r^2 = x^2 + y^2$. This was categorised as an inability to relate variables. Later, the learner realised that both equations can be used to compute the hypotenuse side. The probing technique enabled the learner to notice that any variable can be used to represent side names during the computations of sides.

4.3 Results from learners' responses to the introduction of trigonometry

In this section, learners were required to access relevant properties of the right-angled triangle to discover how and when to apply three basic trigonometric ratios during computations. The use of spatial skills is a prerequisite to acquiring the conceptual structures of the three ratios. During the process of developing the concepts, learners displayed various spatial skills with the four themes. The bar graph is presented below to indicate how many learners operated within the four themes, followed by a table containing codes of the mathematical task and semi-structured interviews. Lastly, learners' results are presented below.



Graph 4. 2 A bar graph summarising how learners exhibited skills in each theme

The bar graph above explains that 11 learners left blank spaces in questions 1.1b.iii. The reason for more blank spaces in Q1.1.b.iii is that the question dictated learners to mentally view the link between the size of an angle and the length of a side, meaning that most learners struggled to apply the three spatial skills to provide a precise conclusion when given questions without spatial figures. In question 1.2.a, learners were expected to try to apply a ratio to check whether the introductory questions assisted or not. However, 2 learners left blank spaces. Like the data on the first task, learners showed that providing answers to questions without spatial figures is an issue. Thus, there are many learners in question 1.1.b.iii. In terms of inappropriate procedures, less than six learners used unsuitable equations. However, question 1.2.a was mostly affected since there are 11 learners. This is because their spatial skills could not assist them to choose the correct ratio. During interviews, the number of learners decreased as they had an opportunity to check their written responses.

In terms of fragmented conceptual structures, less than six learners showed a lack of trigonometric concepts for computing sides and angles in questions 1.1.a.i, 1.1.b.iii,1.2. a. Although there are few of the questions mentioned above, a lot was shown in questions that required spatial skills to respond to sequential questions. Regrettably, learners could not use relevant spatial skills to access relevant trigonometric concepts while developing the concepts of ratios. This affected their ability to handle side names and the BODMAS rule after choosing the correct ratio. Lastly, there is a great number of learners on the first three questions as it did not require compact spatial skills. Furthermore, the bar graph shows that there are fewer than 10 learners who responded to questions that necessities compact spatial skills. Hence, it can be concluded that the number of learners depends on the standard of the question.

Table 2: Codes from learners' responses to the main lesson

Questions	***Blank spaces***	***Inappropriate procedures***	***Fragmentation***	***Appropriate procedures***
1.1.a) I Write side names (AB, AC, and BC) in reference to θ		**IS1.1. A:** Recalled incorrect trigonometric concepts to name side names	**FS.1.1. A. I**: Recalled sides names (hypotenuse, adjacent and opposite) but incorrectly noticed, manipulated, judged the position with reference to the angle.	**1.1.A.I.A**: Recalled side names and correctly noticed, manipulated and judged the position with reference to the angle.
1.1.a) ii Use the right-angled triangle above to define three trigonometric ratios.			**FS.1.1. A. II**: Recalled side names and defined ratios but without using variables, incomplete ratio, wrong representation.	**AS:1.1. A. II**: Recalled side names and correctly used variables while defining ratios, complete ratio, correct representation
1.1.b) i Determine AB		**IS1.1. B. I**: Recalled an incorrect equation to compute the hypotenuse side.	**1.1.B.I. FS**: Recalled the correct equation but made incorrect substitutions, mishandled BODMAS rule, having doubtful thoughts, wrong representation.	**AS.1.1. B. I:** Recalled the relevant equation, correct substitutions, handled BODMAS rule efficiently, correct representation
1.1.b) ii Use the three values to define trigonometric ratios.		**IS.1.1. B. II**: Wrote side names, hypotenuse, adjacent or opposite instead of numerical values	**FS.1.1. B**: Recalled appropriate equation for defining ratios but could not use numeric values (incomplete representation), representing angle as decimals, excluding reference angle while defining ratios	**AS.1.1. B. II**: Recalled correct equation for defining ratios and used values to represent side names in reference the angle.
1.1.b) iii Draw a conclusion based on the length of sides and sizes of an angle.	**BS.1.1B.III:** Blank spaces for various reasons, indolent, clueless (having no ideas of what to do), inarticulation, ignorance (not having skills or not having enough knowledge), incomprehensible	**IS.1.1.B.III**: Drawing a right-angled instead of concluding,	**FS.1.1.B.III**: Recalled correct and incorrect words while drawing conclusions, incomplete responses, and incomprehensible	**AS.1.1.B.III:** Recalled relevant words while drawing conclusions, complete responses, Comprehensible,
1.2.a) Determine the length of AC for the diagrams below: ***If AB= 6 cm and = 49°***	**BS.1.2. A:** Blank spaces for various reasons: incomprehensible, confusion (failure to think clearly).	**IS.1.2. A**: Recalled Unsuitable equation.	**FS.1.2. A:** Recalled and noticed the correct equation but incorrectly substituted values, wrong representation, and failure to handle the **BODMAS** rule efficiently	**AS.1.2. A:** Recalled and noticed the correct equation (comprehensible) and substituted values correctly (correct representation), handled the **BODMAS** rule efficiently, strategic skills.
1.2.b) Calculate the length of CD.		**IS.1.2. B**: Recalled unsuitable equation(doubtful)	**F S.1.2.B:** Recalled and noticed the appropriate equation but Substituted values incorrectly, failure to handle **BODMAS** rule.	**AS.1.2. B:** Recalled and noticed appropriate equations and substituted values correctly, Comprehensible, handled the **BODMAS** rule efficiently, strategic skills.

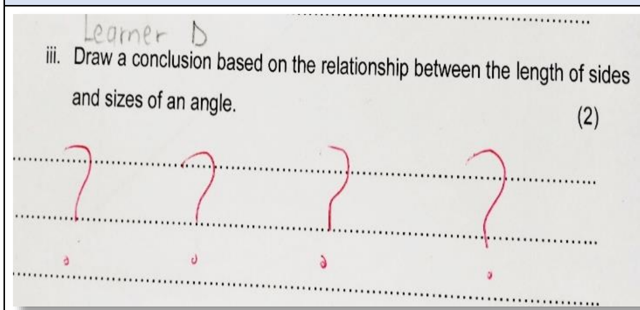
4.3.1 Undocumented trigonometric conceptual structures

In this section, all unanswered questions are documented and transcripts from interviews explaining the reasoning for leaving blank spaces are shown.

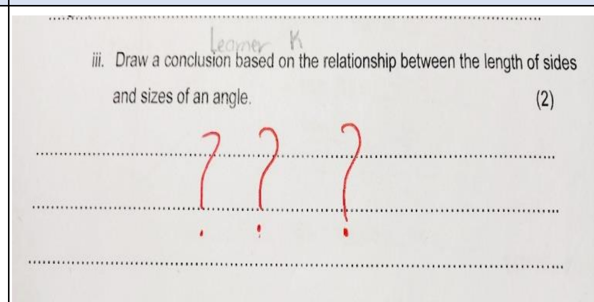
4.3.1.1 Conclusion on relationship between length of sides and sizes of angles

In this unit, the question required learners to use their spatial skills to access conceptual structures, the position of adjacent, opposite and hypotenuse based on the reference angle to conclude the relationship between the length of a side and the sizes of an angle. However, some learners left blank spaces for different reasons. Learners' written responses and transcripts from interviews are shown below:

4.8.a) Learner D written response



4.8.b) Learner K written response



4.8.c) Learner R written response

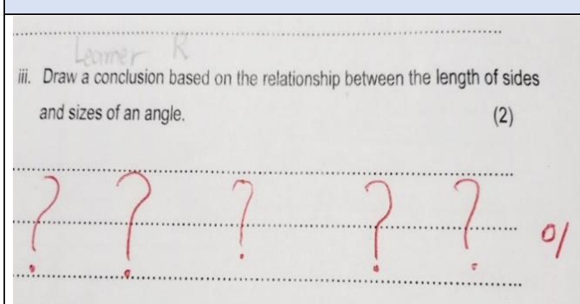


Figure 4. 8 Learner D, K and R question 1.1.b.iii written responses.

Faith : The question required you to write a conclusion based on the relationship between the length of the sides and the size of an angle, but you did not write anything. Why?

Learner D : I did not understand the question.

Faith : Can we read the question together?

Learner D : To be honest, I understood the question, but I was just lazy to think.

Faith : Convince me, please.

Learner D : Remember the question required reasoning skill. I love solving mathematical questions that deal with numbers not words.

Faith : I understand. Imagine if the question paper contained questions that required reasoning skills. Were you going to respond or leave blank spaces?

Learner D : It was going to be a problem. Madam, I do not like questions that require me to think too much.

The transcript above describes that learner D initially indicated that she left a blank space due to failure to link the words within the given question. In addition, the question was silent on drawing a conclusion using answers obtained on the question for computing the hypotenuse labelled AB and using the numeric values to define the ratios. However, Faith could not believe what the learner said as he/she responded in a funny manner. After suggesting a strategy of reading the question together, positively, the learner indicated that she was not willing to think. This was coded **BS1.1.B.III** category indolent. The category indicates that the learner is more interested in solving trigonometric questions that are given in terms of spatial figures. These spatial problems necessitate a weakened mental rotation skill to notice various side names. As a result, explaining how the sides and angles are obtained becomes easier. In contrast, word problems demand sufficient skills to imagine how the size of angles and length of sides are related, and thereafter indicate that the larger the size of an angle the longer the length of a side. Hence, the learner was unable to conclude on the interrelatedness due to failure to notice that the length of the side depends on the size of an angle.

Faith	: You did not answer question iii. Why?
Learner K	: I did not know the answer.
Faith	: Did you understand the question?
Learner K	: Yes, but I did not know what to write. That is the reason I left a blank space but if you provide me with a hint I will start writing.
Faith	: Are the sizes of the three sides on ΔABC equal?
Learner K	: No, AB is the longest side.
Faith	: what about the angle opposite AB?
Learner K	: It is 90°.
Faith	: What do you notice on side AB and 90?
Learner K	: The side is long because of the angle.
Faith	: Perfect.

Generally, the transcript above enlightens that learner K seemed to be terrified to express why she could not conclude the relationship between sides and angles. This was noticed as the learner said ‘‘I did not know the answer’. The response indicates that the learner could not recall and mentally imagine the location of a side and angle. Consequently, failure to imagine the location of a side and an angle could not enable the learner to articulate that the length of the side depends on the size of an angle. This was coded **BS1.1.B.III** category ignorance. Despite this, the learner could not articulate the relationship between the size of an angle and the length of a side. The learner indicated that she managed to link the words used in the question to notice what it requires. However, this was uncertain since an articulation of such nature was not shown. In addition, the learner requested assistance to respond to the given question and all the questions asked helped the learner to notice that the length of the side depends on the size of the angle.

Faith	: You did not answer question iii, why?
Learner R	: I did not understand the question.
Faith	: Can you elaborate?
Learner R	: Some of the words used on the question really confused me.
Faith	: I get you.

The transcript above confirms that the learner could not provide a conclusive statement because of failure to have a clear envision of the words used in the question. This was coded **BS1.1.B.III** category incomprehensible. The question contains familiar words such as ‘draw’. However, it is used to pose multiple meanings in various contexts. The learner could not visualise the word ‘draw’ in the context of justifying the relationship between the size of an angle and the length of a side and use the existing mental image of the word ‘draw’ to meet the demands of the question. This might be one of the challenges that led to confusion.

4.3.1.2 Determining the length of AC and CD

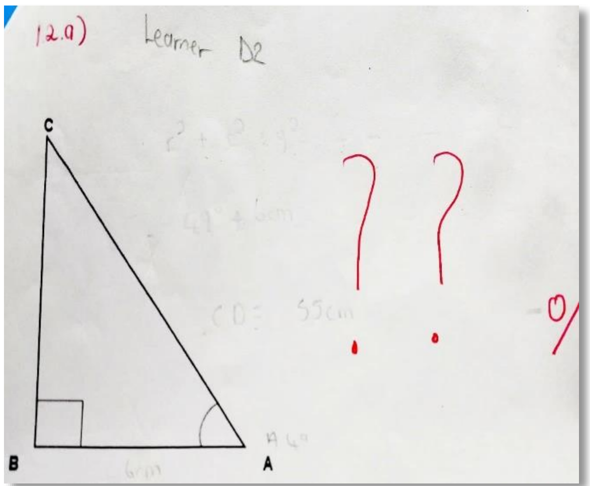
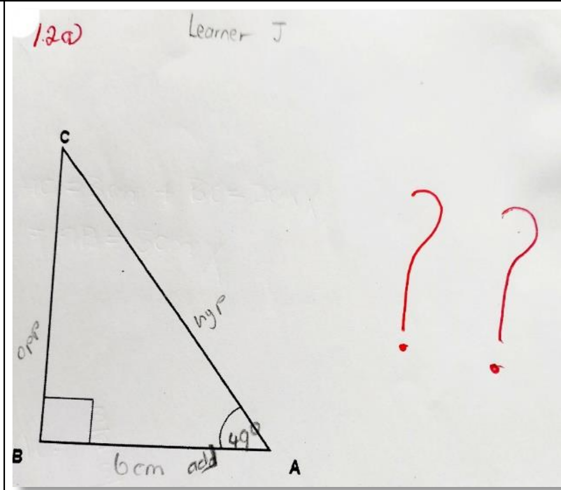
4.9.a) Learner D2 written response	4.9.b) Learner J written response
	

Figure 4. 9 Learner D2 and J question 1.2.a written response

The figures above illustrate that learners D2, and J left blank spaces, and question marks were written to indicate that their scripts were checked. The transcripts to understand why they left blank spaces are shown below:

Faith : You left a blank space in question 1.2.a, why?

Learner D2 : I did not understand the question.

Faith : Do you know what a side is?

Learner D2 : Yes

Faith : Let's use the given figure in question 1.2.a. Between AC, AB and BC which side is a hypotenuse?

Learner D2 : AC is hypotenuse.

Faith : Which one is adjacent?

Learner D2 : AB is an adjacent and BC is the opposite.

Faith : You do know side names; can you please explain why you found it difficult to calculate the length of AC.

Learner D2 : I did not read the question with understanding. I was waiting for you to read and interpret the question for me so that I can start solving the problem.

Faith : Do you think is a good idea?

Learner D2 : Eish, madam I learn best when a teacher gives me an example then tests me later.

Faith : I understand.

The transcript above explains that learner D2 has the skill to rotate a right-angled triangle. This enabled the learner to outline various side names, hypotenuse, adjacent and opposite. Regardless of the side names mentioned above, the learner could not recognise that an angle and adjacent side are given. As a result, computing the length of the hypotenuse labelled AC remained an issue. Later, the learner evidently signified that 'I was waiting for you to interpret the question and provide guidelines to calculate the length of AC'. This was coded **BS1.2. A** category indolent, which shows that the learner requires the teacher to enforce the process of mental rotations. Hence, the category led to failure to outline the given spatial details, angle and adjacent side to compute hypotenuse using a cosine ratio.

Faith : Can you please explain why you left a blank space in question 1.2.a?

Learner J : ……(silence)

Faith : I see you have indicated that AC is hypotenuse, BC is an opposite, AB is an adjacent and $\hat{A} = 49°$. All the information on the figure is correct.

Learner J : Yes madam. I understood the given statement and managed to illustrate the information on the figure. Also, the question required me to calculate the length of AC, but I did not know which ratio to use.

Faith : Okay.

The transcript above explains that the learner initially displayed a negative attitude by keeping quiet when asked a question. This could be a result of trying to figure out a ratio that would assist computing AC. After appraising the learner for imagining how the given figure relates to the given statement, the learner clearly explained that representing the statement implies that the question was understood. Despite that the learner said ‘‘I understood the question’, the existing skill of identifying side names was not enough such that it assists to choose the correct ratio. This was coded **BS1.2. A** category confusion. The category notifies that the learner could not make an informed decision of a ratio to compute the required side.

4.3.2 Inappropriate procedures for computing sides and angles

The process of computing sides and angles requires skills to interpret the trigonometric concepts provided in the given statement and spatial figures. This affords individuals an opportunity to make a proper choice of ratios during computations. However, inappropriate skills lead to failure to choose the relevant equations such as ratios during computations. In detail, the inappropriate procedures were shown when: 1) writing side names; 2) explaining the link between the size of angles and length of sides; and 3) computing AC and CD of different learners as illustrated below.

4.3.2.1 Writing side names and defining trigonometric ratios using spatial figure

4.10 Learner D2 written response

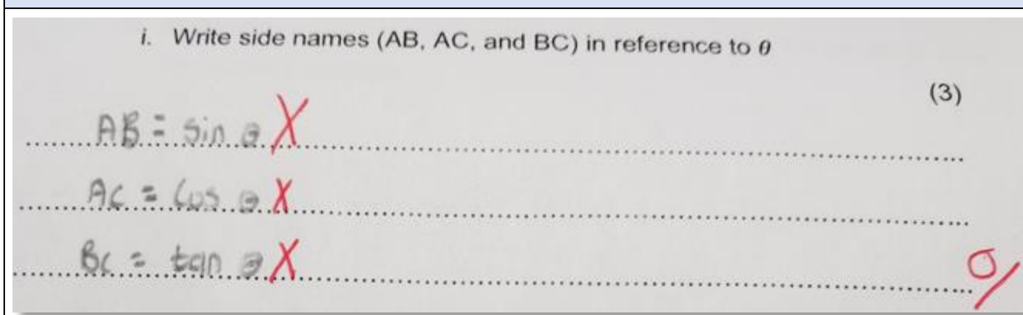


Figure 4. 10 Learner D2 question 1.1.i and ii written responses

Figure 4.10 illustrates that learner D2 did not recognise what the question necessitates. The key concepts, side names and theta on the given question seemed to be clear. However, the learner could not locate sides in reference to ‘’ when given a spatial figure. In addition, at the beginning of the lesson, I explained the objectives of the lesson, “at the end of the lesson, you should be able to define trigonometric ratios and use them to compute sides and angles”. As such, the learner wrote trigonometric ratios instead of side names. Also, the trigonometric ratios were equated to sides, $AB = sin\theta$, $AC = cos\theta$, and $BC = tan\theta$. This indicates that it is inconceivable for the learner to detect when and how sides can be used on ratios. Despite that answers were provided; they were not addressing the given question. This was coded **IS1.1. A,** which forms part of recalling incorrect trigonometric concepts to name side names.

4.3.2.2 Determining the length of AC and CD

Figure 4.11.a) Learner D2 written response	Figure 4.11.b) Learner A written response

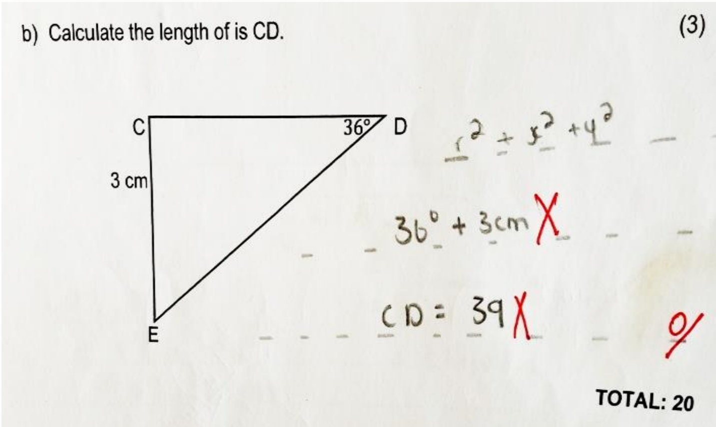
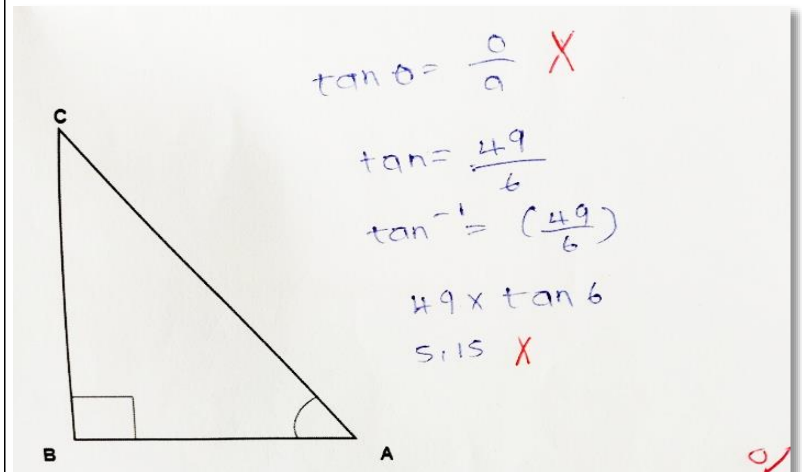

	

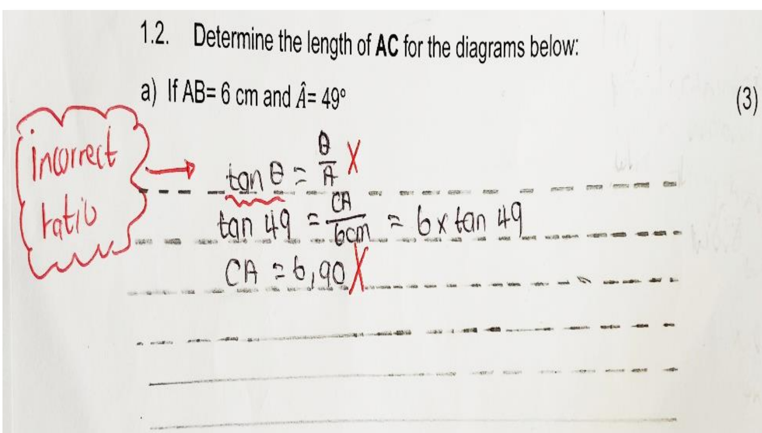
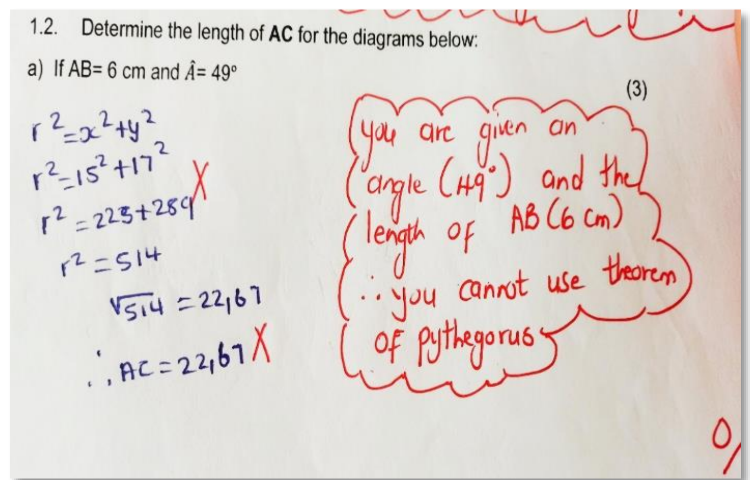
Figure 4.11.c) Learner U written response	Figure 4.11.d) Learner Y written response
	

Figure 4. 11 Learner D2, U question 1.2.b written responses

In figure 4.11.a, learner D2 was required to compute the length of CD using the given angle and the length of CE. Erroneously, the learner recalled the Pythagoras equation and wrote $r^2 + x^2 + y^2$. This forms part of IS.1.2. A category unsuitable equation. Clearly, the learner was given an angle and hypotenuse. Despite that the vital components being given, the learner could not recall and notice that an angle is measured in degrees whereas a side can either be in cm, mm, m, etc. As a result, the learner added angle and length, $36\,° + 3cm = 39$ and this is inappropriate. Transcript from interviews is shown below:

Faith	: Is it the right way of writing a Pythagoras theorem?
Learner D2	: No.
Faith	: Can you please explain why you wrote it like that?
Learner D2	: I could not recall the correct way of writing it.
Faith	: I have noticed that you have added an angle with a side. Are you allowed to add an angle and a side?
Learner D2	: No
Faith	: Is the Pythagoras equation relevant to this equation?
Learner D2	: I am not sure.
Faith	: Think about it.
Learner D2	: I think Pythagoras equation is used when we are given two sides. Meaning, I should have used one of the three ratios.
Faith	: Perfect. Let's check the position of the given angle and the side. Which ratio do you think will be relevant for this question?
Learner D2	: I am struggling to interpret the given right-angled triangle.

The transcript above explains that the learner recalled an unsuitable equation, Pythagoras, and it was written incorrectly. This was coded **IS.1.2. B** unsuitable equation. Although an unsuitable equation was used, Faith requested the learner to recheck the written responses. Excitingly, the learner managed to notice that Pythagoras equation is unsuitable for the given question. Moreover, the right-angled triangle was not given in the usual manner, facing down while the right-angle is on the left-hand side. As a result, it was not easy for the learner to mentally rotate the right-angled triangle. Rotating the figure would have helped the learner to recognise that the side facing the given angle is the opposite. Thus, the learner could not choose a tangent ratio to compute the adjacent side labelled CD.

Figure **4.11.b** illustrates that learner A also manipulated irrelevant properties of a right-angled triangle, side names in reference to θ. The skill of manipulating irrelevant properties mentioned above led to a failure to select a suitable equation which is the cosine ratio. Additionally, the learner was expected to compute the value of AC. However, the figure indicates that the learner thought an angle was required. As a result, the learner substituted $49°$ as an opposite and 6cm as an adjacent side. Extraneously, procedures for computing angles were carried out, $tan^{-1} = \frac{49}{6}$. Thereafter, an irrelevant skill for handling the BODMAS rule was applied as the learner multiplied 49 and tan6, $49 \times tan6$ and obtained 5,15. The transcript of the learner is shown below:

Faith	: You are given the length of AB, which is equal to 6 cm, an angle of A ($\hat{A} = 49°$) and you are required to compute the length of AC. What is the name of AC?
Learner A	: AC is a hypotenuse.
Faith	: but you substituted the value of AC as an adjacent. Is it correct?
Learner A	: No
Faith	: Remember you were supposed to calculate the length of AC, but you followed procedures for calculating an angle. Why?
Learner A	: I did not know the procedures for calculating the length of AC.

Faith	: Oh, I see. But you never raised up a hand for assistance.
Learner A	: I was scared. If you can teach me, I will be able to calculate it.

The transcript above explains that the learner managed to recognise that AC is the hypotenuse. Although the mistake was corrected, learner A maintained that it was challenging to make mental rotations which would have assisted in selecting the correct equation. Subsequently, the difficulty to make mental rotations led to carrying out unsuitable procedures of computing the length of AC. This is also coded **IS1.2. A** category unsuitable equation as well as the inability to handle the BODMAS rule. Hence, the difficulty in selecting the correct equation could not allow the learner to further elaborate on how the ratios are used.

Figure **4.11.c** above proves that the learner utilised tangent ratio instead of cosine to compute the length of AC. The learner managed to recognise that an adjacent side and an angle were given. Nonetheless, the learner could not rotate the figure in such a way that the relevant ratio is noticed. To support the visual data, the transcript from interviews is presented below.

Faith	: Remember you were required to calculate the length of AC, what is the name of AC?
Learner U	: Is adjacent.
Faith	: Is it an adjacent side?
Learner U	: It is the opposite. Oh, I see the ratio is not correct.
Faith	: Then which ratio will be suitable to compute
Learner U	: cosine ratio
Faith	: Thank you.

The transcript above explains that the learner initially assumed that AC is an adjacent whereas AB is hypotenuse side. Having said that, the learner later managed to recognise that AC is a long side opposite a right-angle and it is called opposite. Consequently, the ability to recheck the right-angled triangle assisted in seeing that the applied ratio is incorrect. Hence, a cosine ratio is relevant for computing the length of AC.

In figure **4.11.d**, learner Y is given the hypotenuse as 6 cm and angle A as 49°. Despite that an angle and side is given; the learner applied the Pythagoras theorem instead of a cosine ratio. This was coded **IS.1.2. A** category unsuitable equation. Specifically, the learner could not manipulate the given numeric value of adjacent side which is 6cm and the angle beneath the adjacent side, 49°. Surprisingly, the learner substituted the value of 15 as a numeric value of x and 17 as a numeric value of y. The two numeric values were not given, and the learner did not indicate how they were obtained.

Faith	: You have applied Pythagoras to compute the length of AC.
Learner Y	: Yes.
Faith	: And you substituted 17 as the numeric value of x and 17 as numeric value of y. Can you please explain how the two values were obtained.
Learner Y	: I used a ruler to measure the adjacent and the opposite side, then calculate the hypotenuse side.
Faith	: Oh, I see. But you were given 6 cm as the value of the adjacent. All given values were not used, why?
Learner Y	: It was difficult to decide on the ratio. Therefore, I decided to measure the given right-angle triangle to calculate the hypotenuse easier.

The transcript above explains that learner Y used Pythagoras equation which is unsuitable to the given problem. This is also coded **IS.1.2. A** category unsuitable equation. On the unsuitable equation, the learner could not relate the given side and angle to the three ratios to make the relevant choice. As a result, the learner came up with an incorrect strategy of formulating new values. The learner honesty indicated that a ruler was used to obtain the two numeric values of x and y which were substituted on the Pythagoras equation. This illustrates that the learner is more familiar with the question of computing the hypotenuse using the Pythagoras equation instead of ratios. Hence, irrelevant rotations were made using the irrelevant properties of computing the hypotenuse through Pythagoras within the context of ratios.

4.3.3 Fragmented trigonometric conceptual structures

In this section, learners were required to: 1) write the side names of the given spatial figures; 2) define ratios using variables; 3) use values to define ratios; 4) draw a conclusion on the relationship between the size of an angle and length of sides; and 5) compute the length of hypotenuse labelled AC and adjacent as CD. While working on the questions above, some learners displayed fragmented trigonometric conceptual structures which are shown below:

4.3.3.1 Writing side names and defining ratios using the spatial figure

Figure 4.12.a) given spatial figure	Figure 4.12.b) Learner A written response

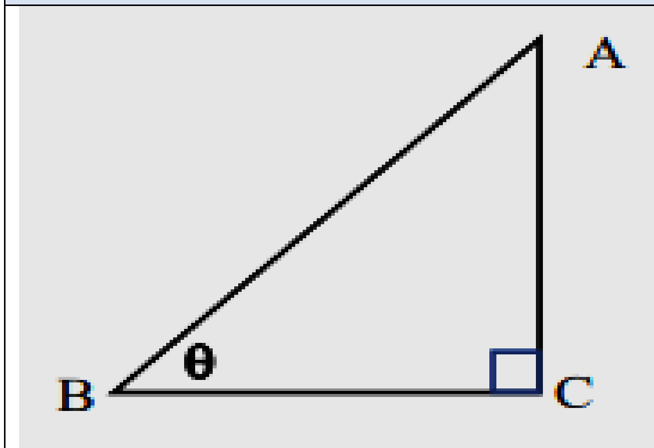
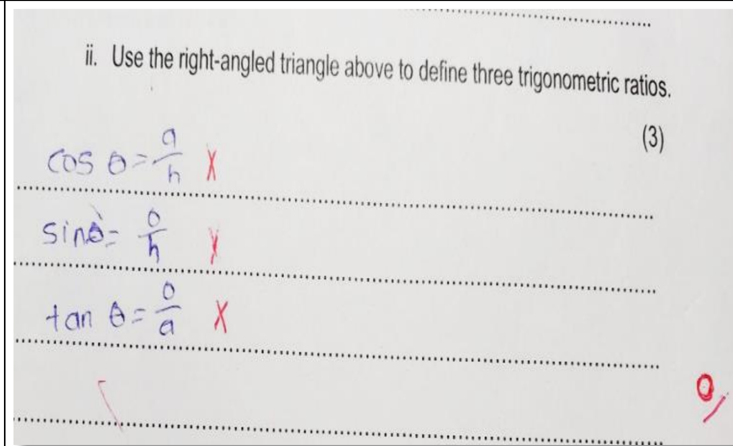

	

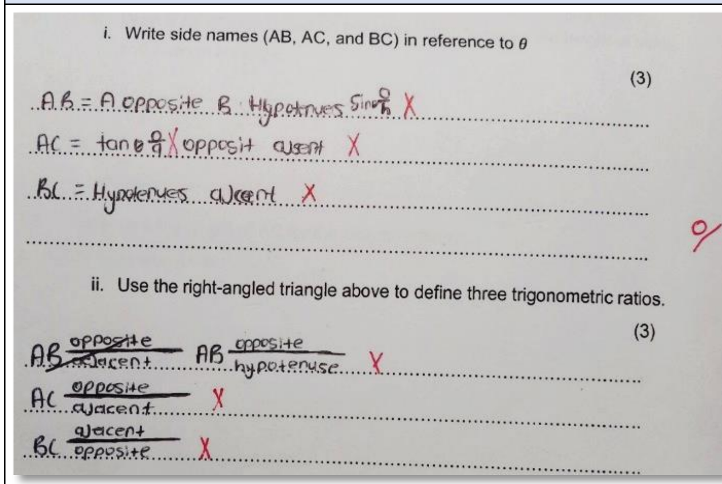
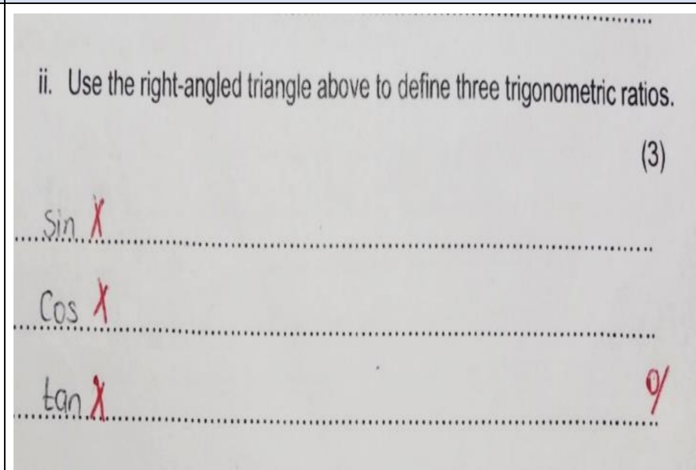
Figure 4.12.c) Learner R written response	Figure 4.12.d) Learner U written response
	

Figure 4. 12 Learner A, R, and U written responses to question 1.1. I and ii

In figure **4.12.a**, the question required the learner to define trigonometric ratios using the given right-angled triangle (ΔACB). Basically, learner A correctly defined the three trigonometric ratios but not in relation to the given spatial figure. Although the question was not fully answered, the learner seemed to know the relationship between side names of the three ratios. Hence, the transcript to understand why the learner could not define the three trigonometric ratios are shown below.

Faith	: You were supposed to define the three trigonometric ratios using the given right-angled triangle. You defined the three ratios using the given right-angled triangle, why?
Learner A	: I knew the question required me to define the trigonometric ratios, but I did not understand how you wanted me to represent them.
Faith	: Can you elaborate further?
Learner A	: I can differentiate between ratios, but the last part of the question was not clear to me.
Faith	: I get you.

The transcript above explains that learner A seemed to have a skill of illustrating the relationship between side names on different ratios. Yet, the acquired knowledge could not be applied efficiently due to insufficient skills of explicitly reading in detail. This was coded **FS.1.1. A. II** under the set of the inability to define the ratio using variables. In addition, the question was phrased in such a way that the word ‘variables’ was contextualised as using the given spatial figure. Hence, the learner might have found it challenging to notice that the vertices of the given spatial figure are represented as variables.

In figure **4.12.c**, learner R was supposed to write the side names of the given right-angled triangle. However, difficulties in differentiating side names of the given right-angled triangle were outlined. For instance, the learner indicated side names incorrectly, AB as an opposite, BC as a hypotenuse side and AC as ‘opposite’ or adjacent. The inability to differentiate side names could be the result of not locating them in terms of theta. In contrast, learners’ responses show that failure to locate side in relation to theta affected the representation while defining trigonometric ratios using variables. As such, the learner wrote $AB\ \frac{opposite}{hypotenuse}$. The written response was not represented correctly since an equal sign was omitted. Despite that the equal sign being omitted, the answers are still not responding to what the question required. The transcript from an interview on this question are shown below:

Faith	: On the question for writing side names, you wrote, $AB\ \frac{opposite}{hypotenuse}$. Do you understand what a side name is?
Learner R	: Yes, for example AB is a hypotenuse.
Faith	: But you did not write the way you are responding now.
Learner R	: I did not know how to present my answers clearly.
Faith	: I get it.

The transcript above explains that the learner can locate sides in reference to theta orally as a side name was specified. This indicates that a skill to imagine and relate side names within the spatial figure is there but representing the ratios symbolically is difficult. This was coded as **FS.1.1. A. II** category wrong representation.

Figure **4.12.d** specifies that learner U recalled how ratios must be presented but seemed to lack few components such as how to locate theta and sides when defining ratios. Consequently, the learner wrote sin, cos and tan. The responses indicate that the answers are not complete since the angle as well as the sides' names are not shown. Below is the transcript from the interviews:

Faith : On the question for defining ratios using the given figure, you wrote sin, cos and tan. Are the ratios defined completely?

Learner U : No

Faith : Did you understand the statement or question?

Learner U : Yes, I was supposed to define trigonometric ratios using that right-angled triangle, but I did not know whether I should use AB, AC or BC to represent the sine, cosine and tangent.

Faith : Can you elaborate further.

Learner U : Actually, I am struggling to understand the link of side names on the ratios?

The snapshot explains that learner U illustrated incomplete ratios as the learner could not recall and manage the existing properties of side names in relation to the ratios. This was coded **FS.1.1. A. II** category incomplete ratio, and it is a result of under-developed skills to extremely alternate and unpack sides involved in each ratio.

4.3.3.2 Determining the length of AC and CD

Figure 4.13 Learner F written response

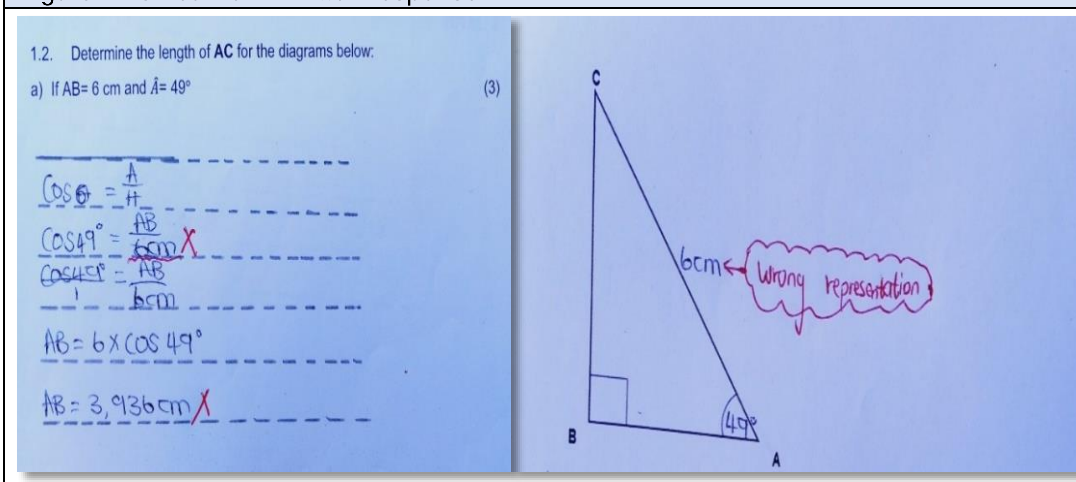


Figure 4. 13 Learner F question 1.2.a written response

The learner was given a statement containing the length of AB as 6 cm and $\hat{A} = 49°$. The given numeric values were to be located using mental rotations on the right-angle triangle to compute the length of the hypotenuse. Essentially, noticing the relationship between the spatial figure and statement is a prerequisite for choosing the correct ratio. However, while handling the computation processes, instead of indicating 6 cm on the adjacent side, the learner indicated on the hypotenuse; thereafter, chose the correct ratio. Although the correct ratio was used, the incorrect representation resulted in learners computing the adjacent side instead of the hypotenuse. Additionally, dealing with fractions necessitates a skill to know how and when to use the BODMAS rule while simplifying the ratio. For instance, instead of multiplying the fraction correctly, the learner multiplied cos$49°$ by 6 which are all numerators, meaning the equal sign was neglected and the multiplication process was not handled efficiently.

Faith	: I see you substituted the numeric value of 6 as hypotenuse side, why?
Learner F	: I am struggling to differentiate side names.
Faith	: Do you know what a side is?
Learner F	: I can give examples, hypotenuse, adjacent and opposite.
Faith	: Do you know what an angle is?
Learner F	: it is a theta.
Faith	: Can you explain how it is formed?
Learner F	: No.
Faith	: Let check how you simplified your answer.
Learner F	: I know that I did not multiply the number and the ratio correctly. I cannot handle operations when dealing with fraction.
Faith	: I get you.

The transcript above explains that the learner used side names interchangeably due to failure to access the properties of an angle. This was coded **FS.1.2. A** category recalled and noticed the correct equation but incorrectly substituted values. The category clearly explains that a relevant ratio was applied, however, dealing with side names mentally in reference to theta was difficult. For instance, the learner was requested to explain what a side is but decided to provide examples. Furthermore, the learner could not articulate how an angle was formed, and this could be evidence that the side names are not known based on the reference angle. Moreover, the learner was required to explain the simplification methods. Honestly, the learner said it is difficult to deal with operations. In contrast, this was coded **FS.1.2. A** category inability to handle BODMAS rule. The category indicates that the learner could not mentally tap the properties of cross multiplication when given fractions. As a result, it was a challenge to organise the ratio simultaneously mentally and the BODMAS rule.

4.3.4 Appropriate procedures for computing sides and angles

4.3.4.1 Computing sides

Figure 4. 14 Learner H question 1.1.bii written response.

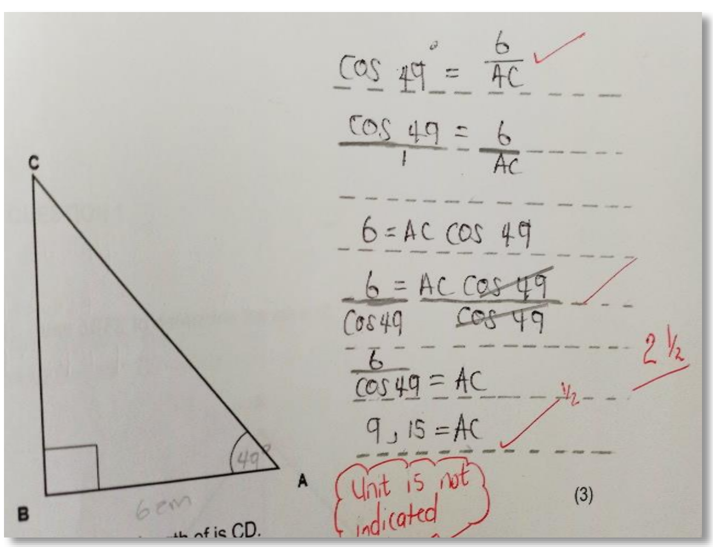


Figure 4. 14 Learner H question 1.1.bii written response

The figure above illustrates that the learner was given a right-angled triangle with a statement containing two numeric values, $\widehat{A}=49°$ and the length of AB= 6 cm. Fundamentally, the learner managed to firstly imagine and notice that the value of AB represents adjacent whereas AC represents hypotenuse. Success in outlining the given information enabled the learner to choose the cosine ratio and substitute all given angles and the length of the adjacent values. In terms of operations, the learner was aware when to multiple and divide. All skills to rotate spatial figures and handle the BODMAS rule were initiated internally and later expressed on the answer sheet. The transcript from the interview is shown below:

Faith : Can you please explain how you obtained the hypotenuse side.

Learner H : On that question, an angle and a side were given. Soo, I noticed that cosine ratio will be relevant as I was required to calculate the length of the hypotenuse.

Faith : Please continue.

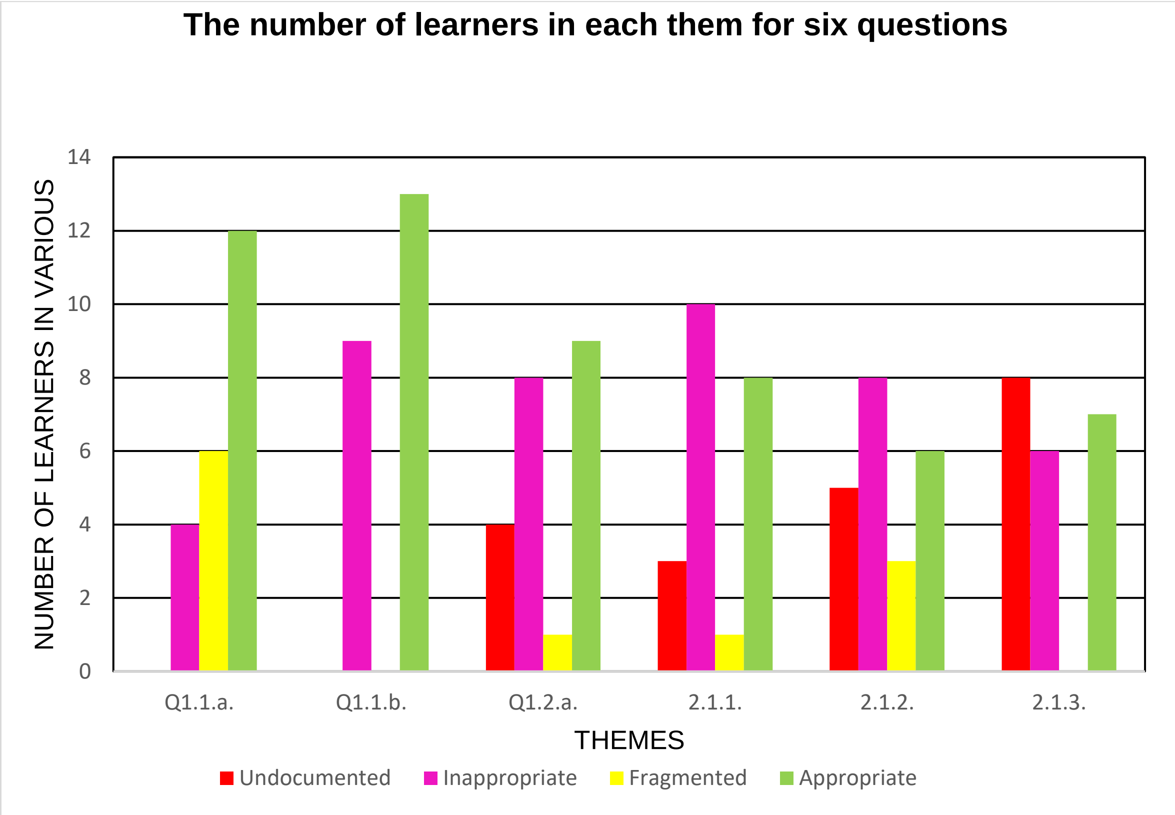
Learner H : It is important to be careful when substituting all the given values. After substituting I had to cross multiply and divided by $\cos 49°$ both sides to obtain the length of AC.

Faith : Thank you.

The transcript above clarifies that learner H is aware that a cosine ratio is utilised strictly when given an adjacent side and a reference angle. Moreover, the learner was able to compute the hypotenuse by simplifying the fraction. The skill to illustrate this prerequisite requirement is coded **AS.1.2. A** category recalled, noticed appropriate equation, substituted values correctly, and handled BODMAS rule efficiently, thus, how a desired solution was obtained.

4.4 Presentation of results on the last task

In this unit, learners presented answers based on their application of familiarity with trigonometric ratios using spatial skills. During the process of presenting answers, most learners struggled to use spatial skills to access the essential concepts that assist in choosing relevant ratios. A bar graph and codes summarising the spatial skills exhibited within the four themes are presented below:



Graph 4. 3 A bar graph summarising how learners exhibited skills in each theme

The bar graph above summarises Grade 10 learners' spatial skills within the four themes. In this case, learners left blank spaces on the last four questions and the highest number of learners was in question 2.1.3. This resulted from the failure to interpret various spatial figures with common sides to obtain correct answers in questions 2.1.1 and 2.1.2. Learners displayed inappropriate procedures in all questions. However, the highest number was in question 2.1.1. Although there are a great number of learners on inappropriate procedure themes, most were correct during interviews. Question number two is sequential, and most learners struggled to use the correct ratio. Moreover, the highest number of learners displayed a fragmented conceptual structure in question 1.1.a whereas there are less than four in other questions but there is none of the question 2.1.3. This indicates that most learners struggled to grasp the concepts of ratios to compute sides and angles. Finally, learners exhibited proper spatial skills whilst computing sides and angles. However, the lowest number of learners was in question 2.1.3 due to its standard.

Table 4 3: Codes from learners' responses to the last task

Questions	***Codes from the mathematical task and semi-structured interviews***			
1.1.a) Use ΔBFE to determine the value of x		**IS1.1. A**: Recalled unsuitable equation.	**FS1.1. A**: Inability to locate sides and angles due to fragmentation, doubtful, unverified procedures and inarticulation.	**AS1.1. A**: Appropriately linking sides and angles through association and retrievals to compute the value of **. .**
1.1.b) use ΔBFE to determine the value of $\hat{E}$		**IS1.1. B:** Recalled unsuitable ratio to compute $\hat{E}$.		**AS1.1. B**: Appropriately ` recalled sine, cosine or tangent ratio to compute $\hat{E}$ and handled the **BODMAS** rule efficiently.
1.2.a) Use ΔBFC to calculate CF	**BS1.2A**: Blank spaces for various reasons, failure to recall Pythagoras equation or divide the length of by base by 2 to get the length of CF.	**IS.1.2. A**: Unsuitable equation or procedure to compute CF.	**FS1.2. A**: Inability to differentiate side names to compute the length of CF; inability to handle **BODMAS** rule.	**AS1.2. A**: Appropriately recalled side names on the Pythagoras equation or divided the numeric value of CF; handled the **BODMAS** rule efficiently.
2.1.1 Calculate Length of AC.	**BS.2.1.1**: Left blank space due to failure to choose the sin30° or cos60° to compute the length of AC	**IS2.1.1**: Recalled unsuitable ratio.	**FS2.1.1**: Recalled sine or cosine but could not substitute a side and an angle correctly.	**AS2.1.1**: Recalled sine or cosine and substituted a side and an angle correctly; handled **BODMAS** rule efficiently.
2.1.2 Size of $C\hat{A}D$	**B.S2.1.2**: Left a blank space due to failure to recognise the given numeric values and chose cosine ratio to compute $C\hat{A}D$	**IS2.1.2**: Recalled unsuitable ratio.	**FS2.1.2**: Recalled cosine ratio but substituted sides incorrectly.	**FS2.1.2**: Recalled cosine ratio and substituted values correctly' handed BODMAS rule efficiently.
2.1.3 Length of DE	**B.S2.1.3**: Left a blank space due to failure to use the 30° and $C\hat{A}D$ to obtain $D\hat{A}E$ which assists to obtain DE	**IS2.1.3**: Recalled unsuitable strategy to compute $D\hat{A}E$ and ratio to compute DE.	**FS2.1.3**: inability to compute $D\hat{A}E$ since wrong answers were obtained from 2.1.2 and 2.1.3; could not handle the BODMAS rules.	**AS2.1.3**: Firstly computed $D\hat{A}E$ using answers from 2.1.1 and 2.1.2 to obtain DE; handled BODMAS rule efficiently.

4.4.1 Undocumented trigonometric conceptual structures

In responding to the last task, some learners left blank spaces due to failure to use spatial skills to apply conceptual structures acquired while working on the second task. Hence, learners' varied responses are presented below.

4.4.1.1 Computing two sides and angles

Figure 4.15.a) Given spatial figure

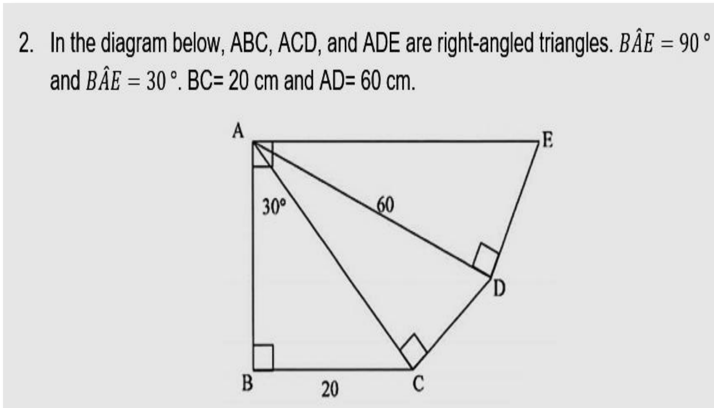


Figure 4.15.b) Learner A response

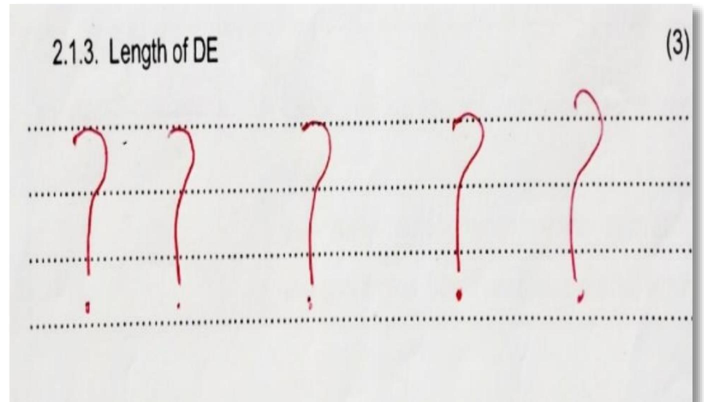


Figure 4.15.c) Learner S response

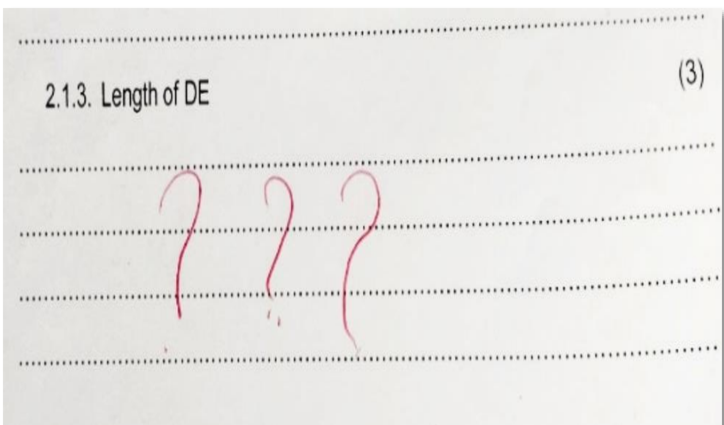


Figure 4.15.d) Learner F response

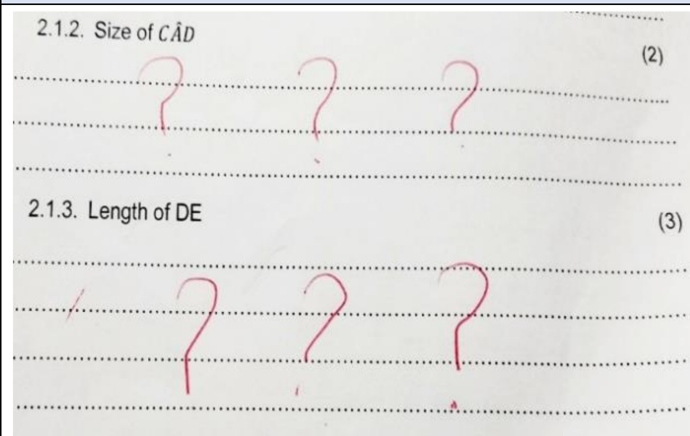


Figure 4. 15 Learner A, S and F question number 2 written responses

Faith : Why did you leave a bank space in question 2.1.3?

Learner A : I did not know how to compute the length of DE.

Faith : Did you understand the question?

Learner A : Yes, but I did not have a strategy to solve the problem.

Faith : Why did you leave a blank space in question 2.1.3.?

Learner S : (Silence)

Faith : why?

Learner S : (silence)

Faith : Why did you leave blank space in question 2.1.2 and 2.1.3?

Learner F : There are many right-angled triangles on one figure, and it is difficult to identify side names.

Faith : Okay.

In question number two, learners were given a spatial figure involving four right-angled triangles with common sides to recall and notice side names in reference to various angles to compute: 2.1.1) the length of DC; 2.1.2 $C\hat{A}D$, and 2.1.3 the length of DE. Moreover, these three questions were sequential, meaning failure to obtain 2.1.1. and 2.1.2. influenced the ability to recognise side names and derive strategies for computing the length of DE. However, learners A, S and F left blank spaces. Generally, the reasons for the three learners to leave blank spaces above are like those that were presented on the first and second tasks. For instance, 1) learner A could not interpret the side names against the angles; 2) learner S could not express mental manipulations; and 3) learner F could not differentiate side names and this category's inability to differentiate side names. Although the learners were coded under blank spaces, learners A and F's responses form part of ignorance, whereas learner S's responses fall under the category of articulation. These could be a result of insufficient skills to aspect a right-angled triangle in one figure. Although the current spatial figure involves numerous right-angled triangles, the difficult to handle and treat the figures based on their reference angles when given one or two spatial figures affected the ability to deal with various spatial figures. Thus, the learners could not compute the required sides and angles.

4.4.2 Inappropriate procedures for computations of sides and angles

4.4.2.1 Computing sides and angles

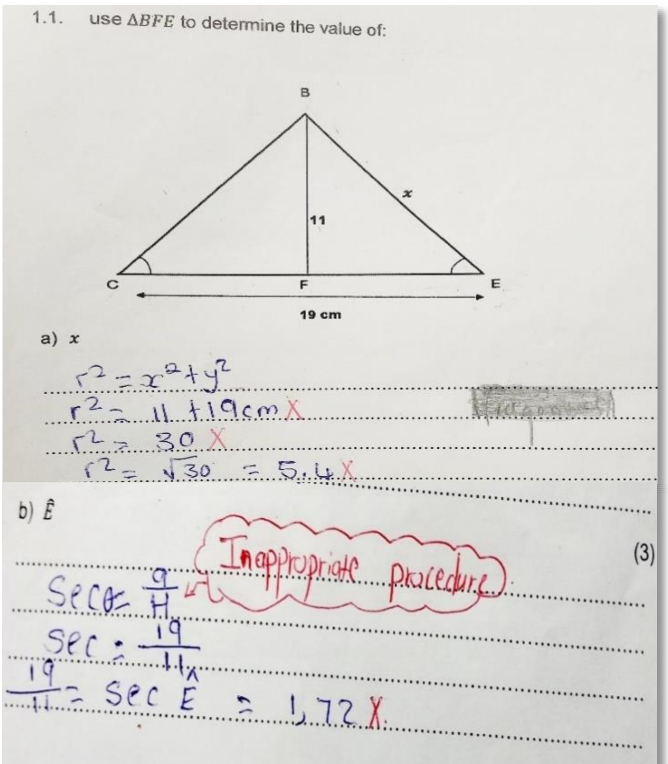
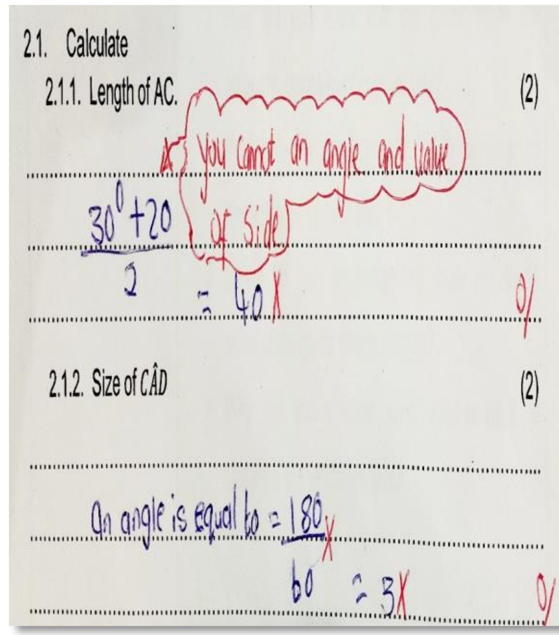
<table>
<tr><th>4.16.a) Learner S written response</th><th>4.16.b) Learner S written response</th></tr>
<tr><td>
</td><td>
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Figure 4. 16 Learner S question 1.1 and 2.1 written response

Faith	: On the question for calculating angle E, you chose sec ratio, why?
Learner S	: (Silence)…………
Faith	: Why you choose secant ratio?
Learner S	: I did not have an idea of what to do.

Question 1.1 required learners to firstly compute the length of BE using the Pythagoras theorem. Consequently, obtaining the length of BE enables learners to select the relevant ratio to compute $\hat{E}$. This means that the previous answer does contribute to the acquisition of $\hat{E}$. In contrast, during the computation of $\hat{E}$, learner S chose an irrelevant equation, secant and this is a reciprocal of cosine. The irrelevant ratio could not enable the learner to obtain the desired solution. This was coded **IS1.1. B** category unsuitable ratio. Full detail on this category was presented both on the first and second tasks. Although the learner chose an incorrect ratio, the written sides are properties of the cosine. On ΔBCE, CE is the length of two right-angled triangles given as 19 cm. Hence, to obtain the adjacent sides of the two figures, the learner was expected to divide 19 cm by 2 since the two base angles are equal. Also, the length of the opposite, 11 cm was given and a visual skill to notice that is a common side for both figures was a requirement. Instead of applying the components above first, the learner substituted the numeric value of 19 as an adjacent and 11 as a hypotenuse, which is incorrect. Thus, the learner could not provide a precise explanation of how the procedures were carried out.

Question two required learners to analyse numerous spatial figures. In responding to question 2.1.1., learners had to check the given sides and make a relevant choice on the ratio to compute the length of AC. Learner S also displayed inappropriate conceptual structures in question 2.1.1. by adding an angle and a side which are all divided by two. The result of failure to notice the units of sides and angles is a contemporary issue as it was discovered on the previous tasks. Similarly, in question 2.1.2., learner S divided an angle of 180 by the length of AD, which is 60 cm. Hence, failure to differentiate sides and angles led to failure to compute the length of AC and $C\hat{A}D$.

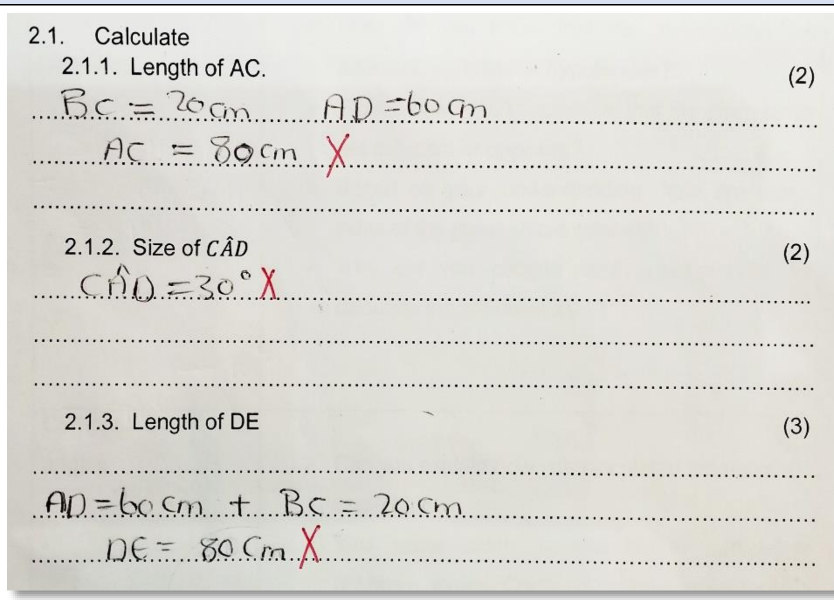
Faith	: In question 2.1.1.you added an angle (90°) and a side (20 cm). Are we allowed to add unlike terms?
Learner S	: No
Faith	: Why did you add an angle with a side?
Learner S	: I do not know.

Faith : Let's check question no. 2.1.2. Where did you get 180?

Learner S : (silence)

The transcript above explains that the learner added an angle and a side. Learner S was asked if it a good idea to add angles and sides and the learner said no. The response indicates that the learner was aware that unlike terms cannot be added together. Moreover, the learner was requested to elaborate on the fragmented idea and the learner could not explain. This is coded **IS21.1** and **IS2.1.2** category inability to notice the differences between sides and angles.

4.17 Learner J written response


2.1. Calculate

2.1.1. Length of AC. (2)

BC = 20cm AD = 60 cm

AC = 80cm X

2.1.2. Size of $C\hat{A}D$ (2)

$C\hat{A}D = 30°$ X

2.1.3. Length of DE (3)

AD = 60 cm + BC = 20 cm

DE = 80 Cm X

Figure 4. 17 Learner J question 2.1.1-2.1.3 written responses

In figure 4.17, learner J could not recognise that BC is the opposite side since the side is opposing 30°, AC is a hypotenuse because it is long in ΔABC, also AC is an adjacent in ΔACD. Although the given numeric values were not recognised based on the location of each right-angled triangle, the learner managed to out and decided to add the given numeric value of BC and AD, $20\text{ cm} + 60\text{ cm} = 80\text{ cm}$. Secondly, an angle was required and the learner was obliged to use a ratio but the learner indicated that $B\hat{A}D =$

$C\hat{A}D = 30°$, and this is incorrect. Lastly, the learner added the length of BC and AC again, which resulted in 80 cm. However, all the numeric values used in the three were given on the spatial figures.

Faith	: Can you please explain how you got the length of AC?
Learner J	: I have added $20\ cm$ and $60\ cm$, $20\ cm + 60\ cm$.
Faith	: But $60\ \mathrm{cm}$ is the length of AD on the second triangle. Is it correct to use $60\ \mathrm{cm}$?
Learner J	: No.
Faith	: Why did you write the size of angle E as 30 degrees?
Learner J	: (silence)

The transcript above explains that learner J does not know how relevant procedures for choosing ratios to compute side and angles. This was categorised as a recalled unsuitable equation.

4.4.3 Fragmented conceptual structures

In this subheading, learners recalled relevant equations but displayed fragmented conceptual structures. The details are shown below:

4.4.3.1 Computing an angle

4.18 Learner N written response

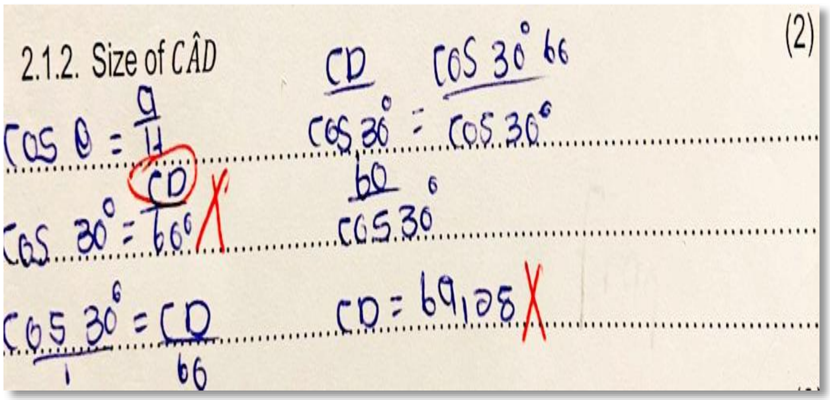


Figure 4. 18 Learner N question 2.1.2 written response

The figure above illustrates that learner N had to compute $C\hat{A}D$ and an answer obtained in question 2.1.1 had to be used. However, learner N managed to visualise and rotate the four right-angled triangles and noticed that on the second right-angled triangle, 60 cm was given as hypotenuse. This enabled the learner to choose the correct ratio, cosine. Although the correct ratio was chosen, the learner substituted the numeric value of a hypotenuse, 30 ° as $C\hat{A}D$ and neglected the adjacent side obtained in question 2.1.2. As a result, the learner ended up computing the length of the CD instead of the $\hat{A}D$. Transcripts from interviews are shown below.

Faith	: Let's check question 2.1.2.
Learner N	: I choose the correct ratio.
Faith	: yes. Check the values you have substituted please.
Learner N	:Yoo, I a mistake. I have substituted the given angle and I was supposed to calculate $C\hat{A}D$ but I have calculated the length of CD.
Faith	: I am happy because you have noticed that you computed the length instead of an angle.

Learner N : And you gave us enough time to write but I did not even confirm my answers.

The transcript above explains that learner N noticed that written responses were not responding to the requirement of the question and computed the length of CD instead of an angle. This indicates that the learner could not fully make a mental rotation of the figure to recognise the given details and clearly pay attention to what the question requires. Although fragmented conceptual structures were made on the written task, the learner managed to notice that a mistake in the wrong manipulation was made. This is coded **FS.2.1.2** category substituting values incorrectly. Hence, more details of incorrect substitution were documented in the previous two tasks.

4.4.4 Appropriate procedures for computing sides and angles

4.4.4.1 Computing four sides and an angle

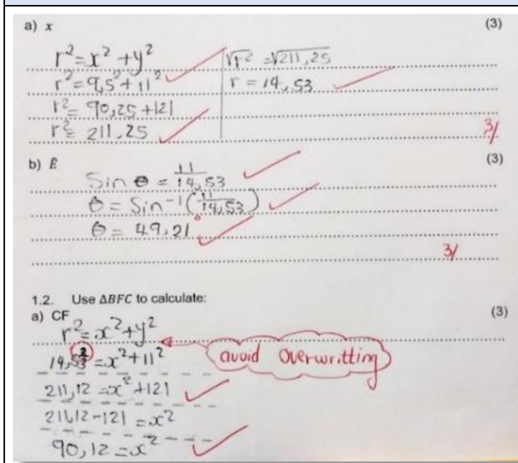
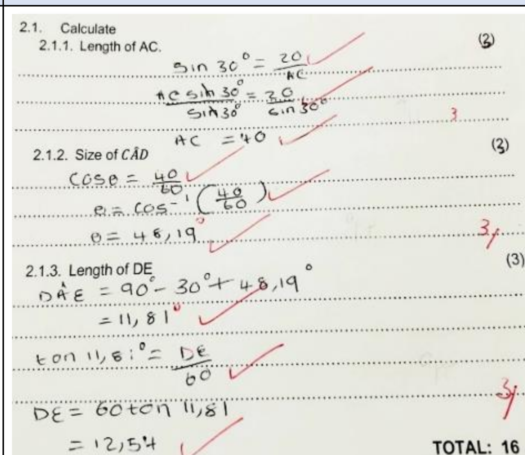
4.19 Learner D written response	4.19 Learner P written response
	

Figure 4. 19 Learners D and P written responses to question 1.1.a, b and 1.2.a

In question 1.1.a., learner D noticed that values for the hypotenuse and the adjacent side are given. Due to the identified details, the learner had the privilege to write the Pythagoras theorem, the BODMAS rule was handled correctly and the value of x was obtained. Question 1.1.b. required the learner to use an answer from the previous question to compute the size of $\hat{E}$. Essentially, the ability to obtain the correct answer in the previous question allowed learner D to choose the relevant ratio, the sine. All features of computing an angle were applied and the correct answer was obtained. Moreover, question 1.2.a became easier for the learner as it was like the first question and the correct answer was obtained.

Faith	: You were required to calculate the value of x and you choose the equation $r^2 = x^2 + y^2$. Why did you choose this equation?
Learner D	: I wanted to calculate the value of the hypotenuse.
Faith	: Can you elaborate further so that I can understand.
Learner D	: That is the equation that could assist in calculating the missing sides.
Faith	: I am still listening (Continue)
Learner D	: ……. (silence)
Faith	: You are struggling to elaborate further?
Learner D	: Yes.
Faith	: Let's move to the second question for calculating $\hat{E}$. Remember we are computing angles and sides using the three basic ratios and you chose sine to calculate $\hat{E}$. Why?
Learner D	: Because we were given the value of the opposite and hypotenuse sides.
Faith	: Why did you write sin^{-1}?
Learner D	: When we calculate an angle using ratios, we must apply an inverse method.
Faith	: Okay. In question 1.2.a you were required to calculate the adjacent side and you have used Pythagoras theorem why?

Learner D : I used the Pythagoras theorem so that I can find the length of CF.

Faith : Can you elaborate further?

Learner D : No.

The transcript above explains that learner S does have the skill of recognising the given detail, the length of BF= 11 cm and CE=19 cm. This helped the learner to apply relevant equations while computing sides and an angle, as well as spatial figures to compute sides and angles. Even though the learner could choose equations wisely, he/she could not use the properties of the equation to justify the written answers further.

In responding to questions 2.1.1 to 2.1.3, learner P had to imagine, retrieve, recall and rotate various right-angled triangles on the given diagram. Essentially, learner P managed to notice that the length of AC can be obtained by using the sine ratio as the opposite side and opposite angles were given. During the process of computing the length of AC, learner P applied the multiplied fractions suitably. Moreover, in question 2.1.2., learner D noticed that the value of AC on the first right-angled triangle is an adjacent side on the second spatial figure. Also, AD was 60 cm and it a hypotenuse. Hence, learner P made the right choice of a ratio, which is cosine. All values were substituted correctly and the size of $C\hat{A}D$ was obtained. Lastly, to obtain the length of DE, the learner started by subtracting the given 30° and 48° obtained from question 2.1.2, which resulted in 11. 81°. Thereafter, a skill to notice that tangent ratio is relevant was used to compute the length of DE. Although answers for all questions are correct, the learner tends to forget writing units for angles and sides, and this should be corrected.

Faith : In question 2 you were given a spatial figure with many right-angled triangles. Question 2.1.1. required you to calculate the length of AC. You calculated the length of AC using sine ratio. Do you think sine is an appropriate ratio?

Learner P : Yes. We are given an angle which is $30°$ and a length of a side opposite the given angle (opposite side). Since we were required to calculate the length of the hypotenuse, sine is a proper ratio. Remember the given information and the required side determines the ratio.

Faith : I am impressed. Continue……...

Learner P : Question 2.1.2. required us to calculate $C\hat{A}D$ in ΔCAD. I used cosine ratio as I was given the hypotenuse and I have obtained the length of the adjacent side in question 2.1.1. The given side and the value of the adjacent obtained in question number 2.1.1. forced me to use cosine ratio.

Faith : What about the last question?

Learner P : $B\hat{A}E = 90°$. From that angle, $B\hat{A}C$ was given as $30°$ and I have calculated $B\hat{A}D = 48{,}19°$. Since I was required to calculate the length of DE, I was obliged to calculate the size of $D\hat{A}E$ so that I can choose relevant ratio which is tangent.

Faith : Well, explained. Well-done

The transcript above explains that learner P was capable of rotating, recalling and noticing side names of the four right-angled triangles. Rotating the figures enabled the learner to use correct ratios during the computation of *AC, size of* $C\hat{A}D$*, and length of DE*. Most importantly, the learner was requested to justify, and all written procedures were clearly explained using the properties of right-angled triangles on ratios.

4.5 Chapter Summary

The chapter presented an analysis of data acquired from three mathematical tasks and semi-structured interviews. Since four themes emerged from the two sets of data, the analysis was shown in each theme as a way of illustrating that Grade 10 learners displayed various spatial skills while responding to the three mathematical tasks and semi-structured interviews. Hence, the next chapter interprets the two sets of data using semantic theory and literature.

CHAPTER FIVE: INTERPRETATION OF RESULTS

5.1 Introduction

The chapter presents the interpretation of results from the three tasks and semi-structured interviews using spatial skills, semantic tenets and the literature review. Firstly, the task of checking learners' prior knowledge. Essentially, this enable teachers to come up with strategies on how to assist learners during the process of developing new knowledge. Secondly, for checking how spatial skills are utilised to access prior knowledge while developing the new knowledge of computing sides and angles using trigonometric ratios. Lastly, the task of checking if learners can use spatial skills to apply new trigonometric ratios when given various figures. In this context, a conceptual structure is the ability to use the SV to access a set of concepts that assist learners to compute sides and angles. An established set of concepts enable learners to use the SR within system representation such as symbols, pictures and words during computations to obtain the desired solutions. The conceptual structure is a prerequisite to navigate between representations using their minds. The three tenets play a vital role in supporting learners to visualise given spatial figures and notice their interrelatedness. This allows them to explicate their responses.

As detailed in chapter two, the theory entails three tenets:1) conceptual structures, which include concepts and procedures that allow learners to compute sides; 2) system representation using symbols, words and figures to support an understanding of conceptual structures; and 3) the sense, mentally navigating between the conceptual structures and system representation. The three tenets support each other. Hence, it is incredible to compute sides and angles without considering them all.

However, computing sides and angles does not require the three tenets only, but also the application of spatial skills to obtain desired solutions. For that reason, the skills and tenets should be intertwined while working on the three trigonometric tasks. For instance, in figure **4.14**, to compute the length of AC, learner H managed to firstly imagine and notice that the value of AB represents adjacent whereas AC represents hypotenuse.

The process of imagining and noticing is considered as application of appropriate spatial skills whereas the noticed angle and sides form part of conceptual structures Success in outlining the given information enabled the learner to choose the cosine ratio and substitute all given angles and the length of the adjacent values. This is considered as operating within system representation. The results obtained through the mathematical tasks and semi-structured interviews were coded using the axial strategy and further analysed thematically. Concisely, the results of the three lessons were categorised into four subheadings, namely: 1) undocumented trigonometric conceptual structures, leaving blank spaces due to: i) difficulties to think in a short period of time, ii) not knowing what to write, and iii) understanding the question but does not know what to write; 2) inappropriate procedures of computing sides and angles by accessing irrelevant concepts; 3) having fragmented trigonometric concepts; and 4) appropriate procedures for computing sides and angles through accessing relevant trigonometric concepts. Hence, the results in each theme are interpreted using the existing literature together with the three tenets of semantic theory below:

5.1.2 Discussion of results on the three tasks

In this section, the four themes are interpreted using semantic theory, and literature on the three pillars as explained in Chapter 2. In each theme, the written responses from the three mathematical tasks as well as the transcripts from interviews are interpreted. However, for similar patterns, one written response is interpreted.

5.2 Undocumented trigonometric conceptual structures

An undocumented conceptual structure is a controversial aspect that should be addressed as it occurs for various reasons. Such controversy was addressed by Huang and Xiao (2019), who postulated that learners leave blank spaces as they are not allocated enough time to think independently on a given task. Similarly, Jakwerth and Stancavage (2003) as well as Hamukwaya (2022) also outlined that learners leave blank spaces either due to lack of time or the inability to figure out answers. In contrast, in this study, learners were given 50 minutes on the first task to check their prior knowledge of the right-angled triangle; 60 minutes on the second task, to develop new trigonometric concepts using spatial skills and 60 minutes on the last task, to check how spatial skills are used to access trigonometric concepts developed on the second task. Hence, the estimated time seemed to be enough as it was based on the aims of the tasks and the number of questions. Although the time seemed to be enough, some learners might have been affected as they often spend a lot of time attempting to make sense of the given questions. For instance, in chapter 4, responses from learner N also support that learners were given enough time to work on the tasks. Essentially, making sense of the given question instinctively leads to a process of organising their thinking before presenting their answers. However, time might have elapsed before learners could present their answers. In addition, Jakwerth and Stancavage (2003) add that some learners leave blank spaces because they do not know what to write. In the current study, the idea mentioned above forms part of being clueless. This is supported by the responses provided by learner R indicating that the question was comprehended but could not figure out concepts such as rays or lines and points that assist in organising the response. Hence, other subsequent spatial skills exhibited during interviews were categorised as incomprehensible, ignorant as well as indolent and are exposed in the following paragraphs.

Firstly, during interviews, some learners presented the category incomprehensible, which explains that words on the given question were not clear as shown in chapter 4. Asbupel et al. (2021) affirm that some learners leave blank spaces as they struggle to link the concepts used in the question. In figure **4.1.a,** learner A is one of the participants who indicated that she left a blank space due to failure to link concepts within the question. The result is in line with that of Putuwwita and Retnawati (2021), who stated that learners have difficulties comprehending or linking concepts to the given question. This forms part of incomprehensible and was coded as **BS1**. However, a probing technique was used for further questioning, and the learner was requested to give examples of angles or sides. Regardless of the negative statement made initially “I do not understand the question”, the learner responded by saying “an angle is where two points meet to make up 90”. The learner tried to give a brief explanation of an angle without being aware. Although the brief explanation was not clear, the learner gave the impression of having little conceptual structures, which are explanations of angles and sides but struggled to present them clearly (Castro-Rodríguez et al., 2022). As a result, it was difficult to use the SV to mentally imagine the interrelatedness between angles and sides (Jaelani, 2021). Besides, the underdeveloped conceptual structures affected the learners' SR to organise their mental imaginations. This was noticed as the learner struggled to verbalise the differences between angles and sides clearly (Mainali, 2021; Moh'd et al., 2022). In the following sense, the role of system representation was not illustrated. Hence, the learner could not consider the sense to illustrate the flexibility of comprehending the question since both conceptual structures and system representation were not proper. Trigonometric literature outlined that it is impossible to operate within the sense if the learner cannot figure out the necessary information on the given question (Nurmeidina & Rafidiyah, 2019). Based on the idea above, it can be concluded that the learner could not exhibit relevant spatial skills due to unlimited trigonometric concepts.

Moreover, in question 1.1.b.iii. learner R's responses are break-taking aspect that shows that it is important to choose words wisely when designing questions. The learner indicated that the word 'draw' confused her since it is used in multiple contexts. The response concurs with the result from Wardhani and Argaswari (2022) together with Khuzwayo (2019), who emphasised that learners struggle to comprehend or link words used in the given statements. As a result, exhibiting the SV, SR and SO is prohibited as that word affected the learner.

Secondly, learner B2 did not illustrate sufficient spatial skills due to a dearth of conceptual structures such as the differences between angles and sides. This forms part of ignorance, lack of trigonometric concepts which are conceptual structures coded **BS1**. Consequently, the dearth of conceptual structures affected the learners' ability to write the differences between angles and sides. Hasanah et al. (2022) referred to it as the inability to operate within system representation. Adhikari and Subedi (2021) supplement that under-developed conceptual structures such as those mentioned above prohibit learners to operate within multiple representations such as computing trigonometric problems. The multiple representations can be verbal, written or words, pictorial and symbolic. According to DeReu (2019), learners experience difficulties in expressing and managing their knowledge of representation. Interpreting the blank spaces of learner B2 using the result, I would say that the learner lacks the SV to mentally imagine the location of sides and angles (Tandon et al., 2022). The absence of the SV prevented the goal of operating within the written or word representation as she could not write the differences between angles and sides. Although the learner could not use the SR effectively to operate within the word tenet on the given task, she tried to use the SO to verbalise mental manipulations during interviews. Therefore, learners' conceptual structures of a right-angled triangle must be enhanced to compute angles and sides given in the form of words, symbols and pictures.

It is known that it is difficult to solve trigonometric questions with inappropriate conceptual structures (Sánchez et al., 2023; Walsh et al., 2017). For instance, a learner can visualise the given details on the question but struggle to recall relevant concepts to present answers which are also coded BS1 category ignorance. In figure **4.1.c**, learner D could not differentiate between an angle and a side. Surprisingly, during interviews, the learner could provide side names verbally such as hypotenuse, opposite and adjacent. Fundamentally, Clements et al. (2018) and Gagnier et al. (2022) argue that learners should develop the three spatial skills in early childhood. In this context, the three spatial skills are crucial in laying a mental foundation when learning the properties of spatial figures. Interpreting learner responses using the idea of Clements et al. and Gagnier et al., the learner seemed to have unsatisfactory trigonometric concepts and spatial skills as it was difficult to clarify what an angle and a side are. This is considered as having unsatisfactory conceptual structures (Castro-Rodríguez et al., 2022). Since side names were provided, it seemed as if they were memorised. As a result, it became difficult to operate within the written and oral representation using the conceptual structures. Moreover, Pfende et al. (2022) attest that teachers should encourage learners to make sense of their strategies and concepts applied when solving problems. In this case, learner B2 was encouraged to mentally navigate between conceptual structures and system representations verbally. However, learner B2's conceptual structure was not enough to operate within the representation. This is referred to as failure to use the SO while operating within the sense (Lowrie et al., 2020). Hence, interviews are vital as they assist in judging learners' conceptual structures although blank spaces are left.

Similarly, on the question for concluding on the size of angles and length of sides, learner D also left a blank space. This implies that the SV, SR and SO were not used to illustrate the three semantic tenets on the answer book. During interviews, it was clearly stated that the learner is more interested in working with numbers than reasoning questions. This was coded **BS1.1.B.III** category indolent. According to Piaget (2016), learning is a process of constructing knowledge by incorporating new knowledge and existing schemas. However, success is based on the relevancy of learners' existing schemas. In relation to the current study, I would say the learner seemed to lack the SV to access existing concepts which would have helped to notice that the length of a side depends on the size of an angle. As a result, the learner could not use the SR as there were no relevant conceptual structures to organise such that the response can be given. Meaning system representation was not considered verbally. Surtees et al. (2013) as well as Rocha et al. (2020) assert that learners should be able to make justification using the SV and SR when solving mathematical problems. Interpreting learner D's responses using the results from these authors, I conclude that it was unbearable to make a justification because the SV and the SR were not shown. Moreover, from the work of Kang and Liu (2018), these responses confirmed that the learner is interested in following procedures, not solving conceptual problems. Hence, following procedures does not really mean that the learner does have sufficient conceptual structures.

Considerate trigonometric conceptual structures that require attention is that of learner K. During conversations, the learner designated that, if I can provide a hint on how to respond to the question, she will write. This is coded **BS1.1.B.III** category ignorance. Vygotsky (1934) encourages teachers to practice an essential Vygotskian principle, scaffolding, which is a technique of providing learners with hints to solve problems. Practising the Vygotskian principle, I tried to explain what the question requires using a spatial figure. Correspondingly, the learner managed to use triangle ABC in question 1.1. to explicate that the greater the size of angles, the longer the sides. Despite that scaffolding was used, learner K seemed to lack properties of a right-angled triangle because a competent learner would not require the teacher to interpret the questions but provide clarity. Also, Kalogirou and Gagatsis (2012) together with Jaelani (2021) explained the SV as the ability to mentally imagine how spatial figures relate to trigonometric concepts such as properties of right-angled triangles as they appear in a variety of orientations. Similarly, learner K's response evidently showed that the learner managed to use the SV to notice that the length of the sides depends on the size of an angle after the Vygotskian principle was applied. Moreover, Fernandez-Baizan et al. (2021) argue that it should be a norm to consider the sense the SR together with the SO were demonstrated as the learner specified that the greater the size of an angle, the longer the side. All these were accomplished using the conceptual structures to represent the conclusion orally. In short, the learner really clarified the relationship using the properties of spatial figures.

Learner J also left a blank space instead of computing the length of AC. However, clarity for leaving blank space was given. During interviews, the learner confirmed that it was difficult to decide on the trigonometric ratio. Evidence that the learner had a problem was observed as the learner said, "I did not know which trigonometric ratio to use". Faturohman and Amelia (2020) postulate that trigonometric ratios are expressed in terms of the sides and angles of right-angled triangles. Therefore, the conceptual structures of the three sides, namely, hypotenuse, opposite and adjacent are compulsory. In this case, the inability to decide on the relevant trigonometric ratio could be a result of lacking the compulsory concepts mentioned above. The compulsory concepts would have allowed the learner to use the SV to mentally analyse the sides and angles formed (Badmus & Jita, 2022). Also, the learner could not use the SR to recognise the relationship between the given side and angle to decide on the ratio(s). The learner was supposed to represent the ratio symbolically such that the given side and angle can be substituted. This is considered a system representation. Consequently, the learner could not apply the SO, and this forms part of the failure to consider the sense (Castro-Rodríguez et al., 2022), since the learner struggled to consider the sense. However, after explaining the pairing of two sides using a right-angled triangle, the learner started to work individually in trying to compute the required side.

In contrast, question two on the last task was given with a spatial figure involving four right-angled triangles. As a result, relevant spatial skills were required for fluently computing the sides and angles. According to Machisi (2023), most learners suffer from mathematical anxiety, which is considered as being afraid or worried when confronted with a mathematical problem. Similarly, some participants like learners A and S seemed to have mathematical anxiety since they could not compute the length of DE. This was a result of learners' failure to apply for the SV. Most importantly, Açikgül et al. (2023) highlighted that the SV is responsible for assisting learners to understand the orientation in which the spatial figure is presented. The idea signifies that the SV would have enabled the participants to notice that there are four right-angled triangles and each of them has specific properties. Although the spatial figures have specific properties, there are some common details shared among them. Evidently, during interviews, learner A indicated that she understood the question but could not compute the required side since the given figure was not visualised.

Moreover, Tandon et al. (2022) asserted the importance of visualisation during the process of solving mathematical problems. From Tandon et al.'s idea, this indicates that the learner could not spatially visualise the given details, which prohibited learner A from operating within the SR to obtain the necessary angle first, which assists in choosing an appropriate trigonometric ratio. This led to the inability to justify her written procedure orally. Similarly, during interviews, learner S could not respond to the questions asked. The literature review highlighted that keeping quiet during interviews indicates that the learner either does not know what to say or is struggling to organise concepts mentally and then to express them orally (Utami et al., 2021). Contrary to learners A and S's interview responses, learner F clearly indicated that it was difficult to mentally process the words used in the question. Based on the idea, failure to understand questions indicates that the words used within the given statement are not clear. This is considered as lacking conceptual structures. Theoretically, lacking conceptual structures prevent learners to operate within the system representation. For instance, illustrating procedures of computing sides or angles using symbolic representation. Consequently, the sense was not considered since the conceptual structures and system representation were not relevant. This means the dearth of side names could not allow the learner to develop skills on when and how to choose trigonometric ratios.

Furthermore, undocumented conceptual structures were also shown by learner D2 while he/she was required to choose a ratio to compute the length of AC and CD. However, during interviews, learner D2's responses indicated that the learner relies on the teacher to solve mathematical problems. This was noticed as the learner said that she did not respond to the question as she was waiting for the teacher to read and interpret the question. The learner seemed to lack confidence in providing answers without assistance from the teacher. Kumatongo and Muzata (2021) assert that sometimes teachers' pedagogical strategies influence learners' ways of responding to questions. For instance, using traditional teaching methods such as chalk and talk while learners remain passive. Similarly, the learner seemed to have a habit of expecting the teacher to provide a clue to reasoning questions. This affects learners' acquisition of skills and knowledge and prohibited the learner from operating within the centeredness approach (Ogunjimi & Gbadeyanka, 2023; Sakata, 2022). Therefore, it is the responsibility of teachers to play their vital roles in early grades.

Thirdly, clueless connotes that learners do comprehend words used on the given questions but lack an idea of what to do. Learner R's responses during interviews provided ample evidence related to the category of clueless. The learner indicated that he comprehended the question but did not know what to write. The ability to outline what the question required and not knowing what to write implies that the learner could not spatially visualise the differences between the angles and sides of a right-angled triangle (Utami et al., 2021). This is considered as a failure to operate within the conceptual structures which are differences between the angles and sides of a right-angled triangle (Martín-Fernández et al., 2021). Consequently, the inability to operate within the conceptual structures led to the failure to use the SR in writing the differences between angles and sides. Lesh et al. (2003) and Johnson (2018) have consistently confirmed that written ideas form part of system representation. Hence, it can be concluded that the learner struggled to illustrate the SR in operating within system representation. Since the learner could not recall the conceptual structures which assist in system representation, it became an issue to use the SO in operating within the sense (Pollitt et al., 2020). The difficulty to consider the sense arose from challenges in processing figures mentally against trigonometric concepts used in the question (Rankweteke, 2020; Gyan et al., 2021). Therefore, learners' trigonometric concepts such as angles and sides should be modified to use spatial skills together with the three tenets to differentiate sides and angles assuredly.

Finally, in terms of the category indolent, learner O also left a blank space on the question for differentiating angles and sides. This implies that the conceptual structure, system representation and sense were not shown in the answer sheet. During interviews, the learner was not interested in responding to the question asked. However, he ended up indicating that he does not know the differences between angles and sides which form part of indolent. Politely, I convinced the learner to give an example of an angle or a side. The learner said it is tricky but explained that 'when two lines combine, they form an angle'. The learner gave the correct answer during interviews using: 1) the SV to imagine relevant words to explain an angle; 2) the SR to connect the words; and 3) the SO to clarify the concept of angle. This was a way of illustrating the articulated properties of an angle (Mix et al., 2017). However, I was curious why the answer was not written in the answer book. Critically, the interviews confirmed that the learner left a blank space not because he does not know the answer but because he was not willing to think. The learners' confidence and spatial skills of responding to the questions made me notice that the learner does have a well-developed trigonometric concept. Well-developed trigonometric concepts form part of appropriate conceptual structures whereas the ability to unpack words of the differences forms part of system representation. As the learner managed to present the answer orally, sense was also considered. Weber (2008) argued that trigonometry is a topic that presents the interrelatedness between angles and sides of spatial figures that should be symbolised. As such, results from learner O’s responses support that the interrelatedness between angles and sides were well articulated verbally. Therefore, it cannot be concluded that learners leave blank spaces because they do not understand or do not know the answer.

5.3 Inappropriate procedures for computing sides and angles

In this theme, learners displayed unsuitable procedures for computing sides and angles. The literature review highlighted that learners tend to use inappropriate procedures due to various reasons. Some of the reasons were addressed by Khuzwayo (2019) as well as Faturohman and Amelia (2020), who indicated that learners confuse side names. This is in line with the results of the current study as some learners could not notice the relationship between the variables used to represent side names. Moreover, some learners used unsuitable procedures due to the following categories doubt, and failure to interpret various spatial figures. Hence, details of the inappropriate categories are documented below.

One of the categories of inappropriate procedures was demonstrated by learner D2 in question 2a and was coded **IS2.A** category inappropriate linkage of variables. For instance, the learner added the variables representing the three side names of a right-angled triangle. Fatmanissa et al. (2020) assert that failure to manage trigonometric schemas leads to difficulty computing sides and angles correctly. Similarly, Figure **4.3.a** indicates that learner D2 seemed to struggle to manage concepts involved in the procedure for computing the length of the opposite side labelled AB. Sánchez et al. (2023) simply referred to this difficulty as a struggle to recall the link between angles and sides, which assists in naming sides such as hypotenuse, opposite and adjacent. In Bell et al. (1983), the side names are referred to as conceptual structures. In addition, interpreting the difficulty using the semantic theory as well as the spatial skills shows that the learner seemed to lack the SV to use the conceptual structures mentioned above to interpret the given spatial figures. Martin-Fernández (2014) contends that the conceptual structures should assist learners to use system representation fluently. However, the existing conceptual structures of learner D2 were insufficient to apply the SR to proceed efficiently further to symbolic representation. This asserts that the symbolic representation was inappropriate, and Hasanah et al. (2022) claimed that it forms part of system representation. Moreover, the struggle to handle the SV and SR affected the SO to operate within the sense of making judgements of the written steps and concepts used during computations (Maknun et al., 2018). Nabie et al. (2018) maintained that learners are taught foundational concepts using procedural instruction without understanding how they work. Similarly, learner D2 might have used inappropriate procedures without understanding how the presented variables and BODMAS rules work. Moreover, during interviews, the learner indicated that trigonometry is complicated since it requires more knowledge from previous grades. The result is like that by Retnawati and Maulidya (2018), who pointed out that solving trigonometric problems is a challenge since does not involve ratios only but other basic concepts such as Pythagoras theorem. I would say that the Pythagoras theorem is a requirement for visualising the properties of a right-angled triangle which assists when dealing with ratios. Hence, the acquisition of the concepts above prevents earners from doubting their thoughts.

In contrast, results from Pfende et al. (2022) maintained that learners tend to write more than two procedures because of doubting their thoughts. The results are in accordance with responses illustrated in figure **4.3.b** by learner K, who wrote trigonometric ratios without indicating the reference angles and Pythagoras theorem. This is ample evidence that the learner could not really apply the SV to notice the appropriate equation to compute the required sides, meaning some of the existing conceptual structures were not relevant to the given question. Buczkowski (2003) purports that SR is regarded as a relationship as it deals with the organisation of visual information. In this context, I would say the learner could not use the SR to organise the given sides such that symbols on the Pythagoras theorem are well handled. During interviews, the learner emphasised the issue of not being sure. As a result, the learner could not provide proper justification because of having some doubts. Shapiro et al. (1997) viewed sense as the ability to use a variety of concepts to navigate between representations. However, learner K could not use relevant Pythagoras concepts to navigate between the symbolic representations while using the SO. This resulted from the application of inappropriate procedures.

Furthermore, during the process of computations on the second task, learners with inadequate conceptual structures tend to portray inappropriate procedures. On the question for writing side names, learner D2 made an internal representation using irrelevant SV to visualise and organise concepts on the given figure. As a result, the use of irrelevant SV triggered the learner to apply irrelevant SR to exhibit incorrectly written representations which are ratios instead of side names (Samsudin & Retnawati, 2018). Writing inappropriate procedures or responses could have been a result of having insufficient conceptual structures which is knowledge of a right-angled triangle. Fahrudin and Pramudya (2019) emphasised how the right-angled triangle is frequently used to explain side names involved in defining the three trigonometric ratios. Thus, having insufficient concepts of a right-angled triangle representation led to the inability to consider the sense to meditate between the trigonometric concepts used to represent the answer. The learner was interviewed to document more details about the written responses. The transcripts from interviews confirmed that the learner had difficulty linking the concepts used in the question such as side names, reference angles and variables. Correspondingly, the result is in line with that by Putuwwita and Retnawati (2021), who argued that learners have difficulties comprehending questions. However, the literature was silent on how the variables are affecting learners during computations of sides. Hence, teachers should have the skill of picking words when designing trigonometric questions such that spatial skills are well exhibited.

It is known that circumventing spatial skills contributes to inappropriate procedures as learners do not acknowledge the important concept within the given statements (Gagnier et al., 2022). In figure **4.11.b**, learner A also struggled to handle the numeric values of an angle and side used within the given statement. This was noticed as the learner chose an incorrect ratio and computed the size of an angle instead of the hypotenuse side. This means wrong written representations were shown. From the idea by Faturohman and Amelia (2020), as explained in the first theme, I would say learner A could not figure out when to compute a side or an angle. Although the spatial skills were circumvented while computing the hypotenuse side, during interviews they were indulged. Evidently, the learner was requested to provide the side names of the given spatial figure. By using the SV to recall and rotate properties of a right-angled triangle, these are called conceptual structures and SR to check their link, the learner managed to explain all side names. This is referred to as the ability to operate efficiently on the oral representation using the SO which is part of system representation. Since explanations were given while checking the written responses, the learner noticed that the side names and angles were swapped. Azizah et al. (2021) are some of the studies that outlined the importance of communication in trigonometry. Thus, giving the learner an opportunity to express her thoughts was proper to clearly understand her spatial skills as well as trigonometric concepts. Essentially, the learner would not have noticed the swapping if she was not given an opportunity to use the SO in operating within the sense. This indicates that the spatial skills utilised while operating within the written representation differ from those exhibited during interviews. Therefore, mathematical communication should be encouraged for learners to consolidate their trigonometric conceptual structures.

Dissimilarly, mathematical communication was not considered by learner U as unrequired spatial skills were exhibited while computing the hypotenuse side as shown in figure **4.11.c**. Initially, the SV was used to organise mental representation of the adjacent side and angle. However, the wrong SR was used to choose an incorrect ratio. Choosing incorrect ratios outlines that an individual lacks an understanding of how spatial figures should be visualised (Rankweteke, 2020). Additionally, a lack of SV to analyse spatial figures is a result of deprived internal representation created mentally and used to establish mathematical meaning (Komala & Suryadi, 2018) and this is referred to as insufficient conceptual structures. Consequently, the external representation was affected which are conventional representations such as symbols or equations (Azmidar & Husan; 2022; Goldin, 2001). Therefore, it is the responsibility of the teacher to assist the learner to develop knowledge of representations. This will greatly reduce the number of learners keeping quiet while asked questions based on their written responses.

Medaille and Usinger (2020) argued that some learners tend to keep quiet when asked questions concerning their written responses. For instance, some learners keep quiet because of fearing social judgement. On the last task, specifically, figure **4.17.a,** learner S used an inappropriate ratio but could not provide reasons for her choice. Contrary to the ideas mentioned, learner S did not keep quiet, due to social judgement because in the previous task, she tried to justify her written procedures. Hence, failure to justify the last task emanates from the difficulty of fully applying the SV to notice the given side and angle (Cam & Kilicer, 2022). As a result, the inappropriate SV influenced the learner to use the inappropriate SR to present answers. Furthermore, the literature review (Mahama & Kyeremeh, 2023) emphasised that learners should be able to represent answers symbolically as well as verbally. Most importantly, verbalising the written responses is marked as the ability to operate within the SO. Contrary to the idea mentioned above, learner S could not verbalise the written responses. Hence, the learners' SO remained doubtful because the written procedures were not justified. It can be concluded that the learner does not have enough spatial skills to visualise spatial figures attentively to access relevant concepts accurately.

Correspondingly, in solving the last three questions (2.1.1-2.1.3), learner J was bounded to use the SV to access the prerequisite details of applying a ratio to compute the required side. Due to the conceptual structures of sides and angles as well as inadequate SR, the learner used erroneous representation, which illustrates the vital idea of Mainali (2021), who states that it is improbable to operate within representation without the necessary concepts. Hence, applying the SO to demonstrate the sense to make relevant comments on the written procedures was difficult. However, the transcript from interviews also emphasised that learner J lacks trigonometric foundational concepts as the learner indicated that the written responses are incorrect. Therefore, it can be settled that a learner should develop foundational concepts to represent foundational concepts in multiple ways.

Contrary to the idea above, learner S is one of the participants who demonstrated lack of foundational concepts by adding an angle and a side. Mathematics is a subject that consists of various topics that are connected. Although the topics are connected, Rohimah and Probawanto (2019) established that most learners still suffer from a weakness in using the SV to connect ideas of the topics when solving the problem. This might be due to the absence of adequate prior knowledge which is part of prior conceptual structures. Additionally, the topics are learned in a sequential style such that the knowledge acquired in the prior topics can be applied in other topics using the SR. For instance, learners were introduced to algebra, where they are taught how to add like terms. Despite that the ideas are taught, mathematical studies revealed that learners tend to overlook the concept of grouping like-terms in other topics (Tastepe & Yanik, 2023). Similarly, learner S seemed to have a deficiency in adding like terms from algebra. This forms part of inappropriate system representation, specifically the symbolic. Algebra is a knowledge context called conceptual structures that would have assisted the learner to comprehend the BODMAS rule in relation to angles and sides. However, failure to comprehend the rules of adding like terms affected the current learning and the learner could not operate within the sense using the SO. Hence, it is the responsibility of teachers to strongly suggest strategies that would be interesting to learners and encourage them to acquire knowledge with understanding.

5.4 Fragmented trigonometric conceptual structures

Computing sides and angles is not easy as learners think; it requires proper conceptual structures and spatial skills. The conceptual structures are core elements of noticing concepts to be considered while representing procedures in computations of sides and angles. When extraneous concepts are used in presenting procedures, fragmentation is noticed. On the three tasks, learners presented various fragmented conceptual structures while differentiating between sides and angles as well as computations of sides and angles. These various fragmented conceptual structures are interpreted below.

Primarily, figure **4.4.a** indicates that learner C was aware that an angle is formed when two sides are combined. However, the concept of the side was not explained, and this forms part of **FS1** category ignorant. The transcripts from interviews confirm that learner C struggles to represent the statement in detail. That is the reason the saying "I do not know what to write" was made. The process of not knowing what to write indicates that the learner lacks the SV to access conceptual structures, which are properties of spatial figures (Maknum et al., 2022). Münzer (2018) asserts that learners with well-developed properties of spatial figures can make a mental rotation of angles and sides in a right-angled triangle. Also, they can differentiate between side names in response to reference angles. In contrast, learners with underdeveloped properties of spatial figures display fragmented conceptual structures. Likewise, learner C's responses confirmed an underdeveloped trigonometric conceptual structure because the SV was not used to explain the concept of side on the answer sheet and during interviews (Cam & Kilicer, 2022). Hence, the underdeveloped conceptual structures relate to fragmented conceptual structures, which forbid the learner to use the SR to express the relationship between sides and angles. Grounded on the literature on trigonometry, it can be concluded that the learner lacks the basics of a right-angled triangle to operate clearly on the written and oral representation (Ngcobo, 2019). This resulted in failure to use the SO to either modify or justify the written responses using conceptual structures as well as system representation, and this forms part of the inability to use the sense. Thus, failure to consider the conceptual structures prohibit learners from operating within the sense.

Conceptual structures were also not considered by learners B and E when differentiating angles and sides. Firstly, learner B indicated that when two lines combine, they form an angle of 90° or more. Secondly, learner E indicated that an angle adds up to 180° while sides are formed next to the angles. This statement lacks appropriate words for clarifying what an angle is, and they are coded FS1 category ignorant. From the knowledge of Cao and Ouyang (2019), it is known that angles range from 0° to 360°. Thus, it is inconsiderate to say an angle can be 90° or more. Most importantly, the transcripts from interviews confirmed that the two learners were not sure why angles were specified in the explanation. Also, the response confirmed that the collection of the concept on angles and sides was not good enough to recall relevant words, and this resulted in using inappropriate ones. The difficulty experienced by learners B and E opposes an important view expressed by Widada et al. (2019), which emphasised that recalling concepts assists learners to obtain the desired solutions. In this case, learners recalled concepts, but they were not relevant to the given question. Despite this, recalling the concepts clearly requires the SV to mentally imagine angles and sides on a spatial figure before representing their differences in the form of words. However, the slight conceptual structures, and knowledge of angles and sides prohibited the learner from considering the SV. Rittle-Johnson et al. (2019) highlighted the importance of the SV within the SR. Since learners B and E struggled to apply the SV, it became difficult to communicate and perceive the position of angles and sides. This is an absence of the SR. The absence of both SV and SR contributed to a failure to justify the written statement orally. This argument is consistent with that by Rocha et al. (2020), who argued that most learners lack the SV, SR and SO. It has been emphasised that learners lack the three spatial skills because teachers use the skills for testing learners' performance not the understanding of concepts. Therefore, it is the duty of the teacher to correct their classroom interactions to enhance learners' spatial skills to compute sides and angles easier.

Also, learners A and B also showed fragmented trigonometric concepts in computing sides due to lack of spatial skills. Learner A was expected to compute the value of the opposite side. In responding to the question, the learner substituted the numeric values of the given side interchangeably. This is coded FS2.B category unverified responses. From the work of DeReu (2019), learners were encouraged to discover the concepts involved in right-angled triangles. In contrast, learner A showed that the process of discovering concepts necessary for computations was an issue. This indicates that the learner cannot use the SV to notice and differentiate side names. Pollitt et al. (2020) postulate that the lack of the SV to access prerequisite knowledge such as side names prohibited the learner from effectively and essentially computing sides. Similarly, in figure 4.5.a, learner A struggled to use the SR to compute the required side due to confusion of side names. The result is in accordance with that by Faturohman and Amelia (2020) as well as Khuzwayo (2019), stating that learners experience difficulties in distinguishing side names. Interpreting the results using semantic theory, the difficulty arose from irrelevant conceptual structures such as differentiating side names of right-angled triangles (Molataola, 2017; Nader, 2021). Subsequently, the challenge of differentiating side names greatly affected the symbolic representation of the Pythagoras theorem as the value of the hypotenuse was substituted as adjacent. The inconsiderate symbolic representation forms part of system representation within the semantic theory (Martin-Fernández et al., 2016). Thus, the SO was not used to operate within the sense to verify the link between the numeric values of the side names and their symbolic representation on the figure and equation. Even though learner A illustrated the irrelevant conceptual structures, system representation and sense on the answer sheet, relevant ones were shown during interviews whereas learner B constantly verbalised side names interchangeably. The learner noticed mistakes made during the computations of the hypotenuse and corrected them during interviews. Therefore, learner A’s response is ample evidence that displaying fragmented conceptual structures on the scripts does not mean that a learner is not enabled in computations as answers can be expressed efficiently and correctly verbally.

Even though some learners can correct their mistakes verbally, other fragmented trigonometric conceptual structures are recognised when learners use spatial skills to access concepts incompletely or incorrectly. Learner A is one of the participants to display fragmented conceptual structures. In particular, the learner was expected to define trigonometric ratios using the given right-angled triangle. However, the three ratios were defined without using the given spatial figure and it is coded FS.1.1. A. II category inability to define the ratio using variables. This shows that the learner could not respond to the question precisely. Although the answers were presented incompletely, the learner tried to provide an understanding of side names as explained in the previous paragraph. Moreover, the interviews attest that learner A only struggled to use spatial skills to grasp the core idea of the words used in the question. Suri et al. (2021) conducted a study emphasising that learners should develop skills for understanding questions. In the current study, the learner showed evidence of lacking the compulsory SV to read the question thoughtfully to gain insight into it. The difficulty to read the question thoughtfully affected the learners' SR to choose the relevant conceptual structures for representing side names accurately (Buczkowski, 2003). In terms of representation, the learner could not use the SR to operate within the internal representation to establish the meaning of side names in reference to θ (Komala & Suryadi, 2018). As a result, to mentally navigate between conceptual structures and the representation remained a challenge since the SO was not shown (Marufi et al 2022). This indicates that it is inconsiderate to compute sides and angles without taping the semantic tenets and conceptual structures such as side names in reference to angles.

According to Williams (2019), trigonometry focuses on studying the relationship between angles and sides of spatial figures. From learner R's responses on figure 4.12.c, I would say the SV was inadequate to establish the relationship between sides and angles. Consequently, the learner used the incorrect SR to write the inapt name of the side labelled AB, which affected ways of using ratios to compute sides and angles. This attests that the learner operated inaptly within the symbolic representation, and this is coded FS1.1A. II category wrong representation (Hasanah et al., 2022). This is consistent with the argument by DeReu (2019), which states that learners experience difficulties in expressing and managing their knowledge of representation. Although the learner wrote an incorrect answer, during interviews correct side names were given. I would say the learner somehow found it easier to operate within the verbal than written representation (Mainali, 2021). An ample of evidence that the learner seemed to struggle with the written representation was shown as the learner clarified that it was difficult to write side names. Therefore, it is certain to say that the learners competently considered the sense because correct answers were given during interviews, and this forms part of SO.

Moreover, in figure 4.12.d, learner U illustrated a skill to comprehend the given statement but could not figure out all essential concepts of defining the ratios. This is coded **FS1.1AII** category incomplete ratio. Translating the responses using semantic theory, I would say that the learner has little conceptual structures of ratios and this prohibited her from using the SV (Capraro et al., 2021). The little conceptual structure innovation led to the inessential use of SR in representing the ratios fluently. Scholars outlined that trigonometry entails various concepts learned previously (Fiorella & Mayer, 2015). Most importantly, these concepts also assist in ensuring that SO is applied while providing an explanation of the written procedures (Edler & Kersten, 2021). Regrettably, when the previous concepts are not understood, they create a knowledge gap in the recent learning of trigonometry (Nanmumpuni & Retnawati, 2021). For instance, in this case, the learner struggled to compute sides and angles due to difficulties to define concepts, angles and sides, and this brought more issues on other questions. This is a result of having insufficient conceptual structures which led to wrong representation. As a result, operating within the SO to verify written procedures was prohibited. Therefore, learners should be assisted to mentally imagine spatial figures during computations.

It is crucial to imagine the position of spatial figures as it assists learners to arrange their thinking. The arranged thinking must be expressed clearly using the language of mathematics. In figure **4.4.d**, learner E seems to have an idea of what an angle and a side is but could not express the ideas in the language of mathematics. The learner was required to elaborate on the given explanation, and she kept quiet, this is coded FS1 category inarticulation. As a result, the learner was given an opportunity to elaborate using the home language. Firstly, the learner used the SV to imagine what an angle and a side are (Münzer, 2018). Secondly, the learner used the SR to organise her thoughts (Rocha et al., 2022). As a result, both concepts were explained. This indicates that the learner does have grounded conceptual structures of sides and angles, which led to a great deal of oral representation, yet struggled to use the language of mathematics to express those thoughts. Lastly, the learner could give justification for her answers, which is referred to as the SO using her home language (Edler & Kersten, 2021). The DBE (2012) in South Africa introduced language policy for both teachers and learners in the learning process. The language policy was introduced to promote an understanding of mathematical concepts. Contrary to the idea above, transcripts from interviews confirmed that learner E cannot use the three spatial skills to exhibit mathematical language on how trigonometric concepts are connected in early grades. Hence, teachers should assist learners to acquire spatial skills such that trigonometric concepts can be used to connect and express trigonometric ideas.

Finally, computations of sides and angles require the SV to outline given details that assist to decide on the relevant ratio. In question 1.2.a and b, learner F could not differentiate side names. The result is in accordance with those by Faturohman and Amelia (2020), who emphasised that their participants had difficulties distinguishing sides when given trigonometric ratios to compute the indefinite side. This might be a result of lacking the SV to access the properties of a right-angled triangle. Similarly, Maknun et al. (2018) explored participants' difficulties in differentiating side names. The knowledge of differentiating side names is considered an insufficient conceptual structure. The insufficient conceptual structures could not allow the learner to efficiently concentrate after choosing the relevant ratio. This relates to failure to operate within the system representation. Although a relevant ratio was chosen, the participant could not use the SR to recognise an opposite and adjacent side. Moreover, the learner could not manage the BODMAS rule to obtain the length of the adjacent instead of the hypotenuse side.

5.5 Appropriate procedures for computations of sides and angles

Applying appropriate procedures is an illustration of being able to operate fluently within the spatial skills in the learning of trigonometry. Learners B2 and J are some of the participants who illustrated appropriate procedures during computations. For instance, learner B2 spatially visualised the given right-angled triangle using the trigonometric concepts of side names (Fujita et al., 2020). Fundamentally, the concepts together with the SV aided the learner to notice that the numeric values of opposite as well as adjacent sides were provided. These concepts fostered the learner to recognise that hypotenuse is the missing side. Based on knowledge of the three spatial skills from Battista (2018) and trigonometric knowledge from Ssebaggala (2019), the ability to identify the given sides enabled the learner to apply SR in seeing the Pythagoras theorem relevant to compute the missing side. The equation was written correctly, and all values were substituted while handling the BODMAS rule efficiently. To be precise, the equation is usually part of the symbolic representation, and Johnson (2018) specified it as a system representation. Hence, the ability to use the BODMAS rule and handle fractions as emphasised by Mahama and Kyeremeh (2023) on the equation led to the correct answer.

Also, during interviews, learner B2 indicated that a right-angled triangle was given with the length of two sides and the missing side can be obtained using the Pythagoras theorem. The learner indicated that $r^2 = x^2 + y^2$ cannot be used, but lately modified her statement by saying the equation is still correct. This indicated that borrowing concepts from other topics is not a matter if they are handled in relation to the given problem. The response forms part of the SO as the learner justified the written procedures using relevant properties of a right-angled triangle (Tartre, 1990). Thus, if learners can have this prerequisite knowledge, solving trigonometric problems will be easier.

The ability to illustrate appropriate procedures relies on how spatial skills are used to access trigonometric concepts. Rocha et al. (2022) is one of the authors to emphasise that spatial skills are effectively influencing learners' trigonometric concepts. Evidently, learner H illustrated how spatial skills contribute to the computations of sides. The learner used relevant SV to recognise the given sides and angles (Jaelani, 2021). Simultaneously, utilising the SV triggered the learner's SR to choose the cosine ratio to compute the hypotenuse side. This is a result of being capable of noticing the concept of the hypotenuse, adjacent, opposite and reference angles and how the concepts are related (Ngcobo, 2019). In addition, the learner operated within the SO as it was easier to elaborate procedures on how the hypotenuse side was obtained. This is referred to as the skill to mentally navigate between the concepts of side names and how they are linked to symbolic representation on the cosine ratio. Hence, it can be concluded that the learner considered the SO since mental manipulations were made and presented verbally. This is a strategy of knowing when and how to apply spatial skills.

Similarly, learner D is one of the participants of knowing when to apply spatial skills on the answer sheet and partially during interviews. The learner used the SV to outline the given information that assists in choosing the right equation, namely, Pythagoras. This is considered a conceptual structure (Martín Fernández, 2021). Most importantly, the learner used the SR to substitute the outlined details from the given spatial figures. Making the substitutions is a great demonstration of operating with the system representation. Moreover, demonstrating precise representation allowed the learner to play a vital role in obtaining the correct answer (Suningsih & Istiani, 2021). Certainly, obtaining the correct value of the side enabled the learner to compute the required angle which is E. Precisely, the learner noticed that the answers obtained in the previous questions should be used to compute the required side. On the turn for demonstrating an understanding of what has been written, learner D could respond to few questions but could not provide more details when the probing technique was applied. The result concurs with those of Fernandez-Baizan et al. (2021), who state that some learners do solve problems but find it challenging to elaborate on how the answers were obtained. Similarly, learner D managed to obtain the required answer but could not operate within the SO fully to explain how the answers were obtained. This is an indication that there are learners who are capable of operating within the written representation only and obtaining correct answers but cannot consider the verbal representation. Therefore, it is vital to learn how to represent trigonometric concepts in numerous representations.

Additionally, learner P also managed to demonstrate appropriate procedures for computing sides and angles when given four spatial figures. Even though the learners demonstrated appropriate procedures in questions 2.1.1-2.1.3, interviews were also conducted to check whether the appropriate procedures are understood or not. Specifically, learner P managed to use the SV to rotate and recall side names in specific right-angled triangles (Corcoran et al., 2012). The SR was used to organise the concepts relevant for computations. Thereafter, explanations were given to illustrate that the specific concepts and BODMAS rule were used logically to obtain desired solutions. Hurrell (2021) confirmed that providing explanations forms part of SO. Hence, the justification showed that the learner can handle the conceptual structures such as recognising concepts for computation; representation as a way of supporting the recognised concepts; and the sense, navigating with the concepts and the representation.

5.6 Synthesis

Chapter five presents an interpretation of the results documented from the answer book and semi-structured interviews. The interpretation was categorised into four themes for better coordination. Even though the results were interpreted within the four categories, the difficulties that hindered learners to compute sides and angles were explored using the three spatial skills, literature review and the semantic theoretical framework. Across the four themes, some learners encountered difficulties providing written representation, especially in questions without spatial figures. Moreover, learners displayed difficulty using spatial skills in accessing concepts that assist during computations of sides and angles. This is an indication of the importance of having relevant prior knowledge and sufficient spatial skills to access it. On the other hand, learners with a habit of not verifying written procedures were noticed as some of the fragmented conceptual structures were corrected. The difficulties clearly outlined that there is a pattern with the four themes.

5.7 Chapter Summary

The chapter presented an interpretation of data acquired from three mathematical tasks and semi-structured interviews. Four themes emerged from the two sets of data. As a result, an interpretation was shown for each theme. This was a way of explaining the meaning of the spatial skills exhibited by Grade 10 learners during computations of sides and angles. Hence, the next chapter discusses the principal findings and recommendations of the study.

CHAPTER SIX: CONCLUSION AND RECOMMENDATION

6.1 Introduction

The study explored Grade 10 learners' spatial skills in computing sides and angles of figures using trigonometric ratios. Numerous studies explained the importance of spatial skills (Açikgül et al. 2023; Dhlamini et al., 2019; Yang et al., 2020), however, they were not in relation to the computation of sides and angles using trigonometric ratios. In this study, spatial skills are core elements for computing sides and angles using trigonometric ratios. Despite the importance mentioned above, learners face challenges in using spatial skills to access concepts such as side names that assist in analysing figures to outline key elements for computing sides and angles. The challenges of computing sides and angles have been discussed by a great number of authors (Nurmeidina & Rafidiyah, 2019; Walsh et al., 2017). However, their methods and techniques for carrying out their study were different from the current study. Essentially, the uniqueness of this study is that the challenges exhibited by Grade 10 learners were analysed thematically and interpreted using the three tenets of semantic theory, including conceptual structure, system representation and sense together with the literature. To provide a synoptic conclusion of this study, the chapter summarises the research design and methods, principal findings of the study, answers to research questions, recommendations, contributions and limitations of the study.

6.2 Reflection on the research design and Methods

This is a qualitative study that used an exploratory case study depicted by Merriam to comprehend how learners exhibited spatial skills during the computations of sides and angles. An exploratory case study was appropriate for this study as it enabled me to identify and comprehend various spatial skills exhibited during computations. To be precise, the study asked two research questions: 1) how do learners' abilities of computing sides and angles of figures reflect their spatial skills? 2) Why do learners compute sides and angles of figures in particular ways? In responding to the first research question, three mathematical tasks were used to document various learners' spatial abilities. Thereafter, semi-structured interviews were conducted on each task for learners to justify their written responses using trigonometric concepts. In a study, two or more methods of collecting data should be used to maintain triangulation. In the current study, mathematical tasks and semi-structured interviews were used to sustain triangulation. As such, this was a strategy for answering the how and why research questions. Furthermore, the two sets of data were analysed thematically through coding, and four themes emerged. Within the four themes, learners' difficulties and abilities were interpreted using semantic theory as well as literature. All the research designs and methods of data collection and analysis are qualitatively relevant to this exploratory study.

6.3 Interpretation of research findings

In this subheading, the study briefly described the principal findings and answers to the two research questions.

6.3.1 Principal findings of the study

The current study outlined the following findings:

6.3.1.1 Conceptual structures of learners who left blank spaces

Leaving blank spaces has been interpreted differently in various studies. In the current study, all learners who left blank spaces were interviewed to document reasons for leaving the spaces. Excitedly, the transcripts confirmed that learners left blank spaces for various reasons. Firstly, some learners left blank spaces due to being clueless, knowing exactly what the question requires but struggling to make mental manipulations or mental rotations of figures such that they can easily select and organise a set of concepts to respond to the given question. The difficulty led to the failure to illustrate procedures for computing sides, which refers to the inability to operate within the symbolic representation. Secondly, some learners struggled to link concepts, side names in reference to together with ratios on the given question and spatial figures. Linking the concepts requires a skill to access keywords that are relevant to computations of sides and angles in various contexts. However, findings highlighted that learners struggled to access the relevant concepts to obtain the desired solutions. Thirdly, some learners were ignorant, lacking a collection of concepts that assist during computations of sides and angles. Although the learners were ignorant to provide exact responses, giving some examples of sides and angles was conceivable. This indicated that the recalled basics were insufficient to oblige computations of sides and angles. Fourthly, some learners confirmed that when given questions related to computations of sides and angles, without spatial figures, they tend to avoid the process of linking concepts, rotating spatial figures and relating questions due to being indolent. Lastly, some learners experienced difficulties in expressing answers in mathematical language. This is referred to as inarticulation. Although utilising mathematical language was difficult, some learners managed to tap into the required concepts relevant to the given question. All these findings confirmed that leaving blank spaces does not mean learners are blank. Hence, it is advisable for teachers to friendly request learners to explain reasons for leaving blank spaces.

6.3.1.2 Lacking skills to access relevant concepts during computations

During computations, most learners had difficulties using their spatial skills to access and make mental rotations of the required concepts. As a result, learners had trouble analysing the given questions and spatial figures. Failure to analyse the given question and spatial figures resulted in difficulties in computing sides and angles. Thus, learners with irrelevant spatial skills to mentally access, manage, manipulate and analyse properties of a right-angled triangle experienced a challenge to justify their written responses.

In addition, the responses showed that feeble spatial skills to develop the concept of computations using ratios were negatively affected by traditional teaching methods employed in prior grades. This was noticed as learners encountered the following challenges: confusing side names, choosing correct ratios but substituting numeric values interchangeably and choosing incorrect ratios during the computation of sides and angles. Thus, classroom instructions are viewed as components that negatively influenced spatial skills for computing sides and angles.

6.3.1.3 Over-reliance on spatial figures

The set of concepts and spatial skills should be appropriate to compute sides and angles given in various representations such as words, symbols and pictures. In this study, some learners showed interest in solving problems given in the form of figures and pictorial representation. However, when a mental manipulation and rotation of spatial figures were required to respond to questions given without spatial figures, learners found it difficult to answer and provide reasons for indicating that the properties of ratios are well acquired. Mostly, those that could solve spatial problems only showed evidence of using memorised procedures that were not clear to them. Hence, it is not appropriate to rely on spatial figures only when computing sides and angles.

6.3.1.4 Difficulty to recognise contextualised concepts in questions

Grade 10 learners encountered challenges in terms of recognising the meaning of the contextualised concepts used in the question. Some words are used in various contexts. When used within the concept of computations, learners tend to get confused. Moreover, the contextualised concepts led to the inability to notice the relationship between sides and angles on ratios. Hence, it is a good idea to design questions in such a way that concepts are easier to visualise and manage them.

6.3.1.5 Verbal representations assist in taping into appropriate conceptual structures

It is essential to engage learners in a conversation based on their written procedures. For this reason, some learners have appropriate conceptual structures but tend to display inappropriate and fragmented conceptual structures. Similarly, in this study, findings confirmed that inappropriate procedures such as choosing incorrect ratios and fragmented conceptual structures such as substituting numeric values interchangeably, and handling the BODMAS rule incorrectly were exhibited in the answer book. However, some were noticed and corrected during interviews. This attests that displaying inappropriate procedures and fragmented conceptual structures on the scripts does not mean that a learner cannot compute sides or angles. Hence, it can be concluded that verbal representation assists learners to tap into appropriate conceptual structures.

6.3.1.6 Inability to justify appropriate computations

During the process of computations, some learners could recognise the given side names or angles. Thereafter, the recognised details allowed them to choose the correct ratios to compute sides or angles. During computations, the BODMAS rule was handled efficiently, and desired solutions were obtained. Although desired solutions were obtained, making justification of their written procedures through recalling and organising relevant trigonometric concepts remained an issue.

6.3.2 Research questions of the study

In responding to the first question: how do learners' abilities of computing sides and angles of figures reflect their spatial skills? Written responses on computations of sides and angles designated the deficiency of spatial skills, which assist in manipulating the required concepts. The procedures as well as reasons provided during computations clearly showed that learners' spatial skills restricted them to flexibly compute sides and angles given in a variety of contexts. An ample of evidence was shown as some learners could not illustrate their spatial skills in the answer book but during interviews. Although most learners still displayed deficient spatial skills in answer books, some learners displayed sufficient ones as they were able to respond to both spatial and conceptual problems in the form of written and verbal representations. Hence, a skill to assist affected learners is required to improve the process of computing sides and angles using ratios.

Moreover, in responding to the second research question: why do learners compute sides and angles of figures in particular ways? Learners computed sides and angles in particular ways due to a lack of spatial skills to access relevant pre-existing concepts and methods of naming sides in reference to angles. Their pre-existing concepts and methods affect their capability to illustrate spatial skills on how to compute sides and angles using ratios. Proper language should be considered while responding to questions in mathematics. However, the principal finding indicated that some learners do have a strategy to compute sides and angles but spatial skills to articulate answers precisely using the mathematical language was an issue. Moreover, the finding of the study also made me aware that some learners are not willing to demonstrate their spatial skills during computations, whereas some struggled to recognise the relationship between the words used in each question. Furthermore, it has been documented that most learners tend to compute sides and angles in particular ways due to over-reliance on spatial figures, meaning if a statement contains necessary numeric values to carry out computations, learners struggled to notice what the question required as a spatial figure is not provided. Some learners were more capable of articulating responses verbally than in the form of writing. This confirmed that learners have different skills for exhibiting their procedures during computations. Lastly, it has shown that they can use spatial skills to access relevant concepts during computations but find it challenging to justify their appropriate procedures. All these reasons were documented after analysing and interpreting both sets of data.

6.4 Recommendations

The study findings provided an overview of the following steps to be taken to improve the learning of trigonometry in Grade 10.

- Learners should be assisted to tap into relevant learners' prior knowledge to notice the link between properties of a right-angled triangle and the given computation questions.
- Chosen classroom instructions should be learner-centered for learners to discover the necessary conceptual structures required during computations of sides and angles using spatial skills. This will be an approach to activating learners' participation mode.

- The study also suggests that more studies should be conducted on the exploration of spatial skills in trigonometry. This will be a way of documenting more difficulties that avert learners to develop key elements of moving flexibly between the three spatial skills.
- The process of assessment is mostly in the form of written representations. This study confirmed that learners express mathematical ideas differently. For instance, some learners are better at verbal representation than written representation. Therefore, I suggest that schools should implement a mathematical policy such that there is a verbal assessment to accommodate learners who tend to leave blank spaces due to failure to operate within the written representation.

6.5 Contributions of the study

Firstly, this study extends the limited research on the exploration of Grade 10 learners' spatial skills in the computation of sides and angles using basic ratios. The study is among the first to consider spatial skills in trigonometry as they are mostly used in Euclidean geometry. In this manner, the study contributes essential steps of using the three spatial skills during the computations of sides and angles using basic ratios. Secondly, the study also adds literature on how spatial skills are exhibited by learners who left blank spaces. In this instance, learners exhibited their spatial skills verbally. Some were correct whereas others indicated a lack of conceptual structures, indolence, inarticulation, etc. Lastly, semantic theory has been found to be a compelling theoretical framework as it links with spatial skills. For instance, the issue of spatially visualising (SV) figures necessities trigonometric concepts which form part of the conceptual structures in the theory. The use of the SR to organise the visualised concepts forms part of the system representation as symbols are used during computations. Also, the use of SO ensures that there is a link between the visualised concepts and the written symbols, and this forms part of the sense. This clearly shows that theory assists to scrutinise the level at which a learner is operating while using spatial skills.

6.6 Limitations of the Study

The current study narrowly concentrated on the spatial skills to compute sides and angles presented in various forms. This was a way of excluding a variety of abilities to compute sides and angles such as problem-solving, critical thinking and mathematical proficiency. Moreover, sides and angles can be computed using trigonometric reciprocals, general solution, and the sine rule. The current study firmly urged learners to employ the three basic trigonometric ratios. Although learners were given some questions that required the use of Pythagoras' theorem, the use of the three basic ratios remained an essence for computations. This shows that questions that required the Pythagoras theorem were used such that learners can discover essential concepts, the relationship between sides on a right-angled triangle in reference while using ratios properties required.

6.7 Conclusion

The purpose of the study was to explore Grade 10 learners' spatial skills in computing sides and angles of figures using trigonometric ratios. To achieve this, an exploratory design was used. The design helped in comprehending Grade 10 learners' spatial skills in computing sides and angles using basic ratios. As a result, semantic theory was proper for exploring their spatial skills. The results of the study indicated that learners leave blank spaces for specific reasons, not willing to mentally organise concepts and struggle to link concepts to the given question. In terms of computations, irrelevant conceptual structures as well as fragmented concepts are utilised during the computations of sides and angles. Findings indicated that learners leave blank spaces mostly on the conceptual question and questions involving the interpretation of various spatial figures. This was found as learners struggled to access concepts relevant to the given question and use mathematical language to express ideas. Moreover, findings highlighted that learners use memorised equations, whereas others struggle to handle the concepts of side names and the BODMAS rule after choosing the correct ratio. Hence, I conclude that learners should be assisted to use spatial skills accurately to access trigonometric concepts while computing sides and angles on both spatial and conceptual problems. Also, learners should be motivated to justify their written procedures as a way of moving flexibly between spatial skills. Lastly, schools should also consider verbal assessment in mathematical classrooms to accommodate learners with difficulties exhibiting spatial skills in written assessments.

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8 APPENDICES

Appendix A: Approval from the University

University of Limpopo
Department of Research Administration and Development
Private Bag X1106, Sovenga, 0727, South Africa
Tel: (015) 268 3935, Fax: (015) 268 2306, Email: anastasia.ngobe@ul.ac.za

TURFLOOP RESEARCH ETHICS COMMITTEE

ETHICS CLEARANCE CERTIFICATE

MEETING: **23 May 2022**

PROJECT NUMBER: **TREC/83/2022: PG**

<u>PROJECT:</u>

Title:	Exploring Grade 10 Learners' Spatial Skills of Computing Sides and Angles in Figures Using Trigonometric Ratios.
Researcher:	R Selowa
Supervisor:	Dr ZB Dhlamini
Co-Supervisor/s:	N/A
School:	Education
Degree:	Master of Education in Mathematics Education

PROF D MAPOSA
CHAIRPERSON: TURFLOOP RESEARCH ETHICS COMMITTEE

The Turfloop Research Ethics Committee (TREC) is registered with the National Health Research Ethics Council, Registration Number: **REC-0310111-031**

Note:

i) **This Ethics Clearance Certificate will be valid for one (1) year, as from the abovementioned date. Application for annual renewal (or annual review) need to be received by TREC one month before lapse of this period.**

ii) **Should any departure be contemplated from the research procedure as approved, the researcher(s) must re-submit the protocol to the committee, together with the Application for Amendment form.**

iii) **PLEASE QUOTE THE PROTOCOL NUMBER IN ALL ENQUIRIES.**

Finding solutions for Africa

Appendix A (1): Approval of Proposal

University of Limpopo
Faculty of Humanities
Executive Dean
Private Bag X1106, Sovenga, 0727, South Africa
Tel: (015) 268 4895, Fax: (015) 268 3425, Email:Satsope.maoto@ul.ac.za

DATE: 22 March 2022

NAME OF STUDENT: **SELOWA, R**
STUDENT NUMBER: **[201648810]**
DEPARTMENT: **MEd – Mathematics Education**
SCHOOL: **Education**

Dear Student

FACULTY APPROVAL OF PROPOSAL (PROPOSAL NO. FHDC2022/2/02)

I have pleasure in informing you that your MEd proposal served at the Faculty Higher Degrees Meeting on 22 February 2022 and your title was approved as follows:

TITLE: Exploring Grade 10 learners' spatial skills of computing sides and angles in figures using trigonometric ratios

Note the following:

Ethical Clearance	**Tick One**
In principle the study requires no ethical clearance, but will need a TREC permission letter before proceeding with the study	
Requires ethical clearance (Human) (TREC) (apply online) Proceed with the study only after receipt of ethical clearance certificate	√
Requires ethical clearance (Animal) (AREC) Proceed with the study only after receipt of ethical clearance certificate	

Yours faithfully

Maoto RS

Prof RS Maoto,
Executive Dean: Faculty of Humanities
Director: Prof MW Maruma
Supervisor: Dr ZB Dhlamini

Finding solutions for Africa

Appendix B: Letter seeking consent from the Department of Education: Limpopo

Selowa R
P O BOX 2571
Tzaneen
0850
16 May 2022

Limpopo Head office
Mafarana circuit
Private Bag X9489
Polokwane
0700

Dear sir/ madam

REQUEST FOR PERMISSION TO CONDUCT A STUDY AT MAFARANA CIRCUIT UNDER MOPANI WEST

1. The above letter has a reference.
2. I, SELOWA R, request the Department of Basic Education Limpopo province for permission to conduct research at a high school around Mafarana circuit under Mopani west, and the title is "**Exploring Grade 10 learners' spatial skills in computing sides and angles using trigonometric ratios**".
3. The research will be conducted after hours to avoid disrupting the program of the school.
4. The research process will involve Grade 10 learners by collecting their responses from the mathematical task and semi-structured.
5. This research will be done under the guidance of Dr Dhlamini from the University of Limpopo.
6. Hope the above is in order.

Kind regards
SELOWA R
0659820685
selowareinhard@gmail.com

Appendix C: Letter of approval: Department of Education: Limpopo Province

LIMPOPO
PROVINCIAL GOVERNMENT
REPUBLIC OF SOUTH AFRICA

DEPARTMENT OF
EDUCATION

CONFIDENTIAL

Ref: 2/2/2 Enq: Makola MC Tel No: 015 290 9448 E-mail:MakolaMC@edu.limpopo.gov.za

Selowa R

Private bag X 1106
Sovenga
0727

RE: REQUEST FOR PERMISSION TO CONDUCT RESEARCH

1. The above bears reference.

2. The Department wishes to inform you that your request to conduct research has been approved. Topic of the research proposal: **" EXPLORING GRADE 10 LEARNERS' SPATIAL SKILLS OF COMPUTING SIDES AND ANGLES IN FIGURES USING TRIGONOMETRIC RATIOS"**

3. The following conditions should be considered:

3.1 The research should not have any financial implications for Limpopo Department of Education.

3.2 Arrangements should be made with the Circuit Office and the School concerned.

3.3 The conduct of research should not in anyhow disrupt the academic programs at the schools.

3.4 The research should not be conducted during the time of Examinations especially the fourth term.

3.5 During the study, applicable research ethics should be adhered to; in particular the principle of voluntary participation (the people involved should be respected).

3.6 Upon completion of research study, the researcher shall share the final product of the research with the Department.

Cnr 113 Biccard & 24 Excelsior Street, POLOKWANE, 0700, Private Bag X 9489, Polokwane, 0700
Tel:015 290 7600/ 7702 Fax 086 218 0560

The heartland of Southern Africa-development is about people

4 Furthermore, you are expected to produce this letter at Schools/ Offices where you intend conducting your research as an evidence that you are permitted to conduct the research.

5 The department appreciates the contribution that you wish to make and wishes you success in your investigation.

Best wishes.

19/04/2023

Mashaba KM
DDG: CORPORATE SERVICES

Date

Cnr 113 Biccard & 24 Excelsior Street, POLOKWANE, 0700, Private Bag X 9489, Polokwane, 0700
Tel:015 290 7600/ 7702 Fax 086 218 0560

The heartland of Southern Africa-development is about people

Appendix D: Assessment of data collection instrument

MATHEMATICAL TASK 1

Data collection tool

MATHEMATICAL TASK

Introduction

Question 1

1. Differentiate between angle and sides. (2)

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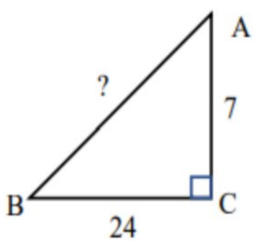
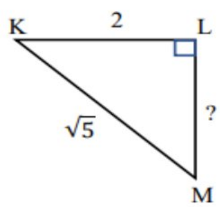
2. Determine the length of the missing sides:

a)



b)



(6)

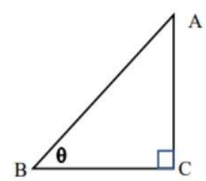
MATHEMATICAL TASK 2

Main lesson

Question 1

1. Trigonometry is a section of mathematics that concentrate on the inter-relationship between the length of sides and sizes of the angles in a triangle.

1.1. a) Study the diagram below and answer the questions that follow:



i. Write side names (AB, AC, and BC) in reference to θ

(3)

……………………………………………………………………………………………..

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ii. Use the right-angled triangle above to define three trigonometric ratios.

(3)

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b. If AC=3 cm, BC=2 cm.

i. Determine AB (3)

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ii. Use the three values to define trigonometric ratios. (3)

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iii. Draw a conclusion based on the relationship between the length of sides and sizes of an angle. (2)

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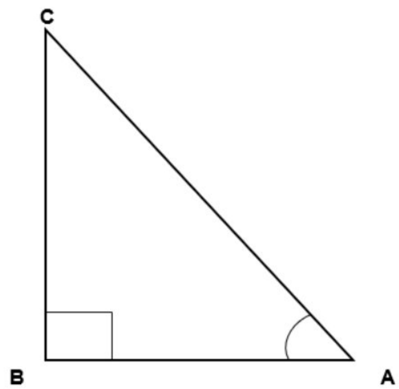
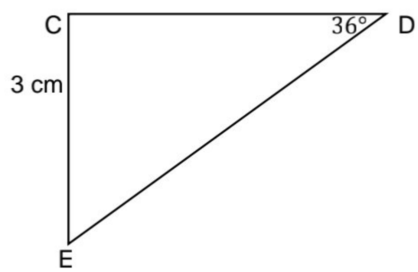
1.2. Determine the length of **AC** for the diagrams below:

a) If AB= 6 cm and $\hat{A}$= 49° (3)



b) Calculate the length of is CD. (3)

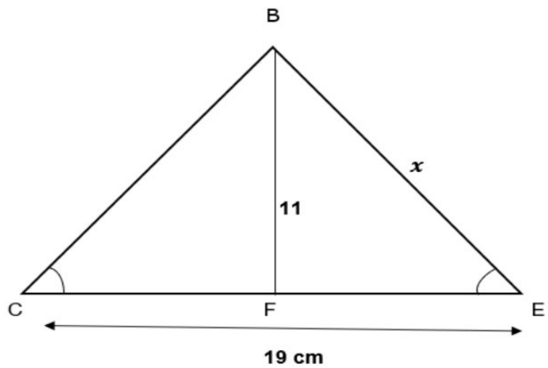


TOTAL: 20

MATHEMATICAL TASK 3

QUESTION 1

1.1. use ΔBFE to determine the value of:



a) x (3)

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…………………………………………………………………………………………………
…………………………………………………………………………………………………
…………………………………………………………………………………………………

b) $\hat{E}$ (3)

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…………………………………………………………………………………………………
…………………………………………………………………………………………………

1.2. Use ΔBFC to calculate:

a) CF (3)

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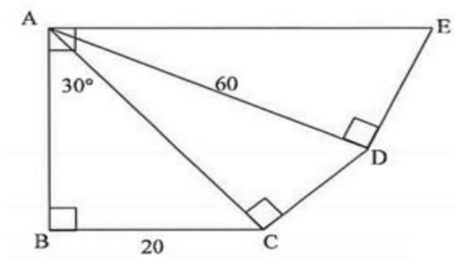
2. In the diagram below, ABC, ACD, and ADE are right-angled triangles. $B\hat{A}E = 90°$ and $B\hat{A}E = 30°$. BC= 20 cm and AD= 60 cm.



2.1. Calculate

2.1.1. Length of AC. (2)

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2.1.2. Size of $C\hat{A}D$ (2)

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2.1.3. Length of DE (3)

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TOTAL: 16

Appendix E: Semi-structured interview schedules for learners

Semi-structured interview schedule for learners

Focus area	Questions and probes
Introduction about the researcher	I am Selowa Reinhard, a master's student from the University of Limpopo. I am conducting research to explore Grade 10 learners' spatial skills in calculating unknown sides and angles of triangles using trigonometric ratios. I would like to ask you questions based on how you responded to the given mathematical task. Would you like to participate?
Computation of sides	➢ Mention the side names you have used to calculate the unknown sides. ➢ How did you know that the given sides are adjacent, opposite or hypotenuse? ➢ What helped you to conclude that the given sides are adjacent or opposite? ➢ Based on your understanding, how are these sides of the given shape related? ➢ Why did you choose $\sin\theta$, $cos\theta\ or\ \tan\theta$ to calculate the unknown side?
Computation of angles	➢ Can you explain how you calculated the unknown angle? ➢ You wrote sides' names before calculating unknown angles. Could you please explain why?

	➢ It is necessary to write the side names down every time when you are asked to calculate unknown angles?
Computation of sides and angles	The question says calculate unknown sides and angle, but you are a given one diagram with more than one triangle. What did you do to identify side names and angles for each triangle? Why are you identifying sides names like that? Do you think is a good idea to identify side names like that? How did you come to this conclusion? Explain how used the information you identified on the diagram to calculate unknown sides and angles.

THE END

Appendix F: Consent form

Consent form

Dear respondent

I MS Selowa Reinhard, student number: 201648810 am a postgraduate doing a master's degree in mathematics in the Department of Education at the University of Limpopo. I am conducting research to explore Grade 10 learners' spatial skills when computing sides and angles of figures using trigonometric ratios. I would like you to sign and return the consent form together with your parent if you agree to participate in this proposed study. This proposed study will therefore assist in developing learners' spatial skills and trigonometric knowledge required in the computation of sides and angles. The data will be collected using mathematical tasks and semi-structured interviews and will be stored in a locked room to protect the privacy of participants. Bear in mind that your participation is voluntary, and you are free to discontinue anytime. However, those who wish to participate, will be treated equally. For queries about the aspects of this proposed study, please do not hesitate to contact me for clarity at 0659820685\0712490692.

Signatures:

Participant
....

Parent
....

APPENDIX G: Approval from the principal

P O BOX 2571
Tzaneen
0850
16 May 2022

Dear Principal

REQUEST FOR PERMISSION TO COLLECT DATA IN THE GRADE 10 MATHEMATICS CLASSROOM

1. The above letter has a reference.
2. I am requesting permission to collect data in Grade 10 through mathematical tasks and semi-structured interviews.
3. The title of my study is "**Exploring Grade 10 Learners' spatial skills in computing sides and angles using trigonometric ratios**".
4. The research will be conducted after hours to avoid disrupting the program of the school.
5. During the process of analysing and interpreting the data, learners' names will be anonymous to protect the personal information of participants.
6. All the data collected will be used for research purposes only.
7. Hope the above is in order.
8. Kindly sign below as a way of permitting me to carry out my study in the Grade 10 class.
9. MABHALATI P J……(the principal) 20220518 (date)

Kind regards
SELOWA R
0659820685
selowareinhard@gmail.com

DEPT. OF EDUCATION - LIMPOPO PROVINCE
PROFESSOR MUHLAVA SHILUVANA HIGH SCHOOL
EMIS NO: 0915530273
18 MAY 2022
We strive to achieve
OFFICE OF THE PRINCIPAL
PRIVATE BAG X 583 LETSITELE 0885
MAFARANA CIRCUIT - MOPANI WEST DISTRICT

Appendix H: Letter for editing

MAFUMISI GROUP

507 Caledon Village, Cell +27794848449, Email: kubayijoe@gmail.com

09 May 2023

Dear Sir/Madam

SUBJECT: EDITING OF DISSERTATION

This is to certify that the dissertation entitled 'Exploring Grade 10 learners' spatial skills of computing sides and angles in figures using trigonometric ratios' by Ms R Selowa has been edited and proofread, and that unless tampered with, I am content with the quality of the dissertation in terms of its adherence to editorial principles of consistency, cohesion, clarity of thought and precision.

Kind regards

Prof SJ Kubayi (DLitt et Phil)