ON P-FRAMES AND THEIR GENERALISATIONS

by

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Declaration

I declare that the dissertation hereby submitted to the University of Limpopo, for the degree of Master of Science in Mathematics has not previously been submitted by me for a degree at this or any other university; that it is my work in design and execution, and that all material contained herein has been duly acknowledged.

Signed: _______________ Date: 31 July 2023

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Dedication

To my late mother; Sinah Ramokone Maema and my late father, Alfred Lesibana Ngoako.

In times of great difficulty,

it is our wisdom to keep our spirit calm,

quiet, and sedate,

for then we are in the best frame both to do our own work,

and to consider the work of God...

-John Wesley (1703-1791).

Abstract

In this dissertation, we study P-frames and their generalisations. On the generalisations of P-frames we consider, in particular essential P-frames, CP-frames, almost P-frames, F-frames, F'-frames and P_F-frames. We show that a frame L is a P-frame if and only if every ideal of $\mathcal{R}L$ is a z-ideal. We also consider R-modules and then show that a frame L is a P-frame if and only if every $\mathcal{R}L$-module is flat. Furthermore, we consider the Artin-Rees property and show that a frame L is a P-frame if and only if $\mathcal{R}L$ is an Artin-Rees ring. Concerning CP-frames we show, analogously to P-frames, that a frame L is a CP-frame if and only if every ideal of $\mathcal{R}_cL$ is a z_c-ideal. It turns out that in CP-frame radical ideals are precisely z_c-ideals. We show, regarding F-frames, that L is an F-frame if and only if $\mathcal{R}L$ is a Bézout ring. We show that L is an F-frame if and only if every ideal of $\mathcal{R}L$ is convex. Finally, we introduce P_F-frames and show that L is a P_F-frame if and only if it is an essential P-frame which is also an F-frame.

Key Words: Frames, P-frames, basically disconnected frames, weakly cozero complemented frames, essential P-frames, CP-frames, almost P-frames, F-frames, F'-frames, P_F-frames.

Contents

Chapter 1

Introduction and Preliminaries

1.1 History of P-spaces and P-frames

The concept of a P-space was introduced by Gillman and Henriksen [50], and provided various characterisations of these spaces (see also [52]). A topological space X is said to be a P-space if every countable intersection of open sets in X is open, which is equivalent to stating that every cozero set in X is closed. Hewitt [57] demonstrated that when X is an almost compact space, it is C^*-embedded in any space in which it is embedded. Similarly, Aull [9] established that a P-space X is C^*-embedded in every P-space in which it is embedded when X is an almost Lindelöf space. Although P-spaces are rare, they play a fundamental role in the study of disconnected topological spaces. The extension of P-spaces to a point-free setting was initiated by Ball and Walters-Wayland [16]. They defined P-frames as frames in which every cozero element is complemented. Ball *et al* [17] demonstrated the existence of P-frames with quotients that are not P-frames. Additionally, Dube [31, 36] established connections between P-frames and ring-theoretic properties of $\mathcal{R}L$ (where $\mathcal{R}L$ is the ring of real-valued continuous functions on a frame L). In particular, he outlined that a frame L is a P-frame if and only if every ideal in $\mathcal{R}L$ is a z-ideal, following the definition given by Mason [71]. Moreover, Dube and Ighedo [37] showed that a frame L is a P-frame if and only if every radical ideal in L is a z-ideal. Recently, Abedi [1] provided some characterizations of P-frames associated with the Artin-Rees property, stating that a frame L is a P-frame if and only if $\mathcal{R}L$ is an Artin-Rees ring.

The notion of essential P-spaces was introduced by Osba *et al* [78], and they defined a space X to be an essential P-space if, with the exception of at most one point, all points in X are P-points. They also established that the ring $C(X)$ is VN-local if and only if X is an essential P-space. Furthermore, Osba and Henriksen [79] studied proper essential P-spaces, stating that a space X falls into this category if and only if $C(X)$ has at least one non-maximal prime ideal, and the non-maximal ideal of $C(X)$ is contained in a single maximal ideal. Dube [36] further extended this concept to frames by introducing the notion of an essential P-frame. A frame L is classified as an essential P-frame if it possesses purely maximal ideals or contains at most one non-pure maximal ideal. Furthermore, proper essential P-frames are defined as frames L for which $\mathcal{R}L$ has at least one non-maximal prime ideal and all of the non-maximal prime ideals of $\mathcal{R}L$ are covered by a single maximal ideal.

Almost P-frames were introduced by Ball and Walters-Wayland [16] and further examined by Henriksen and Walters-Wayland [55] and Dube [36]. In [36], almost P-frames were subjected to various characterisations in terms of ring-theoretic properties of $\mathcal{R}L$. Almost P-frames are extensions of almost P-spaces. Almost P-spaces were introduced by Veksler [85], and further investigated by Kim [63] and Levy [67].

Ball and Walters-Wayland [16] introduced the notion of F-frames and F'-frames. These are extensions of F-spaces and F'-spaces in classical topology. The notion of F-spaces was introduced by Gillman and Henriksen [51]. F-spaces were further investigated by Henriksen and Woods [56], Dow and Förster [30]. F'-spaces were investigated by Comfort *et al* [27] and Dow [29]. Dube [35], Dube and Nsonde-Nsayi [39] studied numerous characterisations of F-frames in designation of $\mathcal{R}L$.

Azarpanah *et al* [14] introduced the notion of P_F-spaces and showed that a weakly cozero complemented P_F-space is precisely a basically disconnected essential almost P-space. The class of CP-frames were introduced by Estaji and Robat Sarpoushi [44].

The motivation for studying P-frames and their generalizations, except essential P-frames, CP-frames, and P_F-frames, originated from the work of Ball and Walters-Wayland [16]. The authors did not only provide the definitions of these frames but also presented a few characterizations for each of them. Subsequently, various authors (see [11, 13, 63, 67]) contributed additional characterizations for these frames. Here, we also consider CP-frames introduced by Estaji and Robat Sarpoushi [44] and we introduce the concept of P_F-frames which captures the notion of P_F-spaces introduced by Azarpanah *et al* [14].

1.2 Synopsis of the dissertation

In this dissertation we study P-frames and their generalisations, namely, essential P-frames, CP-frames, almost P-frames, F- and F'-frames, and P_F-frames.

Here is the outline of the dissertation. Chapter 1 is mainly introductory. In this chapter, we provide the relevant notions pertaining to frames and give the relevant background for the ensuing chapters.

In Chapter 2, we study P-frames and show that the class of P-frames is contained in the class of basically disconnected frames which in turn is contained in the class of weakly cozero complemented frames. We characterise P-frames as those frames which every *coz*-onto quotient maps out of them are C-quotient maps. We also give algebraic characterisation of P-frames and show amongst other characterisations that L is a P-frame if and only if every ideal of $\mathcal{R}L$ is a z-ideal if and only if every $\mathcal{R}L$-module is flat. Furthermore, we study the Artin-Rees property and show that L is a P-frame if and only if $\mathcal{R}L$ is an Artin-Rees ring.

In Chapter 3, we put our attention to essential P-frames and CP-frames. We show that a normal frame is an essential P-frame if and only if $\mathcal{R}L$ is VN-local. We show further that an essential P-frame is strongly zero-dimensional. The class of CP-frames contains the class of

P-frames. We show that L is a CP-frame if and only if every ideal of $\mathcal{R}_cL$ is a z_c-ideal if and only if every radical ideal of $\mathcal{R}_cL$ is a z_c-ideal.

Chapter 4 is entirely almost P-frames. The class of almost P-frames contains the class of P-frames and shows further, that P-frames is the intersection of almost P-frames and basically disconnected frames, also the intersection of almost P-frames with O_z-frames. We show that L is a P-frame if and only if it is an almost P-frame with countable chain condition (ccc). Lastly, we show that every weakly Lindelöf almost P-frame is Lindelöf and we give a few characterisations of almost P-frames in terms of ring-theoretic properties.

Chapter 5 catalogues the characterisations of F-frames and F'-frames. We show that the class of F-frames contains the class of P-frames, essential P-frames, and almost P-frames. We also show that the class of basically disconnected frames is also contained in the class of F-frames. We show that L is an F-frame if and only if $\mathcal{R}L$ is a B$\acute{e}$zout ring if and only if every ideal of $\mathcal{R}L$ is convex. We show that the classes of P-frames, basically disconnected frames and F-frames are contained in the class of F'-frames. We also show that every weakly Lindelöf F'-frame is an F-frame and every zero-dimensional weakly Lindelöf F'-frame is a strongly zero-dimensional F-frame. Lastly, we give some ring-theoretic characterisation of F'-frames.

Chapter 6 catalogues the P_F-frames and we observed in the first section that the class of P-frames is contained in the class of P_F-frames, in turn, is contained in the class of F-frames, and P_F-frames and basically disconnected frames are incomparable. We show that a frame L is a P_F-frame if and only if βL is a P_F-frame. We show amongst other characterisations of P_F-frames, that L is a P_F-frame if and only if it is an essential P-frame which is also an F-frame.

1.3 Preliminaries

In this section, a brief introduction to some needed background material on a frame theory is given. We focus on the definitions, results, and properties required for this dissertation. For the convenience of the reader, full details can be found in [59] and [80] (see also [81]) and we present them with some proofs from the aforementioned texts.

1.3.1 Posets and lattices

Definition 1.3.1. A binary relation $\leq$ on a set L is called a *partial order* if it satisfies the following:

(i) reflexive, that is to say for all $a \in L$, $a \leq a$,

(ii) antisymmetric, that is to say for all $a, b \in L$, $a \leq b$ and $b \leq a$ implies $a = b$, and

(iii) transitive, that is to say for all $a, b, c \in L$, $a \leq b$ and $b \leq c$ implies $a \leq c$.

Furthermore, the set L together with the partial order $\leq$ is called a *partially ordered set* or *poset*.

Definition 1.3.2. If A is a subset of a poset L, then $b \in L$ is called an *upper bound* respectively, (*lower bound*) of A if $a \leq b$ respectively ($a \geq b$), for all $a \in A$. Furthermore, the *join* respectively, (*meet*) of A is the least upper bound respectively when it exists (the greatest lower bound) of A.

We denote the join of A by $\bigvee A$ and the meet by $\bigwedge A$. If $A = \{a, b\}$ has only two elements, then we write $\bigvee A = a \vee b$ and $\bigwedge A = a \wedge b$. In addition, a poset L is:

(i) a *meet-semilattice* (*join-semilattice*), if there exists a meet (join) for any two elements $a, b \in L$,

(ii) a *lattice* if there is a meet and a join for any two elements in L. A lattice L is called:

  (a) *modular* if the implication below holds for all elements $a, b, c \in L$,

$$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c,$$

  (b) *distributive* if the equality below holds for all elements $a, b, c \in L$,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

(iii) a *bounded lattice* whenever all finite subsets of L have a meet and a join. This means that L is a lattice which has a greatest (top) element 1_L and a least (bottom) element 0_L,

(iv) a *complete lattice* if every subset of L has a meet and a join. Note that every complete lattice L is bounded with

$$0_L = \bigvee \emptyset = \bigwedge L \text{ and } 1_L = \bigwedge \emptyset = \bigvee L.$$

Definition 1.3.3. A *complemented lattice* is a bounded lattice, in which every element a has a complement, i.e. an element b such that:

$$a \vee b = 1 \text{ and } a \wedge b = 0.$$

A complemented, distributive lattice is called a *Boolean algebra*.

Definition 1.3.4. A mapping $f : X \to Y$ between two posets X and Y is called *monotone* if:

$$f(x) \leq f(y) \text{ whenever } x \leq y.$$

It is called an *isomorphism* if it is bijective and its inverse is monotone as well. Moreover, we say that f is a *lattice homomorphism* if X and Y are lattices and:

$$f(x \vee y) = f(x) \vee f(y), f(x \wedge y) = f(x) \wedge f(y), \text{ for all } x, y \in X.$$

An *adjunction map* is a pair of monotone maps f and g between two posets such that for all $x \in X$ and $y \in Y$ the relation holds:

$$f(x) \leq y \text{ if and only if } x \leq g(y).$$

Then f is called a *left adjoint* of g and g is the *right adjoint* of f. General theory tells us that:

(i) adjoints are unique,

(ii) a right (left) adjoint preserves all existing meets (joins),

(iii) a monotone map $f : X \to Y$ has a right adjoint g if and only if for all $y \in Y$ the right-hand side in the identity below exists and f preserves all such joins; then the right adjoint of g is given by the formula

$$g(y) = \bigvee\{x \mid f(x) \leq y\},$$

(iv) dually, a monotone map $f : X \to Y$ has a left adjoint g if and only if for all $y \in Y$ the right-hand side in the identity below exists and f preserves all such meets,

$$g(y) = \bigwedge\{x \mid y \leq f(x)\}.$$

Definition 1.3.5. In a complete lattice L, an element a in L is said to have a *pseudocomplement* if there exists a largest (greatest) element x in L such that $a \wedge x = 0$. We denote such an x by a^*. Equivalently, a^* is a pseudocomplement of a if:

$$x \wedge a = 0 \Longleftrightarrow x \leq a^*, \text{ for all } x \in L.$$

More formally

$$a^* = \bigvee\{x \in L \mid x \wedge a = 0\}.$$

The complete lattice L is called pseudocomplemented if every element in L has a pseudocomplement. For example every finite distributive lattice is pseudocomplemented.

Note that lattice homomorphisms do not necessarily preserve pseudocomplements. One has obviously $f(a^*) \leq f(a)^*$ if f is monotone, but the other inequality, generally does not hold.

1.3.2 Frames

Definition 1.3.6. A frame L is a complete lattice such that for any point $a \in L$ and any set $B \subseteq L$ the following infinite distributive law holds:

$$a \wedge \bigvee B = \bigvee\{a \wedge b \mid b \in B\}.$$

Since the De Morgan law for meets does not hold in frames, frames in which this law holds are called *De Morgan frames.*

Definition 1.3.7. A frame L is called a *Boolean frame* if $L = BL$, where BL is the set of all complemented elements of the frame L. A frame L which is a Boolean algebra coincides with a Boolean frame.

Example 1.3.1. *Any complete Boolean algebra is a frame.*

Definition 1.3.8. A *subframe* M of a frame L is a subset $M \subseteq L$ which is a frame under the same operations ($\wedge$ and $\bigvee$) as L with $1_L, 0_L \in M$.

Definition 1.3.9. A notion is said to be a *conservative* notion if whenever it holds in a space X, it also holds in $\mathcal{O}X$ (where $\mathcal{O}X$, denotes the lattice of open subsets of a topological space X) and vice-versa.

Example 1.3.2. *$\mathcal{O}X$ is a complete lattice, hence is a frame.*

Definition 1.3.10. A frame is said to be *spatial* if it isomorphic to a topology.

Example 1.3.3. *$\mathcal{O}X$ is a complete lattice, hence is a spatial frame.*

Definition 1.3.11. An *atom (co-atom)* in a frame is an element a such that $a > 0$ ($a < 1$) such that for each $b \in L$, $a \geq b > 0$ implies that $b = a$ ($a \leq b < 1$ implies that $b = a$). A Boolean algebra is *atomic* if each of its element is a join of atoms.

Example 1.3.4. *A Boolean algebra is atomic if each element of L is a meet of co-atoms.*

The following is an example of a non-spatial frame.

Example 1.3.5. *A Boolean frame without atoms.*

Recall that a pseudocomplement of an element a is a^* such that for $y \in L$, $y \leq a^* \Rightarrow a \wedge y = 0$. However $a \vee a^* = 1$ does not hold in general. In the case where $a \vee a^* = 1$, we say a is *complemented.*

Pseudocomplements, if they exist, satisfy the following properties:

(i) $a \leq a^{**}$,

(ii) $a^* = a^{***}$,

(iii) $a \leq b$ implies $b^* \leq a^*$,

(iv) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,

(v) $(a \vee b)^* = a^* \wedge b^*$ [De Morgan's Law],

(vi) $1^* = 0$ and $0^* = 1$,

(vii) $a \wedge b = 0$ implies $a \leq b^*$ and $b \leq a^*$.

Another property of the pseudocomplement of a frame that we will employ freely is

$$\left(\bigvee_{i \in I} a_i\right)^* = \bigwedge_{i \in I} a_i^*.$$

Definition 1.3.12. For an element $a \in L$, if $a^{**} = a$, then we say a is *regular*. If every element of a frame is regular, then we say the frame is Boolean.

Definition 1.3.13. An element a in L is said to be *dense* if its pseudocomplement is the bottom element. That is, $a \in L$ is dense if $a^* = 0$.

Definition 1.3.14. A frame *homomorphism* is a map $h : L \to M$ between two frames which preserves:

(i) All finite meets, that is, binary meets $h(x \wedge y) = h(x) \wedge h(y)$ for all $x, y \in L$, and the empty meet $h(1) = 1$.

(ii) All arbitrary joins $h(\bigvee X) = \bigvee\{h(x) \mid x \in X\}$, for any $X \subseteq L$, and the empty join $h(0) = 0$.

Note that such an h is automatically order preserving and preserves both the top (that is $h(1_L) = 1_M$) and the bottom (that is $h(0_L) = 0_M$).

Definition 1.3.15. By a *quotient* of a frame L, we mean an onto homomorphic image of L. That is, M is a quotient of L precisely if there is an onto frame homomorphism $h : L \to M$. In such a case h is called a *quotient map*.

When we say a quotient $h : L \to M$ has a property of frames we shall mean that M has that property. Likewise, to say a quotient $h : L \to M$ has a property of homomorphisms means that h has that property.

Definition 1.3.16. A frame homomorphism $h : L \to M$ is said to be:

(i) *dense* if $h(a) = 0 \Rightarrow a = 0$,

(ii) *codense* if $h(a) = 1 \Rightarrow a = 1$,

(iii) a *quotient* map if it is onto,

(iv) an *isomorphism* if it is onto (surjective) and one-to-one (injective).

Because a frame homomorphism h preserves arbitrary joins, h has a right adjoint $h_* : M \to L$ satisfying the property that $x \leq h_*(y)$ in L if and only if $h(x) \leq y$ in M. For $a \in M$

$$h_*(a) = \bigvee\{x \in L \mid h(x) \leq a\}.$$

A homomorphism $h : L \to M$ is called *closed* if $h_*(h(x) \vee y) = x \vee h_*(y)$, for all $x \in L$ and for all $y \in M$.

Example 1.3.6. *If X is a topological space, then the set $\mathcal{O}X$ of all open subsets of X forms a frame ordered by set inclusion. Let $f : X \to Y$ be a continuous map between topological spaces X and Y , the map $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ which is given by*

$$\mathcal{O}f(U) = f^{-1}(U), \forall U \subseteq Y \text{ with } U \text{ open },$$

is a frame homomorphism.

Definition 1.3.17. Let L be a frame. We call $D \subseteq L$ a *downset* if $x \in D$ and $y \leq x$ implies $y \in D$, and $U \subseteq L$ an *upset* if $u \in U$ and $u \leq v$ implies $v \in U$. For any $a \in L$, we write

$$\downarrow a = \{x \in L | x \leq a\},$$

that is a downset, and

$$\uparrow a = \{x \in L \mid a \leq x\},$$

that is an upset. We note that $\downarrow a$ is a frame whose bottom element is $0 \in L$ and top element a. Similarly, $\uparrow a$ is a frame and has $1 \in L$ as the top element and a as its bottom element. These frames are in fact the quotients of L via the maps $L \to \uparrow a$ and $L \to \downarrow a$, given respectively by $x \mapsto a \vee x$ and $x \mapsto a \wedge x$. These quotients are known as the *closed quotients* and *open quotients* respectively.

1.3.3 Cozero elements and cozero map

An element a in a frame L is said to be a *cozero element* if there is a frame homomorphism:

$$\varphi : \mathcal{L}(\mathbb{R}) \to L, \text{ such that } a = \varphi(-,0) \vee \varphi(0,-),$$

where $(-,0) = \bigvee\{(p,0) \mid 0 > p \in \mathbb{Q}\}$ in $\mathcal{L}(\mathbb{R})$ and $(0,-) = \bigvee\{(0,q) \mid 0 < p \in \mathbb{Q}\}$. We recall from [54] that $\mathcal{L}(\mathbb{R})$ may be equivalently defined as the frame generated by the $(p,-)$ and $(-,q)$, where $p, q \in \mathbb{Q}$, subject to the relations:

(i) $(p,-) \wedge (-,q) = 0$ whenever $p \geq q$,

(ii) $(p,-) \vee (-,q) = 0$ whenever $p < q$,

(iii) $(p,-) = \bigvee_{s \in Q, r > p} p(r,-)$,

(iv) $(-,q) = \bigvee_{s \in \mathbb{Q}, s < q} q(-,s)$,

(v) $\bigvee_{q \in \mathbb{Q}} (p,-) = 1$, and

(vi) $\bigvee_{q\in\mathbb{Q}} (-,q) = 1$.

We usually express the cozero element a above as $a = coz\varphi$, for some $\varphi \in \mathcal{L}(\mathbb{R})$.The following results show that cozero elements can be characterised without requiring reference to the frame of reals, $\mathcal{L}(\mathbb{R})$. The properties of the *cozero map* $coz : \mathcal{R}L \to L$, given by

$$coz\varphi = \bigvee\{\varphi(p,0) \vee \varphi(0,q) \mid p,q \in \mathbb{Q}\} = \varphi((-,0) \vee (0,-)),$$

Now it is clear that a *cozero element* of a frame L is an element of the form $coz\varphi$ for some $\varphi \in \mathcal{R}L$. For details about the map *coz*, we refer to [10, 16] and [19]. This map has the following properties:

(i) $coz\gamma\delta = coz\gamma \wedge coz\delta$.

(ii) $coz(\gamma + \delta) \leq coz(\gamma) \vee coz(\delta)$.

(iii) $coz(\gamma + \delta) = coz(\gamma) \vee coz(\delta)$ if $\gamma, \delta \geq 0$.

(iv) $coz\varphi = 0$ if and only if $\varphi = 0$.

(v) φ is invertible if and only if $coz\varphi = 1$.

Regular and completely regular

Definition 1.3.18. An element $a \in L$ is said to be *rather below* $b \in L$, denoted by $a \prec b$, if there exists an element $c \in L$ such that

$$a \wedge c = 0 \text{ and } c \vee b = 1.$$

We call c a *separating element* of a and b. L is pseudocomplemented, then this is equivalent to:

$$a \prec b \Leftrightarrow a^* \vee b = 1_L.$$

A frame L is called *regular* if for every $b \in L$,

$$b = \bigvee\{a \in L \mid a \prec b\}.$$

Regularity for frames is a conservative notion, meaning that for a topological space X, X is regular (as a space) if and only if $\mathcal{O}X$ is regular (as a frame). Next, we turn to some properties of the (rather below) relation $\prec$ that will prove to be useful in chapters to come:

(i) $a \prec b \Rightarrow a \leq b$, and for any $a, 0 \prec a \prec 1$.

(ii) $x \leq a \prec b \leq y \Rightarrow x \prec y$.

(iii) If $a \prec b$, then $b^* \prec a^*$.

(iv) If $a \prec b$, then $a^{**} \prec b$.

(v) If $a_i \prec b_i$ for $i = 1, 2$, then $a_1 \vee a_2 \prec b_1 \vee b_2$ and $a_1 \wedge a_2 \prec b_1 \wedge b_2$.

Definition 1.3.19. For any $a, b \in L$, a is said to be *completely below* b, denoted by $a \prec\!\prec b$, if there is a sequence of elements

$$\{c_r \in L \mid r \in \mathbb{Q} \bigcap [0,1]\}$$

such that $a = c_0$ and $b = c_1$, $c_p \prec c_r$ when $p < r$. We say that the sequence $\{c_r\}$ *witnesses* the relation $a \prec\!\prec b$. A frame L is called *completely regular* in case every b in L is the join of elements completely below it,

$$b = \bigvee \{a \in L \mid a \prec\!\prec b\}.$$

The properties that hold true for the relation $\prec$, also hold true for $\prec\!\prec$:

(i) $a \prec\!\prec b \Rightarrow a \leq b$, and for any $a, 0 \prec\!\prec a \prec\!\prec 1$.

(ii) $x \leq a \prec\!\prec b \leq y \Rightarrow x \prec y$.

(iii) If $a \prec\!\prec b$, then $b^* \prec\!\prec a^*$.

(iv) If $a \prec\!\prec b$, then $a^{**} \prec\!\prec b$.

(v) If $a_i \prec\!\prec b_i$ for $i = 1, 2$, then $a_1 \vee a_2 \prec\!\prec b_1 \vee b_2$ and $a_1 \wedge a_2 \prec\!\prec b_1 \wedge b_2$.

Proposition 1.3.1. [80, Proposition 6.2.3] *For any frame L the following are equivalent for each $a \in L$:*

(1) $a \in CozL$.

(2) $a = \bigvee x_n$ *where* $x_n \prec\prec a$ *for all n=1, 2, ...*

(3) $a = \bigvee a_n$ *where* $a_n \prec\prec a_{n+1}$ *for all n=1, 2, ...*

The authors in [23] have also shown the following as significant consequences of Proposition 1.3.1 for any frame L: The *cozero part* of L, is denoted by $CozL = \{coz\varphi \mid \varphi \in \mathcal{R}L\}$, is the regular sub-$\sigma$-frame consisting of all the cozero elements of L. A frame is completely regular if and only if it is generated by its cozero part. We have the following properties of cozero elements and the cozero part of a frames.

(i) If $a \prec\prec b$, there is cozero element c such that $a \prec\prec c \prec\prec b$.

(ii) If $a \prec\prec b$, there is cozero element c such that $a \wedge c = 0$ and $c \vee b = 1$.

1.3.4 C- and C^*-quotients maps

Definition 1.3.20. A frame homomorphism $h : L \to M$ is said to be:

(i) *coz-codense* if the only cozero element it maps to the top element is the top element.

(ii) *almost coz-codense* if for each $c \in CozL$ such that $h(c) = 1$, there exists $d \in CozL$ such that $c \vee d = 1$ and $h(d) = 0$.

(iii) *coz-onto* if for every $d \in CozM$, there exists $c \in CozL$ such that $h(c) = d$.

Proposition 1.3.2. [40, Proposition 3.3] *For any homomorphism $h : L \to M$, the following are equivalent.*

(1) *h is coz-onto.*

(2) *For all* $a, b \in CozM$ *such that* $a \wedge b = 0$*, there exist* $c, d \in CozL$ *such that* $c \wedge d = 0$*,* $h(c) = a$ *and* $h(d) = b$*.*

(3) *For all* $a, b \in CozM$ *with* $a \wedge b = 0$*, there exist* $c, d \in CozL$ *such that* $c \wedge d = 0$*,* $a \leq h(c)$ *and* $b \leq h(d)$*.*

Definition 1.3.21. A *cover* C of a frame L is a subset C of L such that $\bigvee C = 1$. A *subcover* D of C, we mean a subset $D \subseteq C$ such that $\bigvee D = 1$. A frame L is said to be *normal* if for any two $a, b \in L$ such that $a \vee b = 1$, there exist $c, d \in L$ such that $c \wedge d = 0$ and $c \vee a = 1 = d \vee b$. The $CozL$ is normal (see [23]). In frame theory, the set of all covers of L is denoted by $CovL$. Let $A, B \in CovL$, then we say that A *refines* B (written $A \leq B$) if for any $a \in A$ there is $b \in B$ such that $a \leq b$. Moreover, we say that A *star refines* B (written $A \leq^* B$) if and only if $AA \leq B$ with

$$AA = \{Ax \mid x \in A\} \text{ and } Ax = \bigvee \{q \in A \mid q \wedge x \neq 0\}.$$

A cover A of a frame L is said to be normal whenever there exist a sequence $(A_n)_{n \in \mathbb{N}}$ of covers such that $A = A_1$ and $A_{n+1} \leq^* A_n$, for all n. Then L is called *fully normal* if every cover of it is normal.

Proposition 1.3.3. [16, Theorem 7.1.1] *The following are equivalent for a quotient map* $h : L \to M$*:*

(1) h *is a* C^**-quotient map.*

(2) *Every binary cozero cover of* M *is refined by the image of a binary cozero cover of* L*.*

(3) *Every binary cozero cover of* M *is the image of a binary cozero cover of* L*.*

Proposition 1.3.4. [16, Theorem 7.2.7] *The following are equivalent for a quotient map* $h : L \to M$*:*

(1) h *is a* C*-quotient map.*

(2) *h is a C^*-quotient map and almost coz-codense.*

(3) *h is coz-onto and almost coz-codense.*

Lemma 1.3.1. [16, Corollary 3.2.11] *Every open quotient of a cozero element of a frame is coz-onto.*

We note the following lemma that relates surjective homomorphisms, their adjoints and pseudocomplements.

Lemma 1.3.2. *For a surjective frame homomorphism $h : L \to M$ with $a \in M$ and*

$$h_*(a) = \bigvee\{x \in L \mid h(x) = a\}.$$

If h is dense surjective, then $h_*(a^*) = (h_*(a))^*$.

Lemma 1.3.3. [80, Proposition 2.2.2] *In a regular frame, any dense frame homomorphism is injective.*

Note that the relation $\prec\!\prec$ is the largest interpolative relation contained in $\prec$ (a relation $\mathcal{K}$ is *interpolative* if $a\mathcal{K}b \Rightarrow a\mathcal{K}c\mathcal{K}b$ for some c).

Lemma 1.3.4. [80, Lemma 5.9.1] *In a normal frame L the relation $\prec$ interpolates and coincides with the $\prec\!\prec$ one, which implies that regularity coincides with complete regularity.*

Frame of reals

There are various equivalent ways to introduce the frame of real numbers. We consider the description which is introduced in [19]. Recall that frame of reals, denoted by $\mathcal{L}(\mathbb{R})$, is the frame generated by all ordered pairs (p, q) where $p, q \in \mathbb{Q}$, subject to the relations:

(i) $(p, q) \wedge (s, t) = (p \vee s, q \wedge t)$.

(ii) $(p, q) \vee (s, t) = (p, t)$ where $p \leq s < q \leq t$.

(iii) $(p, q) = \bigvee\{(s, t) \mid p < s < t < q\}$.

(iv) $1_{\mathcal{L}(\mathbb{R})} = \bigvee\{(p,q) \mid p,q \in \mathbb{Q}\}$.

Remark : It follows from (iii) that if $q \leq p$, then $(p,q) = 0$.

Proposition 1.3.5. [80] *The frame $\mathcal{L}(\mathbb{R})$ is completely regular.*

Proof. If $p < s < t < q$, then $(s,t) \prec (p,q)$. Consider $\{(u,v) \mid p < u < s < t < v < q \text{ in } \mathbb{Q}\}$ it is clear that $(s,t) \prec\prec (p,q)$, and then by (iii), it immediately follows that $\mathcal{L}(\mathbb{R})$ is completely regular. □

1.3.5 Coproducts of frames

Let A and B be frames. Then $A \oplus B$ is generated by elements $a \oplus b$ $(a \in A, b \in B)$ such that

(i) $\bigvee_{i \in I}(a_i \oplus b) = \left(\bigvee_{i \in I} a_i\right) \oplus b$ and $\bigvee_{i \in I}(a \oplus b_i) = a \oplus \left(\bigvee_{i \in I} b_i\right)$,

(ii) $(a_1 \oplus b_1) \wedge (a_2 \oplus b_2) = (a_1 \wedge a_2) \oplus (b_1 \wedge b_2)$, and

(iii) for $a, b, c \neq 0, a \oplus b \leq c \oplus d$ if and only if $a \leq c$ and $b \leq d$.

The *coproduct injections or inclusions* $i_A : A \to A \oplus B$ and $i_B : B \to A \oplus B$ are given by $a \mapsto a \oplus 1$ and $b \mapsto 1 \oplus b$ so that $a \oplus b = i_A(a) \wedge i_B(b)$.

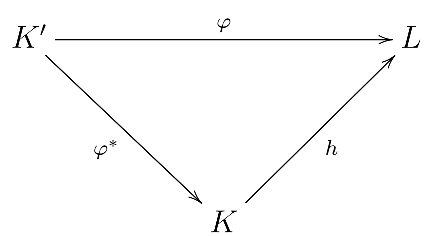
1.3.6 Compactification of frames

Definition 1.3.22. A frame L is *compact* (*countably compact*) if each of its covers (countable covers), admits a finite subcover. Similarly, L is *Lindelöf* if each of its covers admits a countable subcover. In any completely regular Lindelöf frame L, $a \in L$ is cozero if and only if it is Lindelöf.

Definition 1.3.23. A surjective (quotient) dense frame homomorphism $h : L \to M$ is called a *compactification* of M if L is a compact regular frame.

Stone-Čech compactification

For a completely regular frame L, a compact frame K together with a dense (onto) frame homomorphism $h : K \to L$ is called the Stone-Čech compactification of the frame L if for every dense homomorphism with compact domain $\varphi : K' \to L$ there is a unique frame homomorphism $\varphi^* : K' \to K$ such that $\varphi = h \circ \varphi^*$ the following diagram commutes.



The existence of such compactification for completely regular frames is well established in point-free topology. We note the following results without proof.

Proposition 1.3.6. [80]

(1) *Each dense frame homomorphism $h : L \to M$ is injective if M is compact and L is regular.*

(2) *Each compact regular frame is spatial.*

Lemma 1.3.5. [80]

(1) *For every cover $\{a_j \mid j \in \mathbb{N}\}$ of a normal frame L there is a cover*

$$\{b_j \mid j \in \mathbb{N}\} \text{ such that } b_j \prec a_j, \text{ for all } j.$$

(2) *A compact regular frame is normal.*

Note: Compactness in frames is hereditary. This is easy to see because joins in a sub-frame are exactly as in the larger frame.

Ideals

Definition 1.3.24. A non-empty subset I of a frame L is an *ideal* if it satisfies the following:

(i) $0 \in I$,

(ii) a downset ($a \leq b$ and $b \in I \Rightarrow a \in I$), and

(iii) closed under finite joins ($a, b \in I \Rightarrow a \vee b \in I$).

If $1 \notin I$, then an ideal I is said to be *proper*. Let R be a ring, and I and J be ideals in R. The sum of I and J is the ideal $I + J = \{i + j \mid i \in I, j \in J\}$. If $I + J = R$, then I and J are said to be comaximal. The *product of two ideals*, I and J is the ideal consisting of all finite sums of products of the form ij where $i \in I$ and $j \in J$.

Lemma 1.3.6. *Let I and J be ideals in a ring R. Then the following hold:*

(1) *$I \cap J$ is an ideal, and is the smallest ideal of R containing both I and J.*

(2) *$I + J$ is an ideal, and the ideal IJ is contained in $I \cap J$. Furthermore, if $I + J = R$, then $IJ = I \cap J$.*

An ideal I of L is said to be *completely regular* if for each $x \in I$, there is a $y \in I$ such that $x \prec\prec y$.

The set βL of all completely regular ideals of a frame L under set inclusion is a compact completely regular frame, and $\beta : \beta L \to L$, defined by $\beta(I) = \bigvee I$, is a dense surjective frame homomorphism, so that βL is a compactification of L. Furthermore, any frame homomorphism $f : M \to L$ from a compact completely regular frame M factors uniquely through βL, i.e., there exists a unique frame homomorphism $f^* : M \to \beta L$ such that $f = \beta \circ f^*$. The compactification βL is known as the *Stone-Čech compactification* of the frame L. It is clear that βL is finite if and only if L is finite. The right adjoint $\beta_* : L \to \beta L$ of the surjective frame homomorphism β is denoted by r, and $r(a) = \{x \in L : x \prec\prec a\}$ for all $a \in L$. There are several other equivalent descriptions of βL (see [19] and [24]); however, in this dissertation, we shall use the above-mentioned description (see [80]).

A frame L is said to be *pseudocompact* if every continuous real-valued function on L is bounded, that is if $\mathcal{R}L = \mathcal{R}^*(L)$ (denotes, the ring of real-valued continuous functions on L is equals to the ring of bounded real-valued continuous functions on L). Equivalently, $\beta L = \upsilon L$ (Stone-$\check{C}$ech compactification is equals to Hewitt real compactification), for more details (see [77]). Moreover, L is pseudocompact if and only if $CozL$ is a compact σ-frame, that is, if every countable cover of L by cozero elements admits a finite subcover (see [23] for details; [19] contains the result for completely regular frame).

For completely regular frame L, the frame of its completely regular ideals is denoted by βL. If L is normal, then r preserves finite joins as was shown in [15, Lemma 3.1]. We shall frequently use that if $I, J \in \beta L$ and $I \prec\!\prec J$, then $\bigvee I \in J$. An application of Zorn's Lemma shows that in any compact frame every element, but the top, below a maximal element. We denote by $\sum \beta L$ the set of all maximal elements of βL.

Chapter 2

P-frames

In this chapter, we study P-frames. We show that the class of P-frames is contained in the class of basically disconnected frames which in turn is contained in the class of weakly cozero complemented frames. We characterise P-frames as those frames which in every *coz*-onto quotient maps out of them are C-quotient maps. We also give an algebraic characterisation of P-frames. Amongst other characterisations we show that L is a P-frame if and only if every ideal of $\mathcal{R}L$ is a z-ideal if and only if every $\mathcal{R}L$-module is flat. Furthermore, we study the Artin-Rees property and show that L is a P-frame if and only if $\mathcal{R}L$ is an Artin-Rees ring.

2.1 Introduction

We recall that a topological space X is said to be a P-space if every zero-set is open. Since a zero-set is a closed set it follows that in a P-space every zero-set is clopen. Adapting this to frames, Ball and Walters-Wayland [16] have formulated the following:

Definition 2.1.1. A frame L is said to be a P-frame if every $a \in CozL$ is complemented. We emphasise that in P-frames, $CozL = BL$.

That is to say for every $a \in CozL$ there exists $a^* \in L$ such that $a \vee a^* = 1$, which implies that $a^* \in CozL$. Clearly, X is a P-space $\Leftrightarrow$ $\mathcal{O}X$ is a P-frame; so that we have a conservative extension of the topological notion.

Recall that an element a in a frame L is said to be *dense* if $a^* = 0$. The following proposition is taken from [1].

Proposition 2.1.1. [1, Lemma 3] *A frame L is a P-frame if and only if every non-dense cozero element of L is complemented.*

Proof. The right-to-left implication is trivial, since every cozero element is complemented. Conversely, let a be a dense cozero element of L. Therefore $a \neq 0$, and so, by complete regularity, there exists $b \in CozL$ with $b \prec\!\prec a$. If b is dense, then the equality $b^* \vee a = 1$ implies $a = 1$, which is complemented. If b is not dense, then $b \vee b^* = 1$, by hypothesis which makes b^* a cozero element. Since $a \wedge b^* \leq b^*$ and b^* is not dense (lest we have $b = 0$), $a \wedge b^*$ is a non-dense cozero element of L, and so $a \wedge b^* \vee (a \wedge b^*)^* = 1$ which implies $a \vee (a \wedge b^*)^* = 1$. We therefore have $b \prec a$ and $a \wedge b^* \prec a$, which implies $b \vee (a \wedge b^*) \prec a$. But

$$b \vee (a \wedge b^*) = (b \vee a) \wedge (b \vee b^*) = a,$$

and so $a \prec a$, implying that a is complemented. □

Recall from [16] that a frame L is said to be *basically disconnected* if for every cozero element, the join of its pseudocomplement with its double pseudocomplement is the top element. That is to say for all $a \in CozL$, $a^* \vee a^{**} = 1$. We show below that the class of P-frames is contained in the class of basically disconnected frames.

Proposition 2.1.2. *Every P-frame is basically disconnected.*

Proof. Suppose that L is a P-frame and let $a \in CozL$. Then there is $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. Hence $b = a^*$ and $a^{**} = a$. It follows that

$$a^* \vee a^{**} = b \vee a = 1.$$

Thus L is basically disconnected. □

Recall from [38] that a frame L is said to be *weakly cozero complemented* if for each $a \in CozL$ there exists $b \in CozL$ such that $a \wedge b = 0$ and $a \vee b$ is dense. In this dissertation, we call frames with this property weakly cozero complemented in order to distinguish them from P-frames. The class of basically disconnected frames is contained in the class of weakly cozero complemented frames. We show this in the following lemma.

Lemma 2.1.1. [16] *Every basically disconnected frame is weakly cozero complemented.*

Proof. Let $a \in CozL$. The frame L is basically disconnected, so $a^* \vee a^{**} = 1$. Thus a^* is complemented and so, $a^* \in CozL$ with the required property. That is $a \wedge a^* = 0$ and $a \vee a^*$ is dense. Hence L is weakly cozero complemented. □

The converse of the above lemma does not hold in general. We will show in Chapter 5, that the converse holds if L is a weakly cozero complemented F-frame. From Lemma 2.1.1, the following corollary is immediate.

Corollary 2.1.1. [39] *Every P-frame is weakly cozero complemented.*

We give the following proposition without proof.

Proposition 2.1.3. [16, Proposition 3.2.10] *If $a \in CozL$ and $b \in Coz(\downarrow a)$, then $b \in CozL$.*

Ball and Walters-Wayland [16] have shown that a quotient map is a C-quotient if and only if it is *coz*-onto and almost *coz*-codense. In line with this the authors; obtained the following proposition that a frame L is said to be a P-frame if and only if open quotient $h : L \to \downarrow a$ is a C-quotient for each $a \in CozL$.

Proposition 2.1.4. [40, Proposition 4.9] *The following statements are equivalent.*

(1) *L is a P-frame.*

(2) *Every quotient of L is almost coz-codense.*

(3) *Every coz-onto quotient of L is a C-quotient.*

(4) *The open quotient of a cozero element of L is a C-quotient.*

Proof. (1) $\Rightarrow$ (2): Let $h : L \to M$ be a quotient of L and $a \in CozL$ such that $h(a) = 1$. Thus $a^* \in CozL$ such that $a \vee a^* = 1$ and $h(a^*) = h(a^*) \wedge h(a) = h(a^* \wedge a) = h(0) = 0$. Or, alternatively, $h(a^*) \leq h(a)^* = 1^* = 0$. Hence $h : L \to M$ is almost *coz*-codense.

(2) $\Rightarrow$ (3): Recall from Proposition 1.3.4 that a quotient map $h : L \to M$ is a C-quotient map if and only if h is *coz*-onto and almost *coz*-codense. We are done.

(3) $\Rightarrow$ (1): Let $a \in CozL$ and contemplate open quotient $h : L \to \downarrow a$, it is a *coz*-onto homomorphism. Therefore it is C-quotient, so hence L is a P-frame.

(1) $\Leftrightarrow$ (4): Suppose that L is a P-frame, and consider the open quotient map $f : L \to \downarrow a$ for some $a \in CozL$. Then f is a *coz*-onto by Proposition 2.1.3, and is almost *coz*-codense because $a^* \vee b = 1$ for any $b \in CozL$ such that $f(b) = b \wedge a = 1 = a$. Therefore f is a C-quotient map by Proposition 1.3.4, $\downarrow a$ is C-quotient and (4) holds.

Conversely, let $a \in CozL$, then by (4) $h : L \to \downarrow a$ is almost *coz*-codense and this requires the existence of some $b \in CozL$ such that $h(b) = b \wedge a = 0$ and $b \vee a = 1$. That is, b is the complement of a, and (1) holds.

□

2.2 Ring-theoretic characterisations of P-frames

A ring R is *commutative* if the multiplication of its elements is commutative, that is $\varphi\alpha = \alpha\varphi$ for any $\alpha, \varphi \in R$. Let R be a commutative ring with unity element. Throughout, by the term ring, we mean a commutative ring with a unity unless stated otherwise. A set $I \subseteq R$ is said to be an *ideal* in R if $ar \in I$ for each $a \in I$ and $r \in R$. Recall from [53] that an element $\varphi \in R$ is *idempotent* if $\varphi = \varphi\varphi = \varphi^2$. If every element $\varphi \in R$ is an idempotent, then R is a *Boolean ring.*

An element $\varphi \in R$ is a *von Neumann inverse (VN-inverse)*, if $\varphi = \varphi^2\alpha$ for some $\alpha \in R$. Let

R be a ring is called *von Neumann regular ring (VN-regular ring)* if for every $\varphi \in R$ is a VN-inverse. The *pure part* of a ring R is the ideal

$$mI = \{a \in R \mid a = ab \text{ for some } b \in I, \text{ where } I \text{ is an ideal of } R\}.$$

An ideal I is *pure* if $I = mI$. We refer to z-ideals as defined in [71] as z-ideals *á* la Mason. This algebraic definition of z-ideals was coined in the context of rings of continuous functions by Kohls [65] and is also captured by Gillman and Jerison [52]. Dube [31] introduced z-ideals in point-free topology in terms of the cozero map. We shall see however that an ideal of $\mathcal{R}L$ is a z-ideal if and only if it is a z-ideal *à* la Mason. We denote by $Max(R)$ the set of all maximal ideals of R. For any $a \in R$ and $I \subseteq R$, we set

$$\mathcal{M}(a) = \{M \in Max(R)) \mid a \in M\} \text{ and } \mathcal{M}(I) = \{M \in Max(R) \mid M \supseteq I\},$$

and note that, since an ideal contains an element if and only if it contains the principal ideal generated by the element, we have that $\mathcal{M}(a) = \mathcal{M}(\langle a \rangle)$. An ideal I of a ring R is a *z-ideal à la Mason* if whenever $\mathcal{M}(a) \supseteq \mathcal{M}(b)$ and $b \in I$ implies that $a \in I$.
In any ring R, let $vr(R)$ denote the set of elements that have a von Neumann inverse and $nvr(R) = R \backslash vr(R)$.

If $S \subset R$, then the annihilator $A(S)$ of $S = \{x \in R \mid xS = \{0\}\}$, and if $\{s\}$ is a singleton, let $A(s) = A(\{s\})$. The *Jacobson radical* $J(R)$ of R is the intersection of the elements of $Max(R)$ and is given by $\{a \in R \mid (1 - ax)$ is invertible for all $x \in R\}$, $nil(R)$ denotes the ideal of nilpotent elements of R. If $I \subset R$ is an ideal, then I^* will denote $I \backslash 0$. Note that $nil(R)^* \subset J(R)^* \subset nvr(R)$ and that each of these inclusions can be proper. Note that, also $mI = \{a \in R \mid I + A(a) = R\}$. The principal ideal of a ring generated by an element q is denoted by $\langle q \rangle$.

Lemma 2.2.1. [78, Lemma 2.3] *If $a \in R$, then the ideal $\langle a \rangle$ is pure if and only if $\langle a \rangle + A(a) = R$.*

Proof. Suppose that $\langle a \rangle$ is pure, then there exists $x = ar \in \langle a \rangle$ such that $a = ax = a^2 r$. So $a(1 - ar) = 0$ which implies that $1 - ar \in A(a)$. Hence $1 = ar + 1 - ar \in \langle a \rangle + A(a)$.

Conversely, assume that $\langle a\rangle + A(a) = R$. Then there exist $r \in R$, $b \in A(a)$ such that $ar + b = 1$. Multiply both sides by a to get $a^2r = a$. If $c = ay \in \langle a\rangle$, then

$$c = ay = (a^2r)y = (ay)(ar) = c(ar).$$

Hence $\langle a\rangle$ is pure. □

The following theorem together with its proof is taken from [78].

Theorem 2.2.2. [78, Theorem 2.4] *Suppose $a \in R$. Then a has a von Neumman inverse if and only if for each maximal ideal M, $a \in M$ implies $a \in mM$.*

Proof. Suppose that $a = a^2b$ for some $b \in R$. If $M \in Max(R)$ is such that $a \in M$, then $a = a^2b = a(ab) \in mM$.

Conversely, suppose that for each maximal ideal M, $a \in M$ implies $a \in mM$. If for such an $a, A(a) \subset M$ as well, then $a(1 - m) = 0$ for some $m \in M$, in which case $(1 - m) \in M$. Hence $\langle a\rangle + A(a) = R$. So by Lemma 2.2.1, $\langle a\rangle$ is pure. Hence a has a von Neumman inverse. □

Definition 2.2.1. A z-ideal is *strong* if it is the intersection of maximal ideals.

Every ideal I is contained in a least z-ideal, namely $J_z = \{J \subseteq K \mid K \text{ is a } z\text{-ideal}\}$. We have the following property of J_z (see [71]).

(i) $(J^n)_z = J_z$ for all positive integers n;

The following are characterisations of von Neumann regular rings given by Mason [71].

Theorem 2.2.3. *The following are equivalent for a ring R:*

(1) *Every ideal is a strong z-ideal.*

(2) *Every ideal is a z-ideal.*

(3) *Every principal ideal is a z-ideal.*

(4) *R is regular.*

Proof. The following are trivial: (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4): Since $J^2 = (J^2)_z = J_z = J$ for any principal z-ideal, therefore for any $a \in R$, $a^2R = aR$.

(4) $\Rightarrow$ (1): Every ideal in a regular ring is the intersection of the maximal ideals containing it. □

Recall from [53] that an *integral domain* R is a commutative ring with unity having no zero divisors, i.e. for $x, y \in R$, $xy = 0$ implies either $x = 0$ or $y = 0$ (or both). A *field* is a commutative ring with unity, in which every nonzero element has a multiplicative inverse.

The following proposition which gives a characterisation of regular rings is taken from [31] and we supply the proof here for the sake of completeness. The following also follows from Theorem 2.2.2, Theorem 2.2.3, and also from the characterisations of regular rings in [53]. In constructing the proof, we used the following; [8], [41], [42], [52], [68] and [78].

Proposition 2.2.1. [31, Proposition 3.2] *The following are equivalent for a ring R.*

(1) *R is regular ring.*

(2) *Every principal ideal is generated by idempotents.*

(3) *Every prime ideal is maximal.*

(4) *Every ideal is an intersection of maximal ideals.*

(5) *Every ideal is a z-ideal à la Mason.*

(6) *Every principal ideal is a z-ideal à la Mason.*

(7) *Every maximal ideal is pure.*

Proof. (1) $\Rightarrow$ (2): Let R be a von Neumann regular ring and let a be an element that generates a principal ideal $I = \langle a \rangle$ for $a \in R$. Since R is von Neumann regular, there exists an element

x such that $a = axa$. Let $e = ax = axax$. Then, e is idempotent, meaning $e^2 = e$. We will now show that I is generated by e. First, note that every element in I is of the form xa for some $x \in R$. But $xa = exa$, so $xa \in \langle e \rangle$. This shows that $I \subseteq \langle e \rangle$. Next, let r be an element in $\langle e \rangle$. Then, $r = ex$ for some $x \in R$. But $r = axaxx = axr$, so r is in $\langle a \rangle = I$. This shows that $\langle e \rangle \subseteq I$. Therefore, we have shown that I is generated by e, which is an idempotent element.

(2) $\Rightarrow$ (4): In a von Neumann regular ring R where every principal ideal is generated by an idempotent element, we show that every ideal of R is the intersection of maximal ideals. To begin, let I be an ideal of R and use the von Neumann regularity to find an idempotent element $e \in R$ such that $I = eR$. As e is idempotent, we have $e^2 = e$ and hence $e \in I$. Consider the set $\mathcal{M}$ of all maximal ideals of R that contain I. Since every maximal ideal of R is of the form $M = eR$ where e is an idempotent element of R, we can write

$$\mathcal{M} = \{eR \mid e \in E \text{ where } E \text{ is the set of all idempotent elements of } R \text{ that contain } I\}.$$

Note that $\mathcal{M}$ is non-empty since it contains $I = eR$. We claim that $\bigcap_{M \in \mathcal{M}} M = I$. First, it is clear that $I \subseteq M$ for every $M \in \mathcal{M}$. To prove the reverse inclusion, let $x \in \bigcap_{M \in \mathcal{M}} M$. Then, $x \in M = eR$ for every $M \in \mathcal{M}$, so there exists an idempotent element $e_M \in R$ such that $x = ee_M$ for every $M \in \mathcal{M}$. Since $\mathcal{M}$ is non-empty, we can choose a maximal ideal $M_0 \in \mathcal{M}$ and let e_0 be the corresponding idempotent element. Then, we have $e_0 \in E$ and $e_0 \in M_0$, so $e_0 \in I$. Thus, we have $e_0 = e_0 e = e_0 e_{M_0}$, which implies $x = ee_0 e_{M_0} = e_0 e_{M_0} = e_0 \in I$. Therefore, we have shown that $\bigcap_{M \in \mathcal{M}} M = I$. Since $\mathcal{M}$ is the set of all maximal ideals of R that contain I, we have shown that I is the intersection of maximal ideals of R. Therefore, every ideal of R is the intersection of maximal ideals, as desired.

(1) $\Rightarrow$ (3): Suppose R is a von Neumann regular ring, and let P be a prime ideal of R. We want to show that P is a maximal ideal of R. Assume for contradiction that there exists an ideal Q of R such that $P \subsetneq Q$. Since P is prime, Q/P is a proper nonzero ideal of R/P. Because R is von Neumann regular, there exists an element $r \in R$ such that $r - 1 + Q/P = 0$ in R/P. This implies that $r - 1 \in Q$, which in turn implies that r is invertible in R/Q. Since r is invertible

in R/Q, there exists an element $s \in R$ such that $rs - 1 \in Q$. But then, $(rs-1)r \in Q$, and since r is invertible in R/Q, we have $s \in Q$. Therefore, Q is not a proper subset of P, which is a contradiction. Thus, we conclude that every prime ideal of R is maximal.

(3) $\Rightarrow$ (5): Assume that every prime ideal in R is maximal. We want to show that every ideal in R is a z-ideal $\grave{a}$ la Mason. Let I be an ideal of R and let $a, b \in R$ such that $aR \cap bI = abI$. We need to show that $a \in I$. Suppose not, then $I \subsetneq I + (a)$, which means $I + (a)$ is a proper ideal of R. By assumption, there exists a maximal ideal M such that $I + (a) \subseteq M$. Since M is maximal, R/M is a field, and therefore an integral domain. Let $\bar{a}$ be the image of a in R/M, and $\bar{x}$ be the image of x in R/M for any $x \in R$. Since $\bar{a}\bar{x} = 0$ for all $\bar{x} \in R/M$, either $\bar{a} = 0$ or $\bar{x} = 0$. If $\bar{a} = 0$, then $a \in M$, which implies $I + (a) \subseteq M$, a contradiction. Therefore, $\bar{x} = 0$ for all $\bar{x} \in R/M$, which implies $x \in M$ for all $x \in R$. Since M is a maximal ideal, it is a prime ideal, and therefore maximal by assumption. Thus, $aR \subseteq M$. Now, let $y \in I$ be arbitrary. Then $by \in bI$ and $by \in aR$, which implies $by \in abI$. Since $I \subseteq abI$, we have $by \in I$, and so I is a z-ideal $\grave{a}$ la Mason.

(4) $\Rightarrow$ (5): Let I be any ideal of R. By assumption, I is the intersection of all maximal ideals containing it. Hence, for any $a, b \in R$ such that $ab \in I$, we have $ab \in M$ for all maximal ideals M containing I. This implies that $a \in M$ or $b \in M$ for all such M, and hence either $a \in I$ or $b \in I$. Therefore, I is a z-ideal $\grave{a}$ la Mason.

(5) $\Rightarrow$ (6): Let $I = \langle a \rangle$ be a principal ideal of R, and let $x \in R$ such that $ax = a$. Then, $a \in \langle ax \rangle$ and $x \in \langle xa \rangle$. By assumption, I is a z-ideal $\grave{a}$ la Mason, which implies that either $a \in \langle xa \rangle$ or $x \in \langle ax \rangle$. If $a \in \langle xa \rangle$, then there exists $r \in R$ such that $a = rxa$, which implies that $a(1 - rxa) = 0$. Since R is a domain, this implies that $1 - rxa = 0$, and hence xa is a unit in R. Thus, $I = \langle a \rangle$ is a principal ideal z-ideal $\grave{a}$ la Mason. Let R be a von Neumann regular ring. Suppose that every ideal of R is a z-ideal $\grave{a}$ la Mason, and let $I = \langle a \rangle$ be a principal ideal of R. We want to show that I is a z-ideal $\grave{a}$ la Mason. Since R is von Neumann regular, there exists $x \in R$ such that $axa = a$. Then, $a \in \langle axa \rangle$ and $x \in \langle xax \rangle = \langle ax \rangle$, since R is commutative.

By assumption, I is a z-ideal *à* la Mason, which implies that either $a \in \langle xa \rangle$ or $x \in \langle ax \rangle$. If $a \in \langle xa \rangle$, then there exists $r \in R$ such that $a = rxa$, which implies that $a(1 - rxa) = 0$. Since R is an integral domain, this implies that $1 - rxa = 0$, and hence xa is a unit in R. Thus, I is a principal which is a z-ideal *à* la Mason.

(6) $\Rightarrow$ (7): Suppose R is a ring where every principal ideal is a z-ideal *à* la Mason, and let M be a maximal ideal of R. We want to show that M is a pure ideal. Let I be an ideal of R such that $I \subseteq M$. We want to show that I is a pure ideal, i.e., $I = J \cap M$ for some ideal J of R. Since R is a ring where every principal ideal is a z-ideal *à* la Mason, we know that every principal ideal is a z-ideal *à* la Mason. In particular, M is a z-ideal *à* la Mason, which means that M is the intersection of all maximal ideals containing it. Let $\mathcal{M}$ be the set of all maximal ideals of R containing I. Since M is a maximal ideal containing I, we have $\mathcal{M} \neq \varnothing$. Moreover, since R is a ring where every principal ideal is a z-ideal *à* la Mason , every principal ideal is pure. Thus, I is a pure ideal, which means that $I = J_1 \cap J_2 \cap \cdots \cap J_n$ for some ideals $J_1, J_2, \ldots, J_n$ of R.

Now, let $N = M \cap J_1 \cap J_2 \cap \cdots \cap J_n$. We claim that N is a maximal ideal containing I. To see this, suppose for contradiction that there exists an ideal K such that $N \subsetneq K \subseteq R$ and $I \subseteq K$. Since $I \subseteq M$, we have $K \nsubseteq M$, which means that there exists a maximal ideal M' such that $M \subsetneq M' \subseteq K$. But then $M' \in \mathcal{M}$, which means that $I \subseteq M'$, contradicting the assumption that K contains I. Since N is a maximal ideal containing I, and every maximal ideal containing I belongs to $\mathcal{M}$, we have $N \subseteq M$. But since M is a z-ideal *à* la Mason, we have $M = \bigcap_{M' \in \mathcal{M}} M'$. Therefore, $N = M \cap J_1 \cap J_2 \cap \cdots \cap J_n = (J_1 \cap M) \cap (J_2 \cap M) \cap \cdots \cap (J_n \cap M) = J \cap M$ for some ideal J of R, which shows that I is a pure ideal. Therefore, M is a pure ideal.

(1) $\Leftrightarrow$ (7): It follows from Theorem 2.2.2 that $nvr(R) = \emptyset$ if and only if $M = mM$ for each $M \in Max(R)$ and hence the result (see [78, Corollary 2.5]).

□

Definition 2.2.2. An ideal I of a ring $\mathcal{R}L$ is a *prime ideal* if whenever $\tau\varphi \in I$ implies that either $\tau \in I$ or $\varphi \in I$.

A *point* in a frame L is a frame homomorphism $\zeta : L \to 2$ (2 is the frame with two elements $0 \leq 1$). The set of all points of a frame L is denoted by $\sum L = Pt(L)$, this is called the *spectrum* of L. Next, we turn to the characterisation in terms of ideals. We start by recalling from [36] how the **M**- and **O**-ideals are defined. For each $I \in \beta L$, the ideals $\mathbf{M}^I$ and $\mathbf{O}^I$ of $\mathcal{R}L$ are defined by

$$\mathbf{M}^I = \{\varphi \in \mathcal{R}L \mid r(coz\varphi) \subseteq I\} \text{ and } \mathbf{O}^I = \{\varphi \in \mathcal{R}L \mid r(coz\varphi) \prec\!\prec I\}.$$

Clearly, $\mathbf{O}^I \subseteq \mathbf{M}^I$. Since, for any $I \in \beta L$ and $a \in L$, $r(a) \prec\!\prec I$ if and only if $a \in I$, it follows that

$$\mathbf{O}^I = \{\varphi \in \mathcal{R}L \mid coz\varphi \in I\}.$$

The following are shown in [36]:

(i) An ideal of $\mathcal{R}L$ is maximal if and only if it is of the form $\mathbf{M}^I$ for some $I \in \sum \beta L$.

(ii) $\mathbf{M}^I$ is maximal ideal if and only if I is a prime element of βL.

(iii) For any prime ideal P of $\mathcal{R}L$, there is a unique $I \in \sum \beta L$ such that $\mathbf{O}^I \subseteq P \subseteq \mathbf{M}^I$.

(iv) For any $I \in \sum \beta L$, $\mathbf{M}^I$ is the unique maximal ideal containing $\mathbf{O}^I$.

(v) The ideals $\mathbf{M}^I$ (respectively, $\mathbf{O}^I$) are distinct for distinct I for $I \in \sum \beta L$.

(vi) If $\mathbf{M}^I = \mathbf{M}^J$, then $I = J$.

(vii) For any $I \in \sum \beta L$ and $\varphi \in \mathcal{R}L$, $\varphi \in \mathbf{O}^I$ if and only if $\gamma\varphi = 0$ for some $\gamma \notin \mathbf{M}^I$.

Lemma 2.2.4. [35, Lemma 4.8] *Suppose $coz\gamma \prec\!\prec coz\delta$ for some $\gamma, \delta \in \mathcal{R}L$, then there exists an invertible $\tau \in \mathcal{R}L$ such that $\gamma = \gamma\tau\delta^2$. Hence, γ is a multiple of δ.*

Proof. Since $coz\gamma \prec\!\prec coz\delta$, there exists $\tau, \delta \in \mathcal{R}L$ such that $coz\gamma \wedge coz\tau = 0$ and

$$coz\delta \vee coz\tau = 1.$$

Then $coz(\tau^2+\delta^2) = 1$ since $\tau^2, \delta^2 \geq 0$ as squares are nonnegative in any f-ring. In consequence, $\tau^2 + \delta^2$ is invertible. Now we have that $coz(\gamma\tau) = 0$. so $\gamma\tau = 0$. Thus

$$\gamma = \gamma\tfrac{\tau^2+\delta^2}{\tau^2+\delta^2} = \tfrac{\gamma\tau^2+\gamma\delta^2}{\tau^2+\delta^2} = \delta\tfrac{\gamma\delta}{\tau^2+\delta^2},$$

which proves the result. □

The following lemma is taken from [31], we omit the proof.

Lemma 2.2.5. [31, Lemma 3.4] *For any ideal Q of $\mathcal{R}L$,*

$$mQ = \{\alpha \in \mathcal{R}L \mid coz\alpha \prec\prec coz\beta \text{ for some } \beta \in I\}.$$

The pure part of the ideal $\mathbf{M}^I$ is the ideal $\mathbf{O}^I$.

Lemma 2.2.6. [31, Lemma 3.5] *For any $I \in \Sigma\beta L$, $m\mathbf{M}^I = \mathbf{O}^I$.*

Proof. Let $\alpha \in \mathbf{O}^I$. Then $coz\alpha \in I$. Since I is a completely regular ideal, there exists $\gamma \in \mathcal{R}L$ such that $coz\alpha \prec\prec coz\gamma \in I$. But $coz\gamma \in I$ implies that $\gamma \in \mathbf{M}^I$. This shows, by Lemma 2.2.5, that $\alpha \in m\mathbf{M}^I$. Therefore $\mathbf{O}^I \subseteq m\mathbf{M}^I$. Next, let $\alpha \in m\mathbf{M}^I$ and pick $\gamma \in \mathbf{M}^I$ such that $\alpha = \alpha\gamma$. Thus $\alpha(1-\gamma) = 0$. Since $\mathbf{M}^I$ is a proper ideal and $\gamma \in \mathbf{M}^I$, we have that $1-\gamma \notin \mathbf{M}^I$. Now, since α is an annihilated by an element not belonging to $\mathbf{M}^I$, $\alpha \in \mathbf{O}^I$. Therefore $m\mathbf{M}^I \subseteq \mathbf{O}^I$, and hence $m\mathbf{M}^I = \mathbf{O}^I$. □

The following lemma is taken from [31], we omit the proof.

Lemma 2.2.7. [31, Lemma 3.7] *Let Q be an ideal of $\mathcal{R}L$. Then*

$$\cap\mathcal{M}(Q) = \left\{\varphi \in \mathcal{R}L \mid r(coz\varphi) \leq \bigvee_{\alpha\in Q} r(coz\alpha)\right\}.$$

Hence, for any $\gamma \in \mathcal{R}L$, $\cap\mathcal{M}(\gamma) = \{\varphi \in \mathcal{R}L \mid coz\varphi \leq coz\gamma\}$.

Definition 2.2.3. The *radical* of an ideal I of a ring R, denoted by $\sqrt{I}$, is the ideal

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}.$$

If $I = \sqrt{I}$, then I is called a *radical ideal.*

Definition 2.2.4. An ideal I of a ring $\mathcal{R}L$ is a *z-ideal* if $cozf = cozg$ with $g \in I$ implies $f \in I$. Put

$$Rad(I) = \bigcap\{P \mid P \text{ is a prime ideal,} I \subseteq P\}.$$

The following corollary and proof is taken from [31].

Corollary 2.2.1. [31, Corollary 3.8] *An ideal of $\mathcal{R}L$ is a z-ideal if and only if it is a z-ideal à la Mason.*

Proof. Let Q be a z-ideal and suppose $\mathcal{M}(\alpha) \supseteq \mathcal{M}(\beta)$ for some $\beta \in Q$. Then $\alpha \in \cap\mathcal{M}(\alpha) \subseteq \mathcal{M}(\beta)$, and so $coz\alpha \leq coz\beta$, thus $coz\alpha = coz(\alpha\beta)$, and since $\alpha\beta \in Q$, we have that $\alpha \in Q$ because Q is a z-ideal. Therefore Q is a z-ideal *à* la Mason.

Conversely, suppose that $coz\alpha = coz\beta$ with β in the ideal Q. Then by Lemma 2.2.7, $\alpha \in \cap\mathcal{M}(\beta)$. But this implies that $\mathcal{M}(\alpha) \supseteq \mathcal{M}(\beta)$, and consequently that $\alpha \in Q$ by hypothesis. Therefore Q is a z-ideal. □

The following are characterisations of von Neumann inverses that are initiated in [78].

(i) φ has a VN-inverse if and only if for each $\varphi \in \mathbf{M}^I$ implies $\varphi \in m\mathbf{M}^I$.

(ii) φ has a VN-inverse if and only if $\varphi\alpha$ is idempotent for some invertible α.

In classical context we have that a point $p \in \beta X$ is called *P-point* if $\mathbf{O}^p = \mathbf{M}^p$ (see [4]). We now recall from [72] that a point I of βL is a *P-point* if $\mathbf{M}^I = \mathbf{O}^I$. Note that if ν is an idempotent in $\mathcal{R}L$, then $coz\nu$ is complemented because $coz\nu \wedge coz(1-\nu) = 0$ and $coz\nu \vee coz(1-\nu) = 1$ (see [34]).

Proposition 2.2.2. [31, Proposition 3.9] *The following are equivalent for a frame L:*

(1) *L is a P-frame.*

(2) *$\mathcal{R}L$ is a regular ring.*

(3) *Every ideal of $\mathcal{R}L$ is a z-ideal.*

(4) *Every ideal of $\mathcal{R}L$ is an intersection of prime ideals.*

(5) *Every ideal of $\mathcal{R}L$ is an intersection of maximal ideals.*

(6) *Every prime ideal of $\mathcal{R}L$ is an intersection of maximal ideals.*

(7) *Every z-ideal of $\mathcal{R}L$ is an intersection of maximal ideals.*

(8) *For every $\gamma, \delta \in \mathcal{R}L$, $\langle \gamma, \delta \rangle = \langle \gamma^2 + \delta^2 \rangle$.*

(9) *Every principal ideal of $\mathcal{R}L$ is generated by an idempotent.*

(10) $\mathbf{O}^I = \mathbf{M}^I$ *for each* $I \in \sum \beta L$.

Proof. (1) $\Leftrightarrow$ (2): Let $a \in CozL$, then there is $\varphi \in \mathcal{R}L$ such that $a = coz\varphi$ is complemented (since L is a P-frame). Furthermore, if $\varphi \in \mathbf{M}^I$ for some $I \in \Sigma\beta L$, since r preserves the completely below relation, thus $r(a) \prec\prec r(a) \leq I$, showing that $\varphi \in \mathbf{O}^I$. Consequently φ has a von Neumann inverse by Theorem 2.2.6 and first result quoted from [78] of characterisations of von Neumann inverses. Thus $\mathcal{R}L$ is regular.

Conversely, assume $\mathcal{R}L$ is regular. Let $\varphi \in \mathcal{R}L$. Then φ has a von Neumann inverse. Thus by second result quoted from [78] of characterisations of von Neumann inverses, there is $\alpha \in \mathcal{R}L$ such that $\varphi\alpha$ is idempotent and α is invertible. Furthermore $coz(\varphi\alpha)$ is complemented. But $\varphi^2\alpha = \varphi\varphi\alpha = \varphi$ and $coz\alpha = 1$; so that $coz(\varphi\alpha) = coz\varphi \wedge coz\alpha = coz\varphi \wedge 1 = coz\varphi$; so thus $coz\varphi$ is complemented. Therefore L is a P-frame.

The equivalences of (1), (3), (5), (9) and (10) now follow from Proposition 2.2.1.

(3) $\Rightarrow$ (4): Let I be an ideal of $\mathcal{R}L$. It suffices to show that $I = \sqrt{I}$, its radical. So let $\delta \in \sqrt{I}$.

Then there is a natural number n such that $\delta^n \in I$. But $coz\delta = coz(\delta^n)$, and so $\delta \in I$, since I is a z-ideal by (3). This shows that $I = \sqrt{I}$, an intersection of prime ideals.

(4) $\Rightarrow$ (2): Let $\gamma \in \mathcal{R}L$. Any prime ideal of $\mathcal{R}L$ contains γ if and only if it contains γ^2. So the present hypothesis implies that $\langle\gamma\rangle = \langle\gamma^2\rangle$, and consequently $\gamma = \delta\gamma^2$, for some $\delta \in \mathcal{R}L$. We are done.

(5) $\Rightarrow$ (6): It is trivial.

(6) $\Rightarrow$ (2): Since every prime ideal is contained in a maximal ideal. The hypothesis implies that every prime ideal is maximal, so that $\mathcal{R}L$ is regular by Proposition 2.2.1.

(5) $\Rightarrow$ (7): It is trivial.

(7) $\Rightarrow$ (10): For any $J \in \Sigma\beta L$, $\mathbf{O}^J$ is a z-ideal, and $\mathbf{M}^J$ is the unique maximal ideal containing it. So the present hypothesis implies that $\mathbf{O}^J = \mathbf{M}^J$.

(3) $\Rightarrow$ (8): Clearly, $\langle\gamma^2 + \delta^2\rangle \subseteq \langle\gamma + \delta\rangle$. Now let $\phi \in \langle\gamma + \delta\rangle$. Then $\phi = \mu_1\gamma + \mu_2\delta$ for some $\mu_1, \mu_2 \in \mathcal{R}L$. Therefore

$$coz\phi \leq coz\gamma \vee coz\delta = coz(\gamma^2) \vee coz(\delta^2) = coz(\gamma^2 + \delta^2).$$

By (3), $\langle\gamma^2 + \delta^2\rangle$ is z-ideal, it follows that $\phi \in \langle\gamma^2 + \delta^2\rangle$. Thus $\langle\gamma^2 + \delta^2\rangle = \langle\gamma + \delta\rangle$.

(8) $\Rightarrow$ (2): Let $\gamma \in \mathcal{R}L$, and consider the principal ideal $\langle\gamma\rangle$ in $\mathcal{R}L$. By (8), we have

$$\langle\gamma\rangle = \langle\gamma, 0\rangle = \langle\gamma^2\rangle.$$

Therefore $\gamma = \delta\gamma^2$ for some $\delta \in \mathcal{R}L$. We are done.

□

The following lemmas are taken from [58], we omit the proofs.

Lemma 2.2.8. [58, Lemma 2.1.3] *For any $\alpha, \beta \in \mathcal{R}L$, the following are equivalent:*

(1) $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$.

(2) $M_{coz\alpha} = M_{coz\beta}$.

(3) $coz\alpha = coz\beta$.

The class of z-ideals contains the class of maximal ideals and the class of minimal prime ideals.

Lemma 2.2.9. *We have the following for z-ideals:*

(1) *Every maximal ideal is a z-ideal.*

(2) *Every minimal prime ideal is a z-ideal.*

(3) *Intersections of z-ideals are z-ideals.*

Definition 2.2.5. An ideal J of $\mathcal{R}L$ is a *d-ideal* if for any $\alpha \in \mathcal{R}L$ and $\eta \in J$, $coz\alpha \leq (coz\eta)^{**}$ implies $\alpha \in J$. Equivalently if, for any $a \in J$, $P_a \subseteq J$, where P_a is the intersection of all minimal prime ideals of $\mathcal{R}L$ containing a.

The following proposition is taken from [58], we omit the proof.

Proposition 2.2.3. [58, Proposition 4.1.1] *The following are equivalent for a singular ideal Q of $\mathcal{R}L$:*

(1) *Q is a d-ideal.*

(2) *For any $\alpha, \beta \in \mathcal{R}L$, if $\alpha \in Q$ and $(coz\beta)^* = (coz\alpha)^*$ imply $\beta \in Q$.*

(3) *For any $\alpha, \beta \in \mathcal{R}L$, if $\alpha \in Q$ and $(coz\alpha)^* \leq (coz\beta)^*$ imply $\beta \in Q$.*

(4) *For any $\alpha, \beta \in \mathcal{R}L$, if $\alpha \in Q$ and $coz\beta \leq (coz\alpha)^{**}$ imply $\beta \in Q$.*

An *f-ring* R is a lattice-ordered ring such that $(a \wedge b)c = (ac) \wedge (bc)$ holds for every $a, b \in R$ and $c \in R^+ = \{x \in A \mid x \geq 0\}$. It has *bounded inversion* if every element $a \geq 1$ in R is invertible in R. One of special property of f-rings that we note is, for all $a, b \in A$

$$a^2 \geq 0 \text{ and } | ab |=| a || b |.$$

For general information on f-rings (see [26]). We say an element a of an f-ring is *positive* if $a \geq 0$. If a is an invertible positive element in an f-ring, then a^{-1} is also positive. For squares in any f-ring, and $a^{-1} = (a^{-1})^2 a$, which is a product of two positive elements. This one shows that if R is an f-ring with bounded inversion (which is to say any element above the identity of the ring is invertible), then for any $a \in R$, $\frac{1}{1+|a|}$ and $\frac{a}{1+|a|}$ are bounded. A ring ideal I of an f-ring R is an *l-ideal* if $|x| \leq |y|$, $y \in I$ implies $x \in I$. We recall from Larson [66] that minimal prime l-ideals are z-ideals, z-ideals are semiprime and an l-ideal is semiprime if and only if it is an intersection of prime l-ideals.

See also [84], for the following definition.

Definition 2.2.6. An ideal I of a f-ring R is said to be *essential* if it intersects every nonzero ideal of the ring non-trivially. Equivalently, if for any nonzero element a of a ring, there exists a nonzero element $b \in I$ such that b is a multiple of a.

The following lemma and its proof is taken from [36].

Lemma 2.2.10. [36, Lemma 4.3] *An ideal I in $\mathcal{R}L$ is essential if and only if $\bigvee\{coz\varphi \mid \varphi \in I\}$ is dense.*

Proof. ($\Rightarrow$) Let $c \in CozL$ such that $c \wedge \bigvee\{coz\varphi \mid \varphi \in I\} = 0$. We will show that $c = 0$. Let $\tau \in \mathcal{R}L$ such that $coz\tau = c$. Then $\bigvee\{coz\tau \wedge coz\varphi \mid \varphi \in I\} = 0$, which implies that $\tau\varphi = 0$ for each $\varphi \in I$. Therefore the principal ideal $\langle\tau\rangle$ is the zero ideal, for otherwise there exists $\alpha \in \mathcal{R}L$ such that $0 \neq \alpha\tau \in I$; a contradiction because $0 = coz(\alpha\tau)\tau = coz(\alpha\tau)$ implies $\alpha\tau = 0$. Thus $c = 0$, and hence $\bigvee\{coz\varphi \mid \varphi \in I\}$ is dense by complete regularity.

($\Leftarrow$) Let J be a nonzero ideal of $\mathcal{R}L$ and take $0 \neq \alpha \in J$. Then $coz\alpha \wedge \bigvee\{coz\varphi \mid \varphi \in I\} \neq 0$. This implies $0 \neq \bigvee\{coz\alpha \wedge coz\varphi \mid \varphi \in I\} = \bigvee\{coz(\alpha\varphi) \mid \varphi \in I\}$, which in turn implies that $coz(\alpha\delta) \neq 0$ for some $\delta \in I$, and hence $\alpha\delta \neq 0$. So I meets J non-trivially since $\alpha\delta \in I \cap J$. $\square$

Lemma 2.2.11. [35, Lemma 4.4] *For any $I \in \beta L$, we have $\bigvee coz[\mathbf{O}^I] = \bigvee coz[\mathbf{M}^I] = \bigvee I$.*

Proof. The forward implication is trivial, since $\bigvee coz[\mathbf{O}^I] \leq \bigvee coz[\mathbf{M}^I]$. If $\gamma \in \mathbf{M}^I$, then $r(coz\gamma) \subseteq I$. Taking joins yields $coz\gamma \leq \bigvee I$. Thus, $\bigvee coz[\mathbf{O}^I] \leq \bigvee coz[\mathbf{M}^I] \leq \bigvee I$. Now let $c \in CozL$ in I. Say $c = coz\gamma$ for some $\gamma \in \mathcal{R}L$. Then $\gamma \in \mathbf{O}^I$, and so $c \leq \bigvee coz[\mathbf{O}^I]$. Since every element of I is below some cozero element in I, it follows that $\bigvee coz[\mathbf{M}^I] \leq \bigvee I \leq \bigvee coz[\mathbf{O}^I]$. Thus $\bigvee coz[\mathbf{O}^I] = \bigvee coz[\mathbf{M}^I] = \bigvee I$. □

Definition 2.2.7. An ideal I of an f-ring R is called:

(1) *convex* if, for any $a, b \in R$

$$0 \leq a \leq b \text{ and } b \in I \text{ implies that } a \in I.$$

(2) *absolutely convex* if, for any $a, b \in R$,

$$0 \leq |a| \leq |b| \text{ and } b \in I \text{ implies that } a \in I.$$

The following lemma is taken from [58], we omit the proof.

Lemma 2.2.12. [58, Corollary 7.2.1] *An ideal of $\mathcal{R}L$ is a z-ideal if and only if its radical is a z-ideal.*

The following lemma is taken from [37], we omit the proof.

Lemma 2.2.13. [37, Lemma 3.5] *Every radical ideal in $\mathcal{R}L$ is absolutely convex.*

Definition 2.2.8. A ring R is said to be a *z-good* if it has the property that an ideal of R is a z-ideal if and only if its radical is a z-ideal. Equivalently, if every ideal of R whose radical is a z-ideal is itself a z-ideal.

Lemma 2.2.14. [37, Lemma 3.1] *A z-good ring is von Neumann regular if and only if every prime ideal in it is a z-ideal.*

Proof. The left-to right implication follows by Theorem 2.2.3 because in a von Neumann regular ring every ideal is a z-ideal.

Conversely, Let R be a z-good ring in which every prime ideal is a z-ideal. Let J be an ideal of R. Since $\sqrt{J}$ is an intersection of prime ideals, it is an intersection of z-ideals, and is therefore itself a z-ideal. Then J is a z-ideal because R is a z-good, and therefore R is von Neumann regular by Theorem 2.2.3. □

The following lemma is taken from [37], we omit the proof.

Lemma 2.2.15. [37, Lemma 3.4] *$\mathcal{R}L$ is a z-good ring.*

Every d-ideal is a z-ideal, and in von Neumann regular rings d-ideals coincide with z-ideals.

Theorem 2.2.16. [37, Proposition 3.1] *The following are equivalent for a completely regular frame L.*

(1) *L is a P-frame.*

(2) *Every essential ideal in $\mathcal{R}L$ is a z-ideal.*

(3) *Every radical ideal in $\mathcal{R}L$ is a z-ideal.*

Proof. (1) $\Rightarrow$ (2): By (1), it follows that every essential ideal in $\mathcal{R}L$ is a z-ideal, since every ideal in $\mathcal{R}L$ is a z-ideal.

(2) $\Rightarrow$ (3): Let I be a radical ideal in $\mathcal{R}L$. Consider any prime ideal Q containing I. If Q is essential, then it is a z-ideal by hypothesis, and if it non-essential then it is a z-ideal. Thus, I is an intersection of z-ideals, and is therefore a z-ideal.

(3) $\Rightarrow$ (1): Since $\mathcal{R}L$ is a z-good ring, it follows from Lemma 2.2.14 that $\mathcal{R}L$ is a von Neumann regular ring and hence L is a P-frame. □

Dube and Ighedo [37], wrote Spec(R), Rad(R), Abs(R) and Con(R) for the set of primes, radical, absolutely convex, and convex ideals of R. The authors had the following inclusions

$$\text{Spec}(R) \subseteq \text{Rad}(R) \text{ and } \text{Abs}(R) \subseteq \text{Con}(R).$$

If R is a z-good f-ring for which $\text{Rad}(R) \subseteq \text{Abs}(R)$, then R is a von Neumann regular if and only if every ideal in any of the collections above is a z-ideal. For any completely regular frame L, $\text{Rad}(\mathcal{R}L) \subseteq \text{Abs}(\mathcal{R}L)$, which is the result proved by Banaschewski, the proof is recorded in [35, Lemma 3.5], uses uniform frames. It is shown that in $C(X)$ prime ideals are absolute convex, which then applies to all radical ideals (see [52, Theorem 5.5]). Dube and Ighedo also in [37], noted that any convex radical ideal I in an f-ring is actually absolutely convex. The following theorem is immediate.

Theorem 2.2.17. [37, Proposition 3.2] *The following are equivalent for a completely regular frame L.*

(1) *L is a P-frame.*

(2) *Every convex ideal in $\mathcal{R}L$ is a z-ideal.*

(3) *Every absolutely convex ideal in $\mathcal{R}L$ is a z-ideal.*

Remark: The proof of the above Theorem 2.2.17 is also discussed in various other texts on the theory of frames and ordered algebraic structures, such as in [28] and [59]. The characterisations in Theorem 2.2.16 and Theorem 2.2.17 hold with z-ideals replaced by d-ideals.

Next, we give some background about R-modules in order to give a characterisation of P-frames in terms of flat $\mathcal{R}L$-modules. We have the following definition from [82].

Definition 2.2.9. A *right R-module*, where R is a ring, is an additive abelian group M with 0 as the identity, and having a scalar multiplication $M \times R \to M$, denoted by $(m, r) \mapsto mr$ such that, for all $m, \acute{m} \in M$ and $r, \acute{r} \in R$.

(1) $(m + \acute{m})r = mr + \acute{m}r$,

(2) $m(r + \acute{r}) = mr + m\acute{r}$,

(3) $m(r\acute{r}) = (mr)\acute{r}$, and

(4) $m = m1$.

Definition 2.2.10. A *left R-module*, where R is a ring, is an additive abelian group M with 0 as the identity, and having a scalar multiplication $R \times M \to M$, denoted by $(r, m) \mapsto rm$ such that, for all $m, \acute{m} \in M$ and $r, \acute{r} \in R$.

(1) $r(m + \acute{m}) = rm + r\acute{m}$,

(2) $(r + \acute{r})m = rm + \acute{r}m$,

(3) $(r\acute{r})m = (rm)\acute{r}$, and

(4) $m = 1m$.

Remark: Note that if R is commutative ring, then the conditions of *right* and *left R-module* are equivalent. In that case we say M is an *R-module*.

Example 2.2.1.

(1) Every abelian group is a $\mathbb{Z}$-module.

(2) Every ring R is a module over itself if we define scalar multiplication $R \times R \to R$ to be a given multiplication of element of R. More generally, every left ideal in R is a R-module.

Let M and N be two R-modules. An *R-module homomorphism* is a map

$$\phi : M \to N$$

such that

$$\phi(m + n) = \phi(m) + \phi(n) \text{ and } \phi(rm) = r\phi(m).$$

An *R-module isomorphism* is a bijective R-module homomorphism. We will also say that ϕ is R-linear.

Let M be an R-module. A *submodule* N of M is a subset that is a module with inherited addition and scalar multiplication.

If $f : M \to N$ is an R-map between left R-modules, then

$$\textit{kernel } f = \textit{ker} f = \{m \in M \mid f(m) = 0\},$$
$$\textit{image } f = \textit{im} f = \{n \in N \mid \text{ there exist } m \in M \text{ with } n = f(m)\}.$$

The kernel is a submodule.

Let M be an R-module and let X be a subset. The R-module generated by X, denoted by $\langle X \rangle$, is equal to the smallest submodule that contains X. We say that the set X generates M if the submodule generated by X is the whole of M. We say that M is *finitely generated* if it is generated by a finite set. We say M is *cyclic* if it is generated by a single element.

Definition 2.2.11. Let M and N be two R-modules. The *direct sum of M and N*, denoted by $M \oplus N$, is the R-module, which as a set is the Cartesian product of M and N, with addition and multiplication defined coordinate by coordinate:

$$(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2) \text{ and } r(m, n) = (rm, rn).$$

Note that the direct sum is a direct sum in the category of R-modules. Note that the direct sum of R with itself is generated by $(1, 0)$ and $(0, 1)$.

Theorem 2.2.18. *Let $\phi : M \to N$ be a surjective R-linear homomorphism, with kernel K. Then $N \cong M/N$.*

Definition 2.2.12. Let M be an R-module. We say that M is *free* if it is isomorphic to a direct sum of copies (possibly infinite) of R.

Definition 2.2.13. A finite or infinite sequence of R-homomorphism and left R-modules

$$\cdots \to M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \to \cdots$$

is called an *exact sequence* if $imf_{n+1} = kerf_n$, where imf_{n+1} denotes image of f_{n+1} and $kerf_n$ denotes kernel of f_n.

Observe that there is no need to label arrows $0 \xrightarrow{f} A$ and $B \xrightarrow{g} 0$: in either case, there is a unique map, namely $f : 0 \mapsto 0$ or the constant homomorphism $g(b) = 0$ for all $b \in B$. Here are some simple consequences of a sequence of homomorphism being exact. The following proposition is taken from [82], we omit the proof.

Proposition 2.2.4. [82, Proposition 2.18]

(1) *A sequence* $0 \to A \xrightarrow{\varphi} B$ *is exact if and only if* φ *is injective.*

(2) *A sequence* $B \xrightarrow{\varphi} C \to 0$ *is exact if and only if* φ *is surjective.*

(3) *A sequence* $0 \to A \xrightarrow{\varphi} B \to 0$ *is exact if and only if* φ *is isomorphism.*

Next, Let R be a ring. Let A be a right R-module, B be a left R-module, and G be an (additive) abelian group. A function $f : A \times B \to G$ is called R-*biadditive* if, for all $a, a' \in A$, $b, b' \in B$, and $r \in R$, we have:

(i) $f(a + a', b) = f(a, b) + f(a', b)$.

(ii) $f(a, b + b') = f(a, b) + f(a, b')$.

(iii) $f(ar, b) = f(a, rb)$.

If R is *commutative* and A, B and M are R-modules, then a function $f : A \times B \to M$ is called R-*bilinear* if f is R-biadditive and also

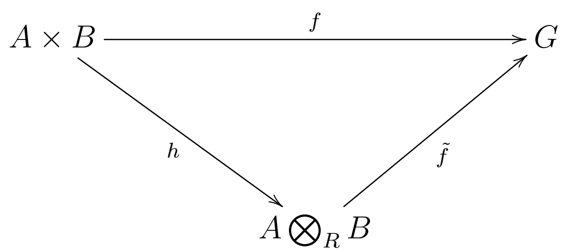
$$f(ar, b) = f(a, rb) = rf(a, b)$$

($rf(a,b)$ makes sense here because $f(a,b)$ now lies in the R-module M).

Given a ring R and modules A and B, then their *tensor product* is an abelian group $A \bigotimes_R B$ and an R-biadditive function

$$h : A \times B \to A \bigotimes_R B$$

such that, for every abelian group G and every R-biadditive $f : A \times B \to G$, there exists a unique $\mathbb{Z}$-homomorphism $\tilde{f} : A \bigotimes_R B \to G$ making the following diagram commutes.



The following definition is culled from Acharyya *et al* [82].

Definition 2.2.14. An ideal I of $\mathcal{R}L$ is called *flat* if the tensor product $I \bigotimes_{\mathcal{R}L}$- is an exact functor, i.e, if

$$0 \to N_1 \xrightarrow{i} N \to N_2 \to 0$$

is an exact sequence of $\mathcal{R}L$-modules, then

$$0 \to M \bigotimes_{\mathcal{R}L} N_1 \xrightarrow{1_M \bigotimes i} M \bigotimes_{\mathcal{R}L} N \xrightarrow{1_M \bigotimes p} M \bigotimes_{\mathcal{R}L} N_2 \to 0$$

is an exact sequence of abelian groups.

Because the functors $I \bigotimes_{\mathcal{R}L}$- : ${}_{\mathcal{R}L}\mathbf{Mod} \to \mathbf{Ab}$ are right exact, we see that a right $\mathcal{R}L$-module I is flat if and only if whenever $i : N_1 \to N$ is an injective, then $1_M \bigotimes i : M \bigotimes_{\mathcal{R}L} N_1 \to M \bigotimes_{\mathcal{R}L} N$ is also injective. Where ${}_{\mathcal{R}L}\mathbf{Mod}$ is the category of all $\mathcal{R}L$-modules and $\mathbf{Ab}$ is the category where the objects are abelian groups, morphisms are homomorphisms, and composition is the usual composition. A quotient of a flat module is not necessarily flat; after all, free modules are flat, and every module is a quotient of a free module.

Lemma 2.2.19. [82, Proposition 3.60] *Let* $0 \to K \to F \xrightarrow{\varphi} A \to 0$ *be an exact sequence of* R*-modules in which* F *is flat. Then* A *is a flat module if and only if* $K \cap FI = KI$ *for every finitely generated left ideal* I*.*

Lemma 2.2.20. [82, Lemma 4.8] *If* R *is a von Neumann regular ring, then every finitely generated left (or right) ideal is a principal, and it is generated by an idempotent.*

In the following theorem, the ring R is not assumed to be commutative ring.

Theorem 2.2.21. [82, Theorem 4.9] *A ring* R *is is von Neumann regular if and only if every right* R*-module is flat.*

Proof. Assume that R is von Neumann regular and B is a right R-module. If

$$0 \to K \to F \to B \to 0$$

is an exact sequence of R-modules with F free, then Lemma says B is flat if $KI = K \cap FI$ for every finitely generated left ideal I. By Lemma 2.2.20, I is principal say $I = Ra$. We must show that if $k \in K$ and $k = fa \in Fa$, then $k \in Ka$. But $k = fa = fa\acute{a} \in Ka$. Therefore, B is flat.

Conversely, take $a \in R$. By hypothesis, the cyclic right R-module R/aR is flat. Since R is free, applies to the exact sequence

$$0 \to aR \to R \to R/aR \to 0$$

to give $(aR)I = aR \cap RI = aR \cap I$ for every left ideal I. In particular, if $I = Ra$, the $aRa = aR \cap Ra$. Thus, there is some $\acute{a} \in R$ such that $a = a\acute{a}a$, and so R is von Neumann regular. □

Recall that an element $a \in L$ is called prime if $a \neq 1$ and $b \wedge c \leq a$ implies either $b \leq a$ or $c \leq a$, and maximal if it is maximal below the top element of L.

Remark : Prime elements are precisely the points of a frame introduced earlier.

It can be shown that in a completely regular frame L, the prime elements are exactly the maximal elements (see [84]). A prime ideal P in R is called an *upper ideal* if the set of all its predecessors in the family of prime ideals partially ordered under set inclusion has a maximal element (see [52, Chapter 14]). Acharyya *et al* [5] proved the following two theorems, in our case we omit the proofs. It is important to note them down.

Theorem 2.2.22. *An upper ideal of $\mathcal{R}L$ is not a z-ideal.*

Theorem 2.2.23. *(Main Theorem on the Existence of an Ascending Chain). If P is a non-maximal prime ideal of $\mathcal{R}L$ and M is the unique maximal ideal extending P, then there exists a strictly ascending chain of upper ideals of $\mathcal{R}L$ that lie between P and M.*

To add the list in Theorem 2.2.16, we give the following result obtained by Acharyya *et al* [6].

Proposition 2.2.5. [6, Theorem 4.1] *A frame L is a P-frame if and only if each prime ideal of $\mathcal{R}L$ is a z-ideal.*

Proof. If L is a P-frame, then each ideal and hence each prime ideal of $\mathcal{R}L$ is a z-ideal by Proposition 2.2.2. Conversely, if L is not a P-frame, then there is a non-maximal prime ideal P of $\mathcal{R}L$ by Proposition 2.2.2. Let M be a unique maximal ideal containing P, then there is an upper ideal U in between P and M by Theorem 2.2.23. Therefore we get a prime ideal U of $\mathcal{R}L$ which is not a z-ideal, as it is an upper ideal by Theorem 2.2.22. Hence the proposition is proved. □

Proposition 2.2.6. [6, Theorem 4.3] *A frame L is a P-frame if and only if every $\mathcal{R}L$-module is flat.*

Proof. Since, L is a P-frame if and only if $\mathcal{R}L$ is a von Neumann regular ring by Proposition 2.2.2, the result follows from Theorem 2.2.21. □

Recall the definition of the right adjoint r, the following lemma.

Lemma 2.2.24. [2, Lemma 4.4] *The following statements hold for a regular frame L.*

(1) *If $p \in Pt(L)$, then either $p^* = 0$ or $p \vee p^* = 1$.*

(2) *If* $p \in Pt(L)$ *and* $p \vee p^* = 1$*, then* $r(p) \vee (r(p))^* = 1_{\beta L}$.

(3) *If* I *is a complemented point of* βL*, then* $\mathbf{O}^I = \mathbf{M}^I$.

Proof. (1). Since $p \leq p \vee p^*$ and every prime element in regular frames is maximal, it follows that $p = p \vee p^*$ or $p \vee p^* = 1$. The former case implies that $p^* \leq p$ which means that $p^* \wedge p^* = 0$ that is, $p^* = 0$.

(2). Since $(r(p))^* = r(p^*)$, $p \in r(p)$, and $p^* \in r(p^*)$, we can conclude that $r(p) \vee (r(p))^* = 1_{\beta L}$.

(3). We are required to show $\mathbf{M}^I \subseteq \mathbf{O}^I$. Suppose $\varphi \in \mathbf{M}^I$. Then $r(coz\varphi) \subseteq I$. Now, $I \prec\prec I$, we have $r(coz\varphi) \prec\prec I$. It follows that $\varphi \in \mathbf{O}^I$. Hence $\mathbf{M}^I \subseteq \mathbf{O}^I$. □

Proposition 2.2.7. *The following are equivalent for a completely regular frame* L*:*

(1) L *is a* P*-frame.*

(2) *If* $I + J$ *is a* z*-ideal, then both* I *and* J *are* z*-ideals.*

(3) *If* $I \cap J$ *is a* z*-ideal, then both* I *and* J *are* z*-ideals.*

Proof. (1) $\Rightarrow$ (2): Let I and J be ideals in $\mathcal{R}L$ such that $I + J$ is a z-ideal. By hypothesis, every ideal in $\mathcal{R}L$ is a z-ideal. Hence I and J are z-ideals.

(2) $\Rightarrow$ (3): If $I \cap J$ is a z-ideal. Now since $I \cap J \subseteq I + J$, by hypothesis I and J are z-ideals.

(3) $\Rightarrow$ (1): Let Q be an ideal in $\mathcal{R}L$. Then either $Q \subset I \cap J$ or $I \cap J \subset Q$ where $I \cap J$ is a z-ideal of $\mathcal{R}L$. If $I \cap J \subset Q$, then by (3), Q is a z-ideal. If $Q \subset I \cap J$, then by (2), Q is a z-ideal. But Q is an arbitrary ideal in $\mathcal{R}L$, so every ideal of $\mathcal{R}L$ is a z-ideal and hence L is a P-frame. □

Lemma 2.2.25. [31, Lemma 3.6] *If* H *is an ideal of* $\mathcal{R}L$ *and* $\alpha \in \mathcal{R}L$ *such that*

$$r(coz\alpha) \prec\prec \bigvee\{r(coz\gamma) \mid \gamma \in H\},$$

then $\alpha \in H$.

Proof. If $r(coz\alpha) \prec\prec \vee\{r(coz\gamma) \mid \gamma \in H\}$, then $r(coz\alpha)^* \prec\prec \vee\{r(coz\gamma) \mid \gamma \in H\} = 1$, and therefore, by compactness of βL, there are finitely many elements $\alpha_1, ..., \alpha_m$ of H such that

$$r(coz\alpha)^* \vee r(coz(\alpha_1)) \vee \cdots \vee r(coz(\alpha_m)) = 1.$$

This implies that

$$r(coz\alpha) \prec\prec r(coz(\alpha_1)^2) \vee \cdots \vee r(coz(\alpha_m)^2) = r(coz({\alpha_1}^2 + \cdots + {\alpha_m}^2)).$$

Acting the join map yields $coz(\alpha) \prec\prec coz(\alpha_m)^2) = r(coz({\alpha_1}^2 + \cdots + {\alpha_m}^2)$. Therefore, by Lemma 2.2.4, α is a multiple of the element ${\alpha_1}^2 + \cdots + {\alpha_m}^2 \in H$, and so $\alpha \in H$. $\square$

Pure ideals of $\mathcal{R}L$ are precisely the ideals $\mathbf{O}^I$ for $I \in \beta L$ (see [33, Proposition 4.3]).

Proposition 2.2.8. [31, Corollary 3.10] *A frame L is a P-frame if and only if every ideal of $\mathcal{R}L$ is pure.*

Proof. If L is a P-frame and H is an ideal of $\mathcal{R}L$, then we claim that $H = \mathbf{O}^I$ for

$$I = \bigvee\{r(coz\alpha \mid \alpha \in H)\}.$$

To see this, let $\gamma \in \mathbf{O}^I$. Then $r(coz\gamma) \prec\prec I$, which implies, by Lemma 2.2.25 that $\gamma \in H$. So $\mathbf{O}^I \subseteq H$. On the other hand, let $\gamma \in H$. Hence, in view of L being a P-frame, $coz\gamma = coz\gamma$, implying that $r(coz\gamma) \prec\prec r(coz\gamma) \leq H$. Therefore $\gamma \in \mathbf{O}^I$, and hence $H \subseteq \mathbf{O}^I$. Consequently, every ideal of $\mathcal{R}L$ is pure. The converse holds because being pure implies $\mathbf{M}^I = m\mathbf{M}^I = \mathbf{O}^I$ for each $I \in \sum \beta L$. $\square$

Abedi [1] abbreviated $\mathbf{M}^{r(a)}$ as $\mathbf{M}_a$ and $\mathbf{O}^{r(a)}$ as $\mathbf{O}_a$, for for $a \in L$. Hence

$$\mathbf{M}_a = \{\varphi \in \mathcal{R}L \mid coz\varphi \leq a\} \text{ and } \mathbf{O}_a = \{\varphi \in \mathcal{R}L \mid coz\varphi \prec\prec a\}.$$

The following definition is culled from [1].

Definition 2.2.15. An ideal I of a ring R has the *Artin-Rees property (AR property)* if for each ideal Q of R there exist $n \in \mathbb{N}$ such that $I^n \cap Q \subseteq IQ$. We say a ring R is an *Artin-Rees ring (AR-ring)* if for every ideal of R has Artin-Rees property. An ideal I of a ring R is called an *Artin-Rees ideal* (*AR-ideal*) if for any two sub ideals M and N of I there exists $n \in \mathbb{N}$ such that

$$M^n \cap K \subseteq MN.$$

Recall the pure part of an ideal of a ring on page 25. This is equivalent to the statement that for every ideal J of R, the equality $I \cap J = IJ$ holds (see also [59]). Johnstone [59] showed that every pure ideal in $\mathcal{R}L$ is a pure ideal, and Dube [31] showed that every pure ideal in $\mathcal{R}L$ is z-ideal. Every pure ideal of a ring R has AR property.

Lemma 2.2.26. [1, Lemma 1] *Every z-ideal of a ring R with the AR property is pure.*

Proof. Let E be a z-ideal of a ring R with the AR property, and $q \in E$. Then there exists $n \in \mathbb{N}$ such that $E^n \cap \langle q \rangle \subseteq E\langle q \rangle$, which implies that $E \cap \langle q \rangle \subseteq E\langle q \rangle$ since E is a z-ideal. Now, $q \in E \cap \langle q \rangle \subseteq E\langle q \rangle$ implies that $q = qf$ for some $f \in E$. Therefore, E is a pure ideal. □

Proposition 2.2.9. [1, Proposition 2] *An ideal of $\mathcal{R}L$ is a z-ideal with the AR property if and only if it is pure.*

Proof. It is trivial, since every AR-ideal has the AR property, and on the other hand we have that every pure ideal in $\mathcal{R}L$ is a z-ideal. □

Lemma 2.2.27. [1, Lemma 4] *If $a = coz\tau$ for some $\tau \in \mathcal{R}L$, then $a \in BL$ if and only if τ is a regular element if and only if there is $\rho \in \mathcal{R}L$ such that $\tau = \rho\tau$ and $coz\rho \leq coz\tau$.*

Proof. Assume that $a = coz\tau \in BL$. Then $coz\tau \prec\prec coz\tau$, and hence there exists $\rho \in \mathcal{R}L$ such that $\tau = \rho\tau^2$, that is, τ is a regular element. If τ is a regular element, then there exists $\upsilon \in \mathcal{R}L$ such that $\tau = \upsilon\tau^2$. If we put $\omega = \upsilon\tau$, then we have $\tau = \omega\tau$ and $coz\omega \leq coz\tau$. Then $\tau(1-0) = 0$, this shows that $coz\tau \wedge coz(1-\omega) = 0$. On the other hand,

$$1 = coz(1-\omega+\omega) \leq coz(1-\omega) \vee coz\omega \leq coz(1-\omega) \vee coz\tau,$$

which implies $coz(1-\omega) \vee coz\tau = 1$. Therefore, $a = coz\tau \in BL$. □

Recall that BL denote the set of all complemented elements of L, that is

$$BL = \{a \in L \mid a \vee a^* = 1\}$$

is a sublattice of L.

Theorem 2.2.28. [1, Theorem 5] *The following are equivalent for a frame L:*

(1) *L is a P-frame.*

(2) *$\mathcal{R}L$ is an AR-ring.*

(3) *Every z-ideal of $\mathcal{R}L$ has the AR property.*

(4) *Every prime d-ideal of $\mathcal{R}L$ has the AR property.*

(5) *Every maximal ideal of $\mathcal{R}L$ has the AR property.*

(6) *Every ideal of $\mathcal{R}L$ with the AR property is an AR-ideal.*

Proof. (1) ⇒ (2): By Propositions 2.2.8 and Proposition 2.2.9, we say $\mathcal{R}L$ is an AR-ring.

(2) ⇒ (3): It is trivial from the definition of AR-ring.

(3) ⇒ (4): Since z-ideals are d-ideals, we are done.

(4) ⇒ (5): It is trivial, since prime d-ideals are maximal ideals.

(5) ⇒ (1): By equivalence of (1) and (10) from Proposition 2.2.2, it remains to show that $\mathbf{O}^J = \mathbf{M}^J$ for every $J \in Pt(\beta L)$. Let $J \in Pt(\beta L)$. Then by (5) it implies that $\mathbf{M}^J$ is a pure ideal. Now, Lemma 2.2.6 shows that $\mathbf{O}^J = \mathbf{M}^J$.

(1) ⇔ (6): If L is a P-frame, thus every ideal is pure. By Proposition 2.2.8, every ideal is an AR-ideal and we are done. Conversely, suppose $a \in CozL$ such that $a = coz\varphi$ for some $\varphi \in \mathcal{R}L$. Proposition 2.1.1 tells us that we can assume $a^* \neq 0$. Then, by completely regularity,

there is $t \in CozL$ such that $t \prec\prec a^*$, and hence $a = coz\varphi \leq a^{**} \prec\prec t^*$, which implies $\varphi \in \mathbf{O}_{t^*}$. Since $\mathbf{O}_{t^*}$ is a pure ideal, it has the AR property, and so, (6) implies that $\mathbf{O}_{t^*}$ is an AR-ideal. Since $\mathbf{M}_a$, $\langle\varphi\rangle \subseteq \mathbf{O}_{t^*}$, we have $\mathbf{M}_a \cap \langle\varphi\rangle \subseteq \mathbf{M}_a\langle\varphi\rangle$. Now, $\varphi \in \mathbf{M}_a \cap \langle\varphi\rangle$ implies $\varphi \in \mathbf{M}_a\langle\varphi\rangle$, it follows that there exist $\mu \in \mathbf{M}_a$ and $\nu \in \mathcal{R}L$ so that $\varphi = \varphi\mu\nu$. Since $coz(\mu\nu) \leq coz\mu \leq coz\varphi$, the Lemma 2.2.27 says $a = coz\varphi \in BL$.

□

We give a brief account of P-ideals and also provide the characterisations of P-frames associated with P-ideals. P-ideals were first introduced and studied in $C(X)$ (denote, the ring of real-valued continuous functions on a completely regular Hausdorff space X) by Rudd [83]. Recall that a nonzero ideal I of $C(X)$ (denotes, the ring of all real-valued continuous functions on a completely regular Hausdorff space X) is called a *P-ideal* if every proper prime ideal of I is maximal in I. We have the following definition of P-ideal in terms of $\mathcal{R}L$ as follows.

Definition 2.2.16. A nonzero ideal I in $\mathcal{R}L$ is said to be a *P-ideal*, if every proper prime ideal in I is maximal in I.

Theorem 2.2.29. *An ideal I in $\mathcal{R}L$ is a P-ideal if and only if every prime ideal of $\mathcal{R}L$ which does not contain I is maximal in $\mathcal{R}L$.*

Proof. Suppose I is a P-ideal and Q is prime in $\mathcal{R}L$ with $Q \not\supseteq I$. Then $I \cap Q$ is a proper prime ideal of I and therefore it is maximal in I, $I \cap Q = I \cap M$ for some maximal ideal M in $\mathcal{R}L$. It follows that $Q = M$. For the converse, if M is a prime ideal in $\mathcal{R}L$ with $I \not\subseteq M$. So $I \cap M$ is a proper prime ideal in I. By hypothesis, M is maximal in $\mathcal{R}L$, so $I \cap M$ is maximal in I. The proof is complete. □

For any ideal Q in $\mathcal{R}L$, the maximal ideals of Q are precisely those ideals of the form $Q \bigcap M$ for $M \not\supseteq Q$.

Lemma 2.2.30. [83, Lemma 1.4] *If I is a P-ideal, then $I = mI$.*

Proof. Let $\varphi \in I$, and assume on contrary that $\varphi \notin mI$. Since mI is a semiprime in $\mathcal{R}L$, there exists a prime ideal Q in $\mathcal{R}L$ with $mI \subset Q$ and $\varphi \notin Q$. But then $I \nsubseteq Q$, and hence Q is maximal in $\mathcal{R}L$. Since $mI \subseteq Q$, we have $Q \bigcap mI = mI$ is a proper prime ideal of I which is maximal in I. But then this contradicts that Q is maximal in $\mathcal{R}L$. Hence $I = mI$ and we are done. □

We recall from [4] that if a space X is a P-space, then every ideal of $C(X)$ is a P-ideal. We have the following as a translation to frame-theoretic property.

Proposition 2.2.10. *If a frame L is a P-frame, then every ideal of $\mathcal{R}L$ is a P-ideal.*

Proof. It follows immediately from Proposition 2.2.8 and Lemma 2.2.30. □

Chapter 3

Essential P-frames and CP-frames

In this chapter, we study essential P-frames and CP-frames. In the first section, we focus on essential P-frames and show that a normal frame is an essential P-frame if and only if $\mathcal{R}L$ is VN (von Neumann)-local. We also show that an essential P-frame is a strongly zero-dimensional frame. In the second section, we show that the class of CP-frames contains the class of P-frames. Furthermore, we show that L is a CP-frame if and only if every ideal of $\mathcal{R}_cL$ is a z_c-ideal if and only if every radical ideal of $\mathcal{R}_cL$ is a z_c-ideal.

3.1 Essential P-frames

We start with a definition of an essential P-frame which originated with Dube [31] and it captures the notion of essential P-spaces in a point-free setting. He was able to do this by combining Theorem 2.6 and Corollary 5.5 of the article of Osba *et al* [78]. Recall from [79] that a space X is called an essential P-space if it has at most one point which fails to be a P-point. We have that X is an essential P-space if and only if all maximal ideals of $C(X)$ except at most one are pure. The following definition, as we can recall from the definition of a P-point, follows nicely.

Definition 3.1.1. A frame L is an *essential P-frame* if there is at most one $I \in \sum \beta L$ such that $\mathbf{O}^I \neq \mathbf{M}^I$. It is a proper essential P-frame if it is an essential P-frame which is not a P-frame. Now $\mathbf{O}^I \neq \mathbf{M}^I$ if and only if there exists $c \in CozL$ such that $c \notin I$, but $r(c) \subseteq I$. Equivalently, if at most one $a \in CozL$ fails to be a cozero complemented element.

The following result follows as a corollary to Proposition 2.2.2 and the above definition.

Corollary 3.1.1. *Every P-frame is an essential P-frame.*

Proposition 3.1.1. [31, Proposition 4.2] *A frame L is a proper essential P-frame if and only if $\mathcal{R}L$ has at least one non-maximal prime ideal, and the non-maximal ideals of $\mathcal{R}L$ are all contained in one maximal ideal.*

Proof. ($\Rightarrow$): Since L is not a P-frame by hypothesis, $\mathcal{R}L$ does have a non-maximal prime ideal. Let Q be such an ideal, and I be an element of $\sum \beta L$ such that $\mathbf{O}^I \neq \mathbf{M}^I$. Since Q is a prime ideal, there exist a unique $J \in \sum \beta L$ such that $\mathbf{O}^J \subseteq Q \subseteq \mathbf{M}^J$. Since $Q \neq \mathbf{M}^I$, as Q is not a maximal ideal, it follows that $\mathbf{O}^J \neq \mathbf{M}^J$, and hence $J = I$, since L is a proper essential P-frame. Therefore all non-maximal ideals are contained in the maximal ideal $\mathbf{M}^I$.

($\Leftarrow$): Let $\mathbf{M}^I$ be the maximal ideal containing all non-maximal prime ideals of $\mathcal{R}L$. We claim that $\mathbf{O}^I \neq \mathbf{M}^I$ and $\mathbf{O}^J = \mathbf{M}^J$ for every $J \in \sum \beta L$ different from I. Let $J \in \sum \beta L$ such that $J \neq I$. The ideal $\mathbf{O}^J$ is a z-ideal, and therefore equals to its radical. Consequently, $\mathbf{O}^J$ is the intersection of the prime ideals containing it. Now there is no non-maximal prime ideal containing $\mathbf{O}^J$, for if there were, then by hypothesis, such an ideal would be contained in $\mathbf{M}^I$, implying that $\mathbf{O}^J \subseteq \mathbf{M}^I$ for $J \neq I$, which is false. Consequently the only prime ideal containing $\mathbf{O}^J$ is $\mathbf{M}^J$, and so $\mathbf{O}^J = \mathbf{M}^J$. As this is true for every $J \in \sum \beta L$ different from I, and since L is not a P-frame, we must have that $\mathbf{O}^I \neq \mathbf{M}^I$. □

The following corollary is immediate.

Corollary 3.1.2. *A Tychonoff space X is a proper essential P-space if and only if $C(X)$ has at least one non-maximal prime ideal, and the non-maximal ideal of $C(X)$ is contained in one maximal ideal.*

According to Osba *et al* [78], a ring R is *von Neumann local* (henceforth abbreviated VN-local) if for each $a \in R$, at least one of a or $1-a$ has a VN-inverse, that is, there is $b \in R$ such that at least one of a or $1-a$, we have $aba = a$ or $(1-a)b(1-a) = (1-a)$.

Lemma 3.1.1. [31, Lemma 4.5] *The following are equivalent:*

(1) *$\mathcal{R}L$ is VN-local ring.*

(2) *For every $f \in \mathcal{R}L$, then $cozf$ or $coz(1-f)$ is complemented.*

(3) *If $c \vee d = 1$ in $CozL$, then c or d is complemented.*

Proof. (1) $\Rightarrow$ (2): By (1), either f or $1-f$ has a VN-inverse. Say f has a VN-inverse. Take an invertible $\tau \in \mathcal{R}L$ such that $f\tau$ is idempotent. Then $coz(f\tau)$ is complemented. But $coz(f\tau) = cozf \wedge coz\tau = cozf$, since $coz\tau = 1$, as τ is invertible.

(2) $\Rightarrow$ (3): Let $c \vee d = 1$ in $CozL$. Take $\gamma, \delta \in \mathcal{R}L$ such that $c = coz\gamma$ and $d = coz\delta$. Then $coz(\gamma^2+\delta^2) = 1$, so that $\gamma^2+\delta^2$ is invertible. Let $f = \gamma^2(\gamma^2+\delta^2)^{-1}$. Then $1-f = \delta^2(\gamma^2+\delta^2)^{-1}$. Thus $cozf = coz\gamma = c$ and $coz(1-f) = coz\delta = d$. Now, (2) implies that either c or d is complemented.

(3) $\Rightarrow$ (1): Let $f \in \mathcal{R}L$. Then $cozf \vee coz(1-f) = 1$. So, by hypothesis, we may assume that $cozf$ is complemented. Now if $f \in \mathbf{M}^I$ for some $I \in \sum \beta L$. Then $f \in \mathbf{O}^I$, since $cozf \prec\prec cozf$. Consequently, f has a VN-inverse. □

It has been shown that a Tychonoff space X is an essential P-space if and only if $C(X)$ is a VN-local ring. Dube extended this result, but for normal frames. An *obstacle* is a point that fails to be a P-point.

Proposition 3.1.2. [31, Proposition 4.6] *If a frame L is an essential P-frame, then $\mathcal{R}L$ is VN-local. The converse holds only if L is normal.*

Proof. If L is a P-frame, then $\mathcal{R}L$ is regular and hence VN-local. Suppose L is a proper essential P-frame with obstacle I. Let f be a non-invertible element of $\mathcal{R}L$ such that $1-f$

is also non-invertible. Since $\mathbf{M}^I$ is a proper ideal, it cannot contain both f and $1-f$. Let it misses f. Since f is not invertible, there is at least one maximal ideal that contains it. Say $\mathbf{M}^J$ be a maximal ideal such that $f \in \mathbf{M}^J$. Then $J \neq I$, and so $\mathbf{M}^J = \mathbf{O}^J = m\mathbf{M}^J$. Therefore f has a VN-inverse.

For the second assertion, suppose, by way of contradiction, that there exist distinct $I_1, I_2 \in \sum \beta L$ such that $\mathbf{O}^{I_i} \neq \mathbf{M}^{I_i}$ for $i = 1, 2$. Then, by what we observed earlier, there exist $c_1, c_2 \in CozL$ such that $r(c_i) \subseteq I_i$ but $c_i \notin I_i$ for $i = 1, 2$. By maximality of I_i, we have $I_1 \vee I_2 = 1_{\beta L}$. Therefore there exist $u_i \in I_i$ such that $u_1 \vee u_2 = 1$. Consequently, $(c_1 \vee u_1) \vee (c_2 \vee u_2) = 1$. So, by hypothesis and Lemma 3.1.1, we may assume that $(c_1 \vee u_1)$ is complemented. By normality of L we have $r(c_1 \vee u_1) = r(c_1) \vee r(u_1) \subseteq I_1$. Since $c_1 \vee u_1$ is complemented, it follows that $c_1 \vee u_1 \in I_1$, implying that $c_1 \in I_1$, and hence a contradiction. □

The following is immediate.

Corollary 3.1.3. [31, Corollary 4.7] *A normal frame L is an essential P-frame if and only if $\mathcal{R}L$ is VN-local.*

As in the case of spaces (see [78, Definition 1.4]), we say a frame L is strongly VN-local if for any $S \subseteq \mathcal{R}L$, $\langle S \rangle = \mathcal{R}L$ implies that some elements of S has a VN-inverse. Osba *et al* [78, Corollary 5.5], showed that for any Tychonoff space X, $C(X)$ is a strongly VN-local ring if and only if it is a VN-local. We show that the results extend in normal frames.

Proposition 3.1.3. [31, Proposition 4.8] *For any normal frame L, $\mathcal{R}L$ is strongly VN-local if and only if it is VN-local.*

Proof. The forward implication is trivial. Conversely, if L is a P-frame, then the result is immediate because $\mathcal{R}L$ is regular and hence VN-local. So, we may assume that there is exactly one $I \in \sum \beta L$ such that $\mathbf{O}^I \neq \mathbf{M}^I$. Now let $S \subseteq \mathcal{R}L$. Take $\varphi_1, ..., \varphi_m \in S$ and $\alpha_1, ..., \alpha_m \in \mathcal{R}L$ such that $\alpha_1\varphi_1, ..., \alpha_m\varphi_m = 1$. Then

$$coz\alpha_1\varphi_1 \vee \cdots \vee coz\alpha_m\varphi_m = 1,$$

which implies that

$$coz\varphi_1 \vee \cdots \vee coz\varphi_m = 1,$$

and hence

$$coz(\varphi_1^2 + \cdots + \varphi_m^2) = 1.$$

As a consequence, $\mathbf{M}^I$ cannot contain all the φ_i, lest it contain an invertible element. So say $\varphi_1 \notin \mathbf{M}^I$. Now, $\mathbf{M}^I$ being the only maximal ideal unequal to its pure part, it follows that if a maximal ideal contains φ, then so it does its pure part. Therefore φ_1 has a VN-inverse. Thus, L is strongly VN-local. □

We recall from Banaschewski and Brümmer [20] that a frame is said to be *strongly zero-dimensional* if its Stone-Čech compactification is generated by its complemented elements, and showed that L is strongly zero-dimensional if and only if $a \prec\prec b$ in L implies the existence of complemented element c in L such that $a \leq c \leq b$. We adopt the definition that a Tychonoff space X is strongly zero-dimensional in case βX has a base of clopen sets. Dube [31] observed that X is strongly zero-dimensional if and only if $\mathcal{O}X$ is strongly zero-dimensional.

Proposition 3.1.4. [31, Proposition 5.1] *Every essential P-frame is strongly zero-dimensional.*

Proof. Let $a \prec\prec b$ in an essential P-frame L. Take $c \in CozL$ such that $a \prec\prec c \prec\prec b$. If c is complemented, then we are done. So suppose c is not complemented. Take $s \in CozL$ such that $a \wedge s = 0$ and $s \vee c = 1$. Since L is an essential P-frame and c is not complemented, thus s is complemented by Lemma 3.1.1. Hence s^* is also complemented. Now $a \wedge s = 0$ implies that $a \leq s^*$, and $s \vee c = 1$ implies that $s^* \leq c$. Consequently, $a \leq s^* \leq b$. Thus L is strongly zero-dimensional. □

Since essential P-frames and strongly zero-dimensional frames are conservative notions, the following is immediate.

Corollary 3.1.4. [31, Corollary 5.2] *An essential P-space is strongly zero-dimensional.*

If L is a P-frame, then every ideal of $\mathcal{R}L$ is generated by idempotents, since $\mathcal{R}L$ is regular; and so, in particular, $\mathbf{O}^I$ is generated by idempotents for each $I \in \beta L$. We show that each ideal $\mathbf{O}^I$ is generated by idempotents precisely when the frame is strongly zero-dimensional. Before proving this let us notice the following about idempotents of $\mathcal{R}L$ and complemented elements of L. Dube [34] observed that $coz\eta$ is complemented for every idempotent η. On the other hand, let c be a complemented element of L and take $\gamma \in \mathcal{R}L$ such that $coz\gamma = c$. Then $coz\gamma \prec\!\prec coz\gamma$, and hence, by Lemma 2.2.4 there is an invertible $\tau \in \mathcal{R}L$ such that $\gamma = \gamma(\gamma^2\tau) = \gamma^3\tau$. This implies that $\gamma\tau = (\gamma\tau)^2$. Since τ is invertible, $coz(\gamma\tau) = coz\gamma \wedge coz\tau = coz\gamma \wedge 1 = coz\gamma = c$. Therefore $\gamma^2\tau$ is an idempotent such that $coz(\gamma^2\tau) = c$.

The following definition is culled from [66].

Definition 3.1.2. A proper ideal I in R is said to be a *semiprime* ideal if, whenever $J^n \subset I$ for an ideal J of R and some positive integer n, then $J \subset I$. Equivalently an l-ideal I of an f-ring R is called *semiprime* if $a^2 \in I$ implies $a \in I$.

Lemma 3.1.2. [7, Lemma 1.1] *Let I be an ideal of R and H be a semiprime ideal in the ideal I, then H is an ideal in R.*

Proof. Suppose that $a \in H$ and $r \in R$, hence $r^2a \in I$ which implies that $(ra)^2 = (r^2a)a \in H$. This shows that $ra \in H$. □

The following results are culled from Abedi [2].

(i) Every element of $\mathcal{R}L$ has an n^{th} root, for any odd $n \in \mathbb{N}$.

(ii) Every position element of $\mathcal{R}L$ has an n^{th} root, for any $n \in \mathbb{N}$.

Since there is a frame map $\rho : \mathcal{L}(\mathbb{R}) \to \mathcal{L}(\mathbb{R})$ such that for every $p, q \in \mathbb{Q}$, $p(p,q) = (p^n, q^n)$. For $\alpha \in \mathcal{R}L$, Abedi defined the frame map $\sqrt[n]{\alpha} : \mathcal{L}(\mathbb{R}) \to L$ to be given as $\sqrt[n]{\alpha} = \alpha \circ \rho$. By the following proposition, $\sqrt[n]{\alpha}$ is an n^{th} root of α.

Proposition 3.1.5. [2, Proposition 3.1] *Let $\alpha \in \mathcal{R}L$ and let $n \in \mathbb{N}$ be an odd number. Then $(\sqrt[n]{\alpha})^n = \alpha$.*

Proposition 3.1.6. [2, Proposition 3.2] *If I and J are prime ideals of $\mathcal{R}L$, then their intersection is equals to IJ.*

Proof. Let $\alpha \in I \cap J$. Since $\alpha^{\frac{2}{3}}\alpha^{\frac{1}{3}} = \alpha \in I \cap J$, we conclude that $\alpha^{\frac{1}{3}} \in I \cap J$. Therefore $\alpha = \alpha^{\frac{2}{3}}\alpha^{\frac{1}{3}} \in I \cap J$, that is , $I \cap J \subseteq IJ$. Now, the proof is complete since the reverse inclusion is always true. □

The following propositions are point-free versions of characterisations of essential P-spaces (see [14]).

Proposition 3.1.7. *A frame L is an essential P-frame if and only if for $a, b \in CozL$ such that $a \vee b = 1$, then at least one of them is complemented.*

Proof. (⇒): Let $a, b \in CozL$ such that $a \vee b = 1$. Then it is immediate by the definition of an essential P-frame that at least one is complemented because L has at most one cozero element which is not complemented.

(⇐): Suppose that $a \vee b = 1$ such that a is complemented (by hypothesis), then $a \vee a^* = 1$. Now, $a \vee b = 1$ implies that $a^* \leq b$. If b is not complemented, then $b \vee b^* \neq 1$. Suppose on contrary that u is another cozero element such that $u \vee a = 1$. Again by hypothesis, a is complemented. Then $a \vee a^* = 1$, and $u \vee a = 1$ implies that $a^* \leq u$. Therefore, $u = b$. Thus L has only one cozero element which is not complemented. Hence L is an essential P-frame. □

The following proposition is a point-free version of [14, Proposition 2.2].

Proposition 3.1.8. *The following statements are equivalent:*

(1) *A frame L is an essential P-frame.*

(2) *Of any two comaximal ideals of $\mathcal{R}(L)$, one is a z-ideal.*

(3) *Of any two comaximal principal ideals of $\mathcal{R}(L)$, one is semiprime.*

(4) *For any $a, b \in CozL$ such that $a \vee b$ is complemented , then one of them is complemented.*

Proof. (1) $\Rightarrow$ (2): Suppose that I and J are ideals of $\mathcal{R}L$ such that $I + J = \mathcal{R}L$. We need to show that either I is a z-ideal or J is a z-ideal. Let $\tau \in \mathcal{R}L$ and $\varphi \in I$ such that $coz\tau = coz\varphi$. L is an essential P-frame so there is a $c \in CozL$ such that $c \notin I$, but $r(c) \subseteq I$. Hence $c \neq coz\varphi = coz\tau$ that does not belong to $coz\varphi$ and $coz\tau$. Since c is the only cozero element which is not complemented, it follows that both $coz\varphi$ and $coz\tau$ are cozero complemented. Therefore, either $\varphi \leq \tau$ or $\tau \leq \varphi$. Then either $\varphi \in J$ or $\tau \in I$ and hence either I or J is a z-ideal.

(2) $\Rightarrow$ (3): Let $\langle\phi\rangle$ and $\langle\varphi\rangle$ be two comaximal principal ideals of $\mathcal{R}L$, one is a z-ideal. Say $\langle\phi\rangle$ is a z-ideal and let $\alpha^2 \in \langle\phi\rangle$. Then for any $\omega \in \mathcal{R}L$ such that $coz(\alpha^2) = coz\omega$. By z-ideal, we have $coz\alpha = coz\alpha \wedge coz\alpha = coz(\alpha^2) = coz\omega$. Hence $\alpha \in \langle\phi\rangle$.

(3) $\Rightarrow$ (1): Let I and J be two comaximal ideals of $\mathcal{R}L$ such that $I = \langle\varphi\rangle$ and $J = \langle\delta\rangle$. Then there exist $\tau = \varphi^n + \delta^m$, for some $n, m \in \mathbb{N}$. We assume without loss of generality that $\tau \neq \varphi^n$ and $\tau \neq \delta^m$. That is, $\tau \notin I$ and $\tau \notin J$. Then $c = coz\tau \notin I$ and $c = coz\tau \notin J$. Now, $c = coz\tau = coz(\varphi^n + \delta^m) = coz|\varphi^n| \vee coz|\delta^m|$. So, we can write $\tau = \delta(\varphi^p + \delta^q)$; where $\delta = \varphi^k$ and $\delta = \delta^s$, so that $\delta \in I \cap J$ and for some $p, q, k, s \in \mathbb{N}$. Hence

$$r(c) = r(coz\tau) = r(coz(\delta(\varphi^p + \delta^q))) = r(coz(\delta(\varphi^p))) \vee r(coz(\delta(\delta^q))).$$

By hypothesis (one is semiprime), say I is semiprime. Hence $c \in CozL$, $c \notin I$ but $r(c) \subseteq I$, showing that L is an essential P-frame.

(1) $\Leftrightarrow$ (4): It is trivial, from Proposition 3.1.7. □

Remark : (2) $\Leftrightarrow$ (3): Also hold for the fact that in a commutative ring with unity, every z-ideal is a semiprime. Hence for any two comaximal principal ideals, one being a z-ideal. Thus the one that is a z-ideal is semiprime.

3.2 CP-frames

In [49], $C_c(X)$ is defined to be the largest subalgebra of $C(X)$ whose elements have countable images. The reader should know that $\mathcal{R}_c(\mathcal{O}X)$ (denotes, the ring of all real-valued continuous functions on a frame $\mathcal{O}X$ that have countable images) the following properties that we shall use (see [45]);

(i) $C_c(X) \cong \mathcal{R}_c(\mathcal{O}X) \cong \mathcal{R}_c(\beta L) \cong \mathcal{R}_c^*(L)$.

A space X is a CP-*space* if $C_c(X)$ is regular. Hence, a topological space X is a CP-space if and only if the frame $\mathcal{O}X$ is a CP-frame. This kind of subalgebra $C_c(X)$ has recently received some more attention (see [12, 25, 43, 48, 60, 61, 73, 74]). Ghadermazi *et al* [49] introduced the concept of CP-spaces as follows: a topological space X is a CP-space if $C_c(X)$ is a regular ring and studied the relation between P-spaces and CP-spaces. Karimi Feizabadi *et al* [62] introduced the ring $\mathcal{R}_cL$ as a point-free version of the ring $C_c(X)$. In 2021, Estaji and Robat Sarpoushi [44] introduced the concept of CP-frames which refers to countable P-frames.

The homomorphism $\tau : \mathcal{L}(\mathbb{R}) \to \mathcal{O}X$ given by $(p, q) \mapsto \S p, q\S$ is an isomorphism, where $(p, q) \in \mathbb{Q}$ and $\S p, q\S = \{x \in \mathbb{R} \mid p < x < q\}$ (see [19]). Recall from [62] that an element $\alpha \in \mathcal{R}L$ is said to be an *overlap* of a subset S of $\mathbb{R}$, denoted by $\alpha \blacktriangleleft S$, if $\tau(u) \cap S \subseteq \tau(v) \cap S$ implies $\alpha(u) \leq \alpha(v)$, for every $u, v \in \mathcal{L}(\mathbb{R})$. Also, we recall from [62, Lemma 3.5] that for any $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$, the following statements are equivalent:

(i) $\alpha \blacktriangleleft S$.

(ii) $\tau(u) \cap S = \tau(v) \cap S$ implies $\alpha(u) = \alpha(v)$, for $u, v \in \mathcal{L}(\mathbb{R})$.

(iii) $\tau(p, q) \cap S = \tau(v) \cap S$ implies $\alpha(p, q) = \alpha(v)$, for $v \in \mathcal{L}(\mathbb{R})$ and any $p, q \in \mathbb{Q}$.

(iv) $\tau(p, q) \cap S \subseteq \tau(v) \cap S$ implies $\alpha(p, q) \leq \alpha(v)$, for $v \in \mathcal{L}(\mathbb{R})$ and any $p, q \in \mathbb{Q}$.

We say that $\alpha\mathcal{R}L$ has a point-free countable image if there exists a countable subset S of $\mathbb{R}$ such that $\alpha \blacktriangleleft S$. We write $\mathcal{R}_cL$ for the set of all $\alpha\mathcal{R}L$ such that α has the point-free countable image (see [46]). It is shown that for each completely regular frame L, the set $\mathcal{R}_cL$ is the sub-f-ring of $\mathcal{R}L$ (see [62]). The following definition is culled from [44].

Definition 3.2.1. A frame L is said to be a CP-frame if $\mathcal{R}_cL$ is regular.

We need to show that the class of CP-frames contains the class of P-frames. To show this we need the next few results. We make use of an interesting function introduced by Ball and Walters-Wayland [16]. Let a be a complemented element of L. Define the frame map $e_a : \mathcal{L}(\mathbb{R}) \to L$ by

$$e_a(p,q) = \begin{cases} 1 & \text{if } p < 0 < q, \\ a^* & \text{if } p < 0 < q \leq 1, \\ a & \text{if } 0 \leq p < 1 < q, \\ 0 & \text{otherwise,} \end{cases}$$

for each $p, q \in \mathbb{Q}$. Then $e_a \in \mathcal{R}L$, $coz(e_a) = a$, and $coz(1 - e_a) = a^*$. Also, for each $\alpha \in \mathcal{R}L$, $\alpha e_a \in \mathcal{R}L$ and it is easy to check that

$$\alpha(e_a)(p,q) = \begin{cases} \alpha(p,q) \vee a^* & \text{if } p < 0 < q, \\ \alpha(p,q) \wedge a & \text{otherwise,} \end{cases}$$

for each $p, q \in \mathbb{Q}$. It is well known that a map α from the subbase of $\mathcal{L}(\mathbb{R})$ into a frame L defines a frame homomorphism $\mathcal{L}(\mathbb{R}) \to L$ if and only if it sends the relations (i)-(vi) from page (16 to 17) to identities in L..

Proposition 3.2.1. [44, Proposition 3.2] *Let $\alpha \in \mathcal{R}_cL$ such that $coz\alpha \vee (coz\alpha)^* = 1$. Then $\iota_\alpha : \mathcal{L}(\mathbb{R}) \to L$ given by*

$$\iota_\alpha(p,q) = \begin{cases} (coz\alpha)^* \vee \alpha((-,\tfrac{1}{p}) \vee (\tfrac{1}{q},-)) & \text{if } p < 0 < q, \\ \alpha(\tfrac{1}{q},\tfrac{1}{p}) & \text{if } p < q \leq 0 \text{ or } 0 \leq p < q, \end{cases}$$

for each $p, q \in \mathbb{Q}$, determines a real-valued continuous functions in $\mathcal{R}_cL$.

Proof. First, it should be pointed out that when $p < q \leq 0$ if $q = 0$, then we have

$$\iota_\alpha(p,q) = \alpha\left(-,\frac{1}{p}\right)$$

and also when $0 \leq p < q$ if $p = 0$, then we have $\iota_\alpha(p,q) = \alpha(\frac{1}{q}, -)$. Since

$$coz\alpha \vee (coz\alpha)^* = 1.$$

Hence $\iota_\alpha \in \mathcal{R}L$, by Lemma 4.9 in [3]. So, we need only to show that ι_α is an overlap of a countable subset T of $\mathbb{R}$. Since $\alpha \in \mathcal{R}_cL$, it follows that we can select a countable subset S of $\mathbb{R}$ such that $\alpha \blacktriangleleft S$. If we put $T = \{\frac{1}{x} \mid x \in S, x \neq 0\} \cup \{0\}$, then it is routine to show that $\iota_\alpha \blacktriangleleft T$. □

The foregoing proposition tells us that $\iota_\alpha \in \mathcal{R}_cL$, and $coz(\iota_\alpha) = coz\alpha$ when $\alpha \in \mathcal{R}_cL$ and $coz\alpha \vee (coz\alpha)^* = 1$. We recall from [45] that for a frame L, $Coz_cL = \{coz\alpha \mid \alpha \in \mathcal{R}_cL\}$. Then the following is immediate:

Corollary 3.2.1. [44, Corollary 3.3] *If $a \in L$ is such that $a \vee a^* = 1$, then $a, a^* \in Coz_c(L)$.*

Proposition 3.2.2. [44, Proposition 3.4] *Let L be a P-frame. For every $\alpha \in \mathcal{R}_cL$, $\iota_\alpha\alpha = e_{coz\alpha}$, where ι_α is the real-valued continuous function in the* Proposition 3.2.1.

Proof. Using the fact that if L is a regular frame and $f, g : L \to M$ are frame homomorphisms such that $f(x) \leq g(x)$ for every $x \in L$, then $f = g$ (see [80]), it suffices to show that, for each $p, q \in \mathbb{Q}$, $\alpha\iota_\alpha(p,q) \leq e_{coz\alpha}(p,q)$ since $\mathcal{L}(\mathbb{R})$ is a regular frame. Let $p, q \in \mathbb{Q}$. We consider four cases.

Case 1: Assume $0, 1 \notin \tau(p,q)$. If there exist $u, v, w, z \in \mathbb{Q}$ such that $\langle u,v\rangle\langle w,z\rangle \subseteq \langle p,q\rangle$, then $(\frac{1}{v}, \frac{1}{u}) \wedge (w,z) = 0$, which implies that

$$\begin{aligned}\iota_\alpha\alpha(p,q) &= \bigvee\{\iota_\alpha(u,v) \wedge \alpha(w,z) : \langle u,v\rangle\langle w,z\rangle \subseteq \langle p,q\rangle\} \\ &= \alpha\left(\bigvee\left\{\left(\frac{1}{v},\frac{1}{u}\right) \wedge (w,z) : \langle u,v\rangle\langle w,z\rangle \subseteq \langle p,q\rangle\right\}\right) \\ &= 0 \\ &= e_{coz\alpha}(p,q).\end{aligned}$$

Case 2: Assume $0, 1 \in \tau(p,q)$. Since $e_{coz\alpha}(p,q) = 1$, it follows

$$\alpha\iota_\alpha(p,q) \leq e_{coz\alpha}(p,q).$$

Case 3: Assume $0 \notin \tau(p,q)$ and $1 \in \tau(p,q)$. Then

$$\alpha\iota_\alpha(p,q) \leq coz(\alpha\iota_\alpha) = coz\alpha = e_{coz\alpha}(p,q).$$

Case 4: Assume $0 \in \tau(p,q)$ and $1 \notin \tau(p,q)$. Then

$$\begin{aligned}
\alpha\iota_\alpha(p,q) \wedge coz\alpha &= \alpha\iota_\alpha(p,q) \wedge coz(\alpha\iota_\alpha) \\
&= \alpha\iota_\alpha(p,q) \wedge \alpha\iota_\alpha((-,0) \vee (0,-)) \\
&= \alpha\iota_\alpha(p,q) \wedge ((-,0) \vee (0,-)) \\
&= \alpha\iota_\alpha((p,q) \wedge ((-,0) \vee (0,-))) \\
&= \alpha\iota_\alpha((p,0) \vee (0,q)).
\end{aligned}$$

Now, considering case 1, we can conclude that $\alpha\iota_\alpha(p,q) \wedge coz\alpha = 0$, which implies

$$\alpha\iota_\alpha(p,q) \leq (coz\alpha)^* = e_{coz\alpha}(p,q).$$

□

The class of CP-frames contains the class of P-frames as it is observed below.

Proposition 3.2.3. [44, Proposition 3.5] *Every P-frame is a CP-frame.*

Proof. Let L be a P-frame. We want to show that $\mathcal{R}_cL$ is regular. Consider $\alpha \in \mathcal{R}_cL$. Since $coz\alpha \vee (coz\alpha)^* = 1$, it follows by Proposition 3.2.1 that $i_\alpha \in \mathcal{R}_cL$ and hence by Proposition 3.2.2, we have $\alpha i_\alpha = e_{coz\alpha}$, which implies $\alpha^2 i_\alpha = \alpha e_{coz\alpha}$. Furthermore

$$coz(\alpha(1 - e_{coz\alpha})) = coz\alpha \wedge coz(1 - e_{coz\alpha}) = coz\alpha \wedge (coz\alpha)^* = 0,$$

so that $\alpha = \alpha e_{coz\alpha}$. Thus, $\alpha^2 i_\alpha = \alpha$. This shows that $\mathcal{R}_cL$ is regular.

□

Estaji and Robat Sarpoushi [44] gave an example of a CP-space which is not a P-space. Now, using that P-frames and CP-frames are conservative notions, we, therefore, conclude that there is a CP-frame which is not a P-frame. Next, we give some properties of CP-frames. We start with a lemma.

Lemma 3.2.1. [44, Lemma 4.1] *Let L be a frame and A be a regular subring of $\mathcal{R}L$. Then for every $\alpha \in A$, $coz\alpha$ is complemented in $CozL$.*

Proof. Consider $\alpha \in A$. Since A is regular, it follows that there exists an element $\beta \in A$ such that $\alpha = \alpha^2\beta$. So

$$coz\alpha = coz(\alpha^2\beta) = coz(\alpha^2) \wedge coz\beta = coz\alpha \wedge coz\beta = coz(\alpha\beta).$$

Since $\alpha\beta$ an idempotent element of $\mathcal{R}L$, we conclude that $coz\alpha$ is complemented. □

Proposition 3.2.4. [44, Proposition 4.2] *A frame L is a CP-frame if and only if every $coz\alpha \in Coz_cL$ is complemented in Coz_cL.*

Proof. Let L be a CP-frame. Then $\mathcal{R}_cL$ is regular and by Lemma 3.2.1, we are done. Conversely, suppose the lattice Coz_cL is complemented and let $\alpha \in \mathcal{R}_cL$. Since $coz\alpha \vee (coz\alpha)^* = 1$, it follows that $\iota_\alpha \in \mathcal{R}_cL$ by Proposition 3.2.1. Now, similar to Proposition 3.2.3, we would have $\alpha^2\iota_\alpha = \alpha$. Therefore $\mathcal{R}_cL$ is regular, that is, L is a CP-frame. □

The concept of z_c-ideals introduced by Estaji *et al* [45]. We recall the following definitions from [45].

Definition 3.2.2. An ideal I in a ring $\mathcal{R}_cL$ is called a z_c-ideal if, for every $\alpha \in \mathcal{R}_cL$ and $\beta \in I$, $coz\alpha = coz\beta$ implies $\alpha \in I$.

Definition 3.2.3. For every $a \in L$, let $\mathbf{M}_a^c = \{\alpha \in \mathcal{R}_cL \mid coz\alpha \leq a\}$.

We are now ready for the following proposition.

Proposition 3.2.5. [45, Proposition 3.5] *The following statements are equivalent for an ideal I of $\mathcal{R}_cL$.*

(1) *I is a z_c-ideal.*

(2) *For any $\alpha, \beta \in \mathcal{R}_cL$, $\beta \in I$ and $coz\alpha \leq coz\beta$ implies $\alpha \in I$.*

(3) $I = \bigcup\{\mathbf{M}^c_{coz\alpha} \mid \alpha \in I\}$.

Proof. (1) $\Rightarrow$ (2): Suppose $\alpha \in I$ and $coz\beta \leq coz\alpha$. Then

$$coz\beta = coz\alpha \wedge coz\beta = coz(\alpha\beta).$$

Since $\alpha\beta \in I$, by (1), we conclude that $\beta \in I$.

(2) $\Rightarrow$ (3): It is clear that $I \subseteq \bigcup\{\mathbf{M}^c_{coz\alpha} \mid \alpha \in I\}$, since for every $\gamma \in \mathcal{R}_cL$, $\gamma \in \mathbf{M}^c_{coz\gamma}$. To see the inverse inclusion, let $\alpha \in I$ and consider $\beta \in \mathbf{M}^c_{coz\alpha}$. This means that $coz\beta \leq coz\alpha$, so that, by hypothesis, $\beta \in I$. Therefore $\mathbf{M}^c_{coz\alpha} \subseteq I$, hence we are done.

(3) $\Rightarrow$ (1): Let $\alpha \in I$ and $\beta \in \mathcal{R}_cL$ with $coz\alpha = coz\beta$. Then $\beta \in \mathbf{M}^c_{coz\beta} = \mathbf{M}^c_{coz\alpha} \subseteq I$, and hence we are done.

□

Lemma 3.2.2. [45, Lemma 3.8] *Every maximal ideal of $\mathcal{R}_cL$ is a z_c-ideal.*

The following lemma will be useful in the sequel. We omit the proof.

Lemma 3.2.3. [45, Lemma 3.13] *Let $\alpha \in \mathcal{R}L$ and $\rho_3 : \mathcal{L}(\mathbb{R}) \to \mathcal{L}(\mathbb{R})$ by $\rho_3(p, q) = (p^3, q^3)$. Then*

(1) $\rho_3 \in \mathcal{R}(\mathcal{L}(\mathbb{R}))$.

(2) ${\rho_3}^3 = id_{\mathcal{L}(\mathbb{R})}$.

(3) $(\alpha \circ \rho_3)^3 = \alpha$.

(4) $coz(\alpha \circ \rho_3) = coz\alpha$.

(5) *if $\alpha \in \mathcal{R}_cL$, then $\alpha \circ \rho_3 \in \mathcal{R}_cL$.*

The following proposition shows that if I and J are z_c-ideals in $\mathcal{R}_cL$, then the product IJ is a z_c-ideal.

Proposition 3.2.6. [45, Proposition 3.14] *If I and J are z_c-ideals in $\mathcal{R}_cL$, then $IJ = I \cap J$.*

Proof. It is trivial that $IJ \subseteq I \cap J$, we show the reverse inclusion. Let $\alpha \in I \cap J$. Suppose that ρ_3 be the same in Lemma 3.2.3. Then, by Lemma 3.2.3 (3, 5), we have $\alpha^{\frac{1}{3}} \in \mathcal{R}_cL$ and $\alpha^{\frac{1}{3}}\alpha^{\frac{1}{3}} \in \mathcal{R}_cL$. Also, $\alpha = (\alpha^{\frac{1}{3}})^3 = \alpha^{\frac{1}{3}}\alpha^{\frac{2}{3}}$ and $coz\alpha = coz(\alpha^{\frac{1}{3}})$. Now, since $\alpha \in I \cap J$ and I and J are z_c-ideals, we infer that $\alpha^{\frac{1}{3}} \in I$ and $\alpha^{\frac{1}{3}} \in J$. Therefore $\alpha = \alpha^{\frac{1}{3}}\alpha^{\frac{2}{3}} \in IJ$ and we are done. □

If we relax the strict equality in Lemma 2.2.8, we have the following.

Lemma 3.2.4. [45, Lemma 3.26] *For $\alpha, \beta \in \mathcal{R}_cL$, the following are equivalent.*

(1) $coz\beta \leq coz\alpha$.

(2) $\mathbf{M}^c_{coz\beta} \subseteq \mathbf{M}^c_{coz\alpha}$.

(3) $\mathcal{M}_c(\alpha) \subseteq \mathcal{M}_c(\beta)$.

The following proposition shows that z_c-ideals in $\mathcal{R}_cL$ are precisely z-ideals *á* la Mason.

Proposition 3.2.7. [45, Proposition 3.27] *An ideal I in $\mathcal{R}_cL$ is a z_c-ideal if and only if it is a z-ideal á la Mason.*

Proof. Let I be a z_c-ideal and suppose that $\alpha, \beta \in \mathcal{R}_cL$ such that $\mathcal{M}(\alpha) \subseteq \mathcal{M}(\beta)$ and $\alpha \in I$. Since $\mathcal{M}_c(\alpha) \subseteq \mathcal{M}_c(\beta)$, we conclude by Lemma 3.2.4 that $coz\beta \leq coz\alpha$, which follows that $\beta \in I$, because I is a z_c-ideal (by Proposition 3.2.5). Therefore I is a z-ideal *á* la Mason.

Conversely, let I be a z-ideal *á* la Mason. Suppose that $coz\beta \leq coz\alpha$ and $\alpha \in I$. Then, by Lemma 3.2.4, $\mathcal{M}_c(\alpha) \subseteq \mathcal{M}_c(\beta)$, which follows that $\mathcal{M}(\alpha) \subseteq \mathcal{M}(\beta)$. Therefore, we have $\beta \in I$ because I is a z-ideal *á* la Mason. □

Proposition 3.2.8. [45, Proposition 4.6] *Every prime ideal of $\mathcal{R}_cL$ is contained in a unique maximal ideal.*

Resulting from [53], [71], and from Proposition 2.2.2, we now give a characterisation of CP-frames. We noticed the mistake made by Estaji and Robat Sarpoushi [44] by citing the wrong article for Proposition 2.2.2.

Theorem 3.2.5. [44, Theorem 4.3] *The following are equivalent for a frame L.*

(1) *L is a CP-frame.*

(2) *Each ideal of $\mathcal{R}_cL$ is a z_c-ideal.*

(3) *Each ideal of $\mathcal{R}_cL$ is an intersection of a set of prime ideals.*

(4) *Each ideal of $\mathcal{R}_cL$ is an intersection of a set of all maximal ideals.*

(5) *Each prime ideal of $\mathcal{R}_cL$ is an intersection of a set of maximal ideals.*

(6) *For each $\alpha, \beta \in \mathcal{R}_cL$, $\langle \alpha, \beta \rangle = \langle \alpha^2 + \beta^2 \rangle$.*

(7) *Each principal ideal of $\mathcal{R}_cL$ is generated by an idempotent.*

(8) *Each prime ideal of $\mathcal{R}_cL$ is maximal.*

(9) *For each $\alpha \in \mathcal{R}_cL$, $coz\alpha \vee (coz\alpha)^* = 1$.*

Proposition 3.2.9. [45, Proposition 3.20] *Let Q be an ideal of $\mathcal{R}_cL$, and $\alpha \in \mathcal{R}_cL$. If $\mathbf{M}^c_{coz\alpha} \subseteq \sqrt{Q}$, then $\mathbf{M}^c_{coz\alpha} \subseteq Q$.*

Corollary 3.2.2. [45, Corollary 3.21] *An ideal of $\mathcal{R}_cL$ is a z_c-ideal if and only if its radical ideal is a z_c-ideal.*

Proof. $(\Rightarrow)$: By Proposition 3.2.5 and 3.2.9, it follows that its radical ideal is a z_c-ideal.

$(\Leftarrow)$: Let Q be an ideal of $\mathcal{R}_cL$. Suppose that for $\alpha, \beta \in \mathcal{R}_cL$, $\alpha \in Q$ and $coz\alpha = coz\beta$. Since $\sqrt{Q}$ is a z_c-ideal, $\beta \in \sqrt{Q}$. By Proposition 3.2.9, $\mathbf{M}^c_{coz\beta} \subseteq \sqrt{Q}$ and hence $\mathbf{M}^c_{coz\beta}$. Since $\beta \in \mathbf{M}^c_{coz\beta} \subseteq Q$, it implies that $\beta \in Q$. Therefore Q is a z_c-ideal. $\square$

The following lemma is immediate from Corollary 3.2.2.

Lemma 3.2.6. [44, Lemma 4.9] *$\mathcal{R}_c L$ is a z-good ring.*

The two lemmas, Lemma 2.2.14 and Lemma 3.2.6 are sufficient to prove the following results in an analogous way with the P-frames.

Corollary 3.2.3. [44, Corollary 4.10] *The following statements are equivalent for a frame L.*

(1) *L is a CP-frame.*

(2) *Each essential ideal in $\mathcal{R}_c L$ is a z_c-ideal.*

(3) *Each radical ideal in $\mathcal{R}_c L$ is a z_c-ideal.*

Proof. It is evident. □

Every radical ideal in $\mathcal{R}_c L$ is absolutely convex in an analogous way with the P-frames. With this fact, we arrive at the following characterisation of CP-frames.

Corollary 3.2.4. [44, Corollary 4.11] *The following statements are equivalent for a frame L.*

(1) *L is a CP-frame.*

(2) *Each convex ideal in $\mathcal{R}_c L$ is a z_c-ideal.*

(3) *Each absolutely convex ideal in $\mathcal{R}_c L$ is a z_c-ideal.*

Chapter 4

Almost P-frames

In this chapter, we study almost P-frames as another generalisation of P-frames. Almost P-frames originated in [16] as point-free extensions of almost P-spaces and were further examined in [55] and [36]. We show that the class of P-frames is contained in the class of almost P-frames, and a frame is a P-frame if and only if it is a basically disconnected (weakly cozero complemented) almost P-frame. We also show that a frame is a P-frame if and only if it is both an O_z-frame and an almost P-frame in turn if and only if it is an almost P-frame with countable chain condition (*ccc*). Lastly, we show that every weakly Lindelöf almost P-frame is Lindelöf. We also give a few characterisations of almost P-frames in terms of ring-theoretic properties.

4.1 Basic Concepts

We start with a definition culled from [16].

Definition 4.1.1. A frame L is said to be an *almost P-frame* if for every $a \in CozL$, $a = a^{**}$.

We observe that a basically disconnected frame is not necessarily an almost P-frame.
Let $a \in CozL$. If L is a basically disconnected frame, then $a^* \vee a^{**} = 1$. It follows that a^* and a^{**} are complemented elements of L and hence are cozeros. By normality of $CozL$ there exist

$u, v \in CozL$ such that $u \wedge v = 0$, and $u \vee a^* = 1 = a^{**} \vee v$. Since $a \leq a^{**}$, implies

$$a \vee a^* \leq a^{**} \vee a^* = 1.$$

Thus 1 is not the only dense cozero element. Therefore L is not an almost P-frame. Unlike P-frames, almost P-frames are not necessarily basically disconnected. However, we do not have an example at the moment to illustrate this. Next, we show that the class of almost P-frames contains the class of P-frames.

Proposition 4.1.1. [39] *Every P-frame is an almost P-frame.*

Proof. Let $a \in CozL$. Then $a^* \in CozL$, $a \wedge a^* = 0$ and $a \vee a^* = 1$. Thus $a = a^{**}$, as required. □

The following proposition is an extension of [63, Corollary 2.7] to a point-free setting.

Proposition 4.1.2. [16, Proposition 8.4.7] *A frame is a P-frame if and only if it is basically disconnected almost P-frame.*

Proof. The left to right implication is trivial. Conversely, suppose that L is a basically disconnected almost P-frame and let $a \in CozL$. Then by basically disconnectedness of L, we have $a^* \vee a^{**} = 1$. The frame L is an almost P-frame, so $a = a^{**}$. Thus $a^* \vee a = 1$, as required. □

The following result is stronger than the result just proved. To see this, we observe from Matlabyana [72] that basically disconnected frames are weakly cozero complemented. See also from Dube and Nsonde-Nsayi [39] that every weakly cozero complemented almost P-frame is a P-frame.

Proposition 4.1.3. *A frame is a P-frame if and only if it is a weakly cozero complemented almost P-frame.*

Proof. The left to right implication is immediate from Proposition 4.1.2. Conversely, suppose that $a \in CozL$. We want to show that a is complemented. Since L is weakly cozero complemented, it follows that there exists $b \in CozL$ such that $a \wedge b = 0$ and $a \vee b$ is dense. Then

$a^* \wedge b^* = (a \vee b)^* = 0 \Rightarrow (a^* \wedge b^*)^* = 0^* = 1$. Since L is an almost P-frame, it follows that $a = a^{**}$ and so, $a \vee b^{**} = 1$. Thus a is complemented. Therefore L is a P-frame. □

O_z-frames were introduced by Banaschewski and Gilmour [22] and we recall from [21] that $s \in L$ is *coz-embedded* if $s \wedge - : L \to \downarrow s$ is a coz-onto. A frame L is an *O_z-frame* if every $a \in L$ is *coz*-embedded (see [21] and [22]). It is shown in [21, Proposition 2.2] that a frame L is an *O_z- frame* if and only if every regular element of L is a cozero element. Using this characterisation Matlabyana [72] has shown that an O_z-frame is weakly cozero complemented. Recall also that a frame L has the *countable chain condition* (abbreviated *ccc*) if every collection of pairwise disjoint elements of L is countable. He then showed that every *ccc* is weakly cozero complemented. Therefore, the following corollaries are immediate.

Corollary 4.1.1. *A frame is a P-frame if and only if it is both an O_z-frame and an almost P-frame.*

Corollary 4.1.2. *A frame is a P-frame if and only if is an almost P-frame with ccc.*

The following proposition is taken from Dube [36] as an example, that if the coproduct of two frames is an almost P-frame, then each summand is an almost P-frame.

Proposition 4.1.4. [36, Example 3.2] *If $A \oplus B$ is almost P-frame, then A and B are both almost P-frames.*

Proof. Let i_A and i_B be coproduct inclusions for almost P-frames A and B, respectively. Let $a \in CozA$ implies $c \oplus 1 \in Coz(A \oplus B)$ as $a \oplus 1 = i_A(a) \wedge i_B(1)$, meet of two cozero elements. Since $(a \oplus b)^{**} = a^{**} \oplus b^{**}$ for all $a \in CozA$ and for all $b \in CozB$. So $a^{**} \oplus 1^{**} = (a \oplus 1)^{**} = a \oplus 1$ (since, $A \oplus B$ is almost P-frame). Hence $a = a^{**}$. □

Theorem 4.1.1. [36, Proposition 3.3] *For any frame L, the following are equivalent:*

(1) *L is an almost P-frame.*

(2) *The only dense cozero element of L is the top element 1.*

(3) *Every dense onto homomorphism* $h : L \to M$ *is coz-codense.*

(4) *Every dense onto, coz-onto homomorphism* $h : L \to M$ *is a* C*-quotient map.*

(5) *For every dense* $c \in CozL$*, the homomorphism* $h : L \to \downarrow a$ *is a* C*-quotient map.*

Proof. (1) $\Leftrightarrow$ (2): Let $a \in CozL$ be a dense cozero element. Then $a^* = 0$ and $a^{**} = 0^* = 1$. But L is an almost P-frame, so $a = a^{**} = 1$. Conversely, let $a \in CozL$, we must show that $a = a^{**}$. Let $x \prec\prec a^{**}$. Take $s \in CozL$ such that $x \wedge s = 0$ and $s \vee a^{**} = 1$. Now $(s \vee a)^* = s^* \wedge a^* = (s \vee a^{**})^* = 0$ since $s \vee a^{**} = 1$. This shows that $s \vee a$ is dense. Therefore $s \vee a = 1$, showing that $x \prec a$. Therefore $a = a^{**}$.

(2) $\Rightarrow$ (3): Let $a \in CozL$ such that $h(a) = 1$. Then $0 = h(a) \wedge h(a^*)$ implies that $h(a^*) = 0$, in addition hence $a^* = 0$ (h is dense). Therefore, $a = 1$.

(3) $\Rightarrow$ (4): It follows from Proposition 1.3.4, that an onto homomorphism that is both *coz*-onto and *coz*-codense is a C-quotient map.

(4) $\Rightarrow$ (5): If $a \in CozL$ is dense, then $Coz(\downarrow a) = (\downarrow a) \bigcap CozL$. Hence $L \to \downarrow a$ is both dense and *coz*-onto.

(5) $\Rightarrow$ (2): Let $a \in CozL$ such that a is dense and $f : L \to \downarrow a$ be homomorphism $x \mapsto x \wedge a$. By hypothesis it implies that f is almost *coz*-codense. Since $f(a) = 1_{\downarrow a}$, it follows that there exists $b \in CozL$ such that $a \vee b = 1_L$ and $f(b) = 0$. Hence $a \wedge b = 0$, and therefore $b = 0$ since a is dense. Consequently, $a = 1_L$.

□

Next, we show that dense frame homomorphisms preserve almost P-frames, and also that almost P-frames are reflected by *coz*-codense frame homomorphisms.

Lemma 4.1.2. [36, Lemma 3.4]

(1) *Let* $h : L \to M$ *be a dense onto, coz-onto homomorphism. If* L *is an almost* P*-frame, then so is* M*.*

(2) *Let $h : L \to M$ be a dense onto, coz-codense homomorphism. If M is an almost P-frame, then so is L.*

Proof. (1). Let $a \in CozM$ and take $b \in CozL$ such that $h(b) = a$. Since a dense onto homomorphism commutes with pseudocomplementation, we have that

$$a = h(b) = h(b^{**}) = h(b)^{**} = a^{**}.$$

Therefore M is an almost P-frame.

(2). Let a be a dense cozero element of L. Then $h(a)$ is a dense cozero element of M, and therefore $h(a) = 1$ since M is an almost P-frame. Since h is *coz*-codense, this implies that $a = 1$, and so L is an almost P-frame. □

As a consequence of Lemma 4.1.2, we have the following.

Theorem 4.1.3. [36, Proposition 3.5] *L is an almost P-frame if and only if υL is an almost P-frame.*

Proof. If υL is almost P-frame, then $h : \upsilon L \to L$, is given by join, is dense onto and *coz*-onto. Then L is an almost P-frame. Conversely, suppose L is an almost P-frame and $h : L \to \upsilon L$ is given by join, it is a dense onto and *coz*-codense. Then υL is an almost P-frame. □

We recall from [38] that a frame L is said to be *weakly Lindelöf* if every cover of L has a countable subset that is dense. For the following proposition, we relax the complete regularity condition and consider a regular frame. This is a frame version of [86, Theorem 16.8].

Proposition 4.1.5. *Every regular Lindelöf frame is normal.*

Proof. Let $a, b \in L$ be such that $a \vee b = 1$. Then the set $\{a, b\}$ is a binary cover of L. Put $a = \bigvee\{x \mid x \prec a\}$ and $b = \bigvee\{y \mid y \prec b\}$. By Lindelöfness there is a countable subcover T of L, where $T = \{x_i \vee y_i \mid x_i \prec a, y_i \prec b\}_{i \in I}$. Then

$$\left(\bigvee_{i \in I} T\right)^* = \left(\bigvee_{i \in I}\{x_i \vee y_i \mid x_i \prec a, y_i \prec b\}\right)^* = 1^* = 0.$$

$$\Rightarrow \bigwedge_{i \in I}\{{x_i}^* \wedge {y_i}^* \mid x_i \prec a, y_i \prec b\} = 0.$$

$$\Rightarrow \bigwedge_{i \in I}\{{x_i}^* \mid x_i \prec a\} \wedge \bigwedge_{i \in I}\{{y_i}^* \mid y_i \prec b\} = 0.$$

Put $c = \bigwedge_{i \in I}\{{x_i}^* \mid x_i \prec a\}$ and $d = \bigwedge_{i \in I}\{{y_i}^* \mid y_i \prec b\}$ are such that $c \wedge d = 0$ and $c \vee a = 1 = d \vee b$. Now, c and d are the required elements to satisfy the condition for normality. Hence L is normal as required. □

Next, we show that in almost P-frames the property of weakly Lindelöfness and the property of Lindelöfness coincide.

Proposition 4.1.6. *Every weakly Lindelöf almost P-frame is Lindelöf.*

Proof. Let $a \in CozL$. L is almost P-frame, so the only dense element of L is 1. Since $1 = (1^*)^*$, we have

$$1 = (1^*)^* = 0^* = \bigvee\{x \in L \mid x \wedge 0 = 0\}.$$

Thus $\bigvee\{x \in L \mid x \wedge 0 = 0\}$ is a cover of L. L is weakly Lindelöf, so there is a countable subset $T = \{x_i\}_{i \in I}$ whose join is dense. Therefore L is Lindelöf. □

The following corollary is a consequence of Proposition 4.1.6 and Proposition 4.1.5.

Corollary 4.1.3. *Every weakly Lindelöf almost P-frame is normal.*

4.2 Ring-theoretic characterisations of almost P-frames

In this section, we give characterisations of almost P-frames in terms of ideals in rings. The annihilator of $S \subseteq R$ is an ideal

$$Ann(S) = \{a \in R \mid as = 0 \text{ for every } s \in S\}.$$

If S is a singleton, say $S = \{a\}$, $Ann(S)$ will be abbreviated as $Ann(a)$, for more detail (see [77]). We start with lemmas culled from [36], and we omit the proof for the first and the third lemma. The proof of the second lemma is also found in [72], using a different approach.

Lemma 4.2.1. [36, Lemma 4.1] *Let $\varphi, \psi \in \mathcal{R}L$. Then $(coz\varphi)^* \leq (coz\psi)^*$ if and only if $Ann(\varphi) \subseteq Ann(\psi)$.*

Lemma 4.2.2. [36, Corollary 4.2] *Any $\varphi \in \mathcal{R}L$ is a zero divisor if and only if $coz\varphi$ is not dense.*

Proof. We know that $coz\varphi$ is dense if and only if $(coz\varphi)^* = 0$, if and only if

$$(coz\varphi)^* \leq 0 = (coz1)^*,$$

if and only if $Ann(\varphi) \subseteq Ann(1) = \{0\}$, if and only if $Ann(\varphi) = \{0\}$, if and only if φ is not a zero divisor. □

Lemma 4.2.3. [36, Lemma 4.4] *Suppose $coz\gamma \prec\prec coz\psi$ for some $\gamma, \psi \in \mathcal{R}L$. Then there exists $\alpha \in \mathcal{R}L$ such that $\gamma = \alpha\psi$.*

We have the following from [36].

Proposition 4.2.1. [36, Proposition 4.5] *The following are equivalent for a frame L:*

(1) *L is almost P-frame.*

(2) *Every non-zero divisor in $\mathcal{R}L$ is invertible.*

(3) *Every proper ideal of $\mathcal{R}L$ contains only zero-divisors.*

(4) *Every proper principal ideals of $\mathcal{R}L$ is non-essential.*

(5) *Every fixed countably generated ideal of $\mathcal{R}L$ is contained in a non-essential principal ideal.*

Proof. (1) ⇒ (2): Let $\varphi \in \mathcal{R}L$ be a non-zero divisor. Then $a = coz\varphi$ is a dense cozero element of L by Lemma 4.2.2. So the homomorphism $h : L \to \downarrow a$ is a C-quotient map by Theorem 4.1.1. It is therefore almost coz-codense by Proposition 1.3.4, hence coz-dense because of denseness. But $a = coz\varphi = 1$, thus φ is invertible.

(2) ⇒ (3): Because non-zero divisors are invertible, if an ideal contains one such then it contains the ring unity and is therefore improper.

(3) ⇒ (4): Let $\langle\varphi\rangle$ be a proper principal ideal of $\mathcal{R}L$. By (3), thus φ is a zero-divisor, so $coz\varphi$ is not dense by Lemma 4.2.2. But $\bigvee\{coz\varphi \mid \varphi \in I\} = coz\varphi$, so $\langle\varphi\rangle$ is not essential by Lemma 2.2.10.

(4) ⇒ (5): Let $I = \{\varphi_n \mid n \in \mathbb{N}\}$ be fixed. Let $a = \bigvee coz(\varphi_n)$ and note that

$$1 \neq \bigvee\{coz\alpha \mid \alpha \in I\} = a \in CozL.$$

Let $\tau \in \mathcal{R}L$ such that $a = coz\tau$. Then the principal ideal $\langle\tau\rangle$ is proper, and is therefore non-essential by hypothesis. So, thus a is not dense. Take $0 \neq b \in CozL$ such that $a \wedge b = 0$. Let u be a non-zero cozero element of L such that $u \prec\!\prec b$. Pick $s \in CozL$ such that $u \wedge s = 0$ and $s \vee b = 1$. Let $\psi \in \mathcal{R}L$ such that $s = coz\psi$. Since s is not dense, $\langle\varphi\rangle$ is not essential. For any $n \in \mathbb{N}$, $b \wedge coz(\varphi_n) = 0$ and therefore $coz(\varphi_n) \prec\!\prec coz\psi$ (since $b \vee coz\psi = 1$). There exists $r_n \in \mathcal{R}L$ for each n such that $\varphi_n = \psi r_n$. Therefore $I \subseteq \langle\psi\rangle$.

(5) ⇒ (1): Let $1 \neq a \in CozL$, and let $\varphi \in \mathcal{R}L$ such that $a = coz\varphi$. Therefore the principal ideal $\langle\varphi\rangle$ is proper, $\langle\varphi\rangle$ is not essential and thus φ is not dense. Hence L is an almost P-frame (by Theorem 4.1.1). □

The following proposition is culled from [36].

Proposition 4.2.2. [36, Proposition 4.13] *The following are equivalent for a frame L:*

(1) *L is almost P-frame.*

(2) *Every z-ideal of $\mathcal{R}L$ is a d-ideal.*

(3) *Every maximal ideal of $\mathcal{R}L$ is a d-ideal.*

Proof. (1) $\Rightarrow$ (2): It is immediate since $coz\alpha = (coz\alpha)^{**}$ if L is an almost P-frame.

(2) $\Rightarrow$ (3): Since every maximal ideal is a z-ideal in $\mathcal{R}L$, we are done.

(3) $\Rightarrow$ (1): Let a be a dense cozero element of L. Suppose on contrary, that $a \neq 1$. Then set $J = \{\alpha \in \mathcal{R}L \mid coz\alpha \leq a\}$ is a proper ideal of $\mathcal{R}L$ since $coz(\tau + \upsilon) \leq coz\tau \vee coz\upsilon$ and $coz(\alpha\rho) = coz\alpha \wedge coz\rho$. Let I be a maximal ideal of $\mathcal{R}L$ such that $J \subseteq I$. Take $\eta \in \mathcal{R}L$ such that $a = coz\eta$. Now, by (3), I is a d-ideal containing η. But $coz1 \leq (coz\eta)^{**}$, which implies that $1 \in I$; a contradiction since I is a proper ideal. $\square$

Theorem 4.2.4. [36, Proposition 4.14] *A frame L is almost P-frame if and only if*

$$\bigcup\left\{\mathbf{O}^I \mid I \in \sum \beta L\} = \bigcup\{\mathbf{M}^I \mid I \in \sum \beta L\right\}.$$

Proof. Let a be a dense cozero element of L. Suppose on contrary that $a \neq 1$. Let $\alpha \in \mathcal{R}L$ such that $a = coz\alpha$. Pick $I \in \sum \beta L$ such that $r(coz\alpha) \subseteq I$. Then $\alpha \in \mathbf{M}^I$, and so, by hypothesis, there exists $J \in \sum \beta L$ such that $\alpha \in \mathbf{O}^J$. But this implies that $coz\alpha \in J$, which is not possible since $J \neq 1$.

Conversely, the one inclusion is trivial since $\mathbf{O}^I \subseteq \mathbf{M}^I$ for each I. Let $I \in \sum \beta L$ and let $\alpha \in \mathbf{M}^I$. Since $\mathbf{M}^I$ is a proper ideal, it follows from Proposition 4.2.1 that α is a zero-divisor. Pick a nonzero $\varphi \in \mathcal{R}L$ such that $\alpha\varphi = 0$. Since $\varphi \neq 0$, $r(coz\varphi) \neq 0$. By spatiality of βL it follows that there exists $J \in \sum \beta L$ such that $r(coz\varphi) \nsubseteq J$. Consequently $r(coz\varphi) \vee J = 1$, by maximality of J. But $r(coz\varphi) \wedge r(coz\alpha) = 0$, so $r(coz\alpha) \prec\prec J$, which implies that $\alpha \in \mathbf{O}^J$. This establishes the other inclusion. $\square$

We close this chapter with the following proposition, which is culled from [31].

Proposition 4.2.3. [31, Proposition 4.4] *Let L be a proper essential P-frame with obstacle I. Then the following are equivalent.*

(1) *L is an almost P-frame.*

(2) *For every* $f \in \mathbf{M}^I \backslash \mathbf{O}^I$*, there exists* $J \in \sum \beta L$*,* $J \neq I$ *such that* $f \in \mathbf{O}^J$*.*

(3) *For every* $c \in CozL$ *such that* $r(c) \subseteq I$ *and* $c \notin I$*, there exists* $J \in \sum \beta L$*,* $J \neq I$*, such that* $c \in J$*.*

Chapter 5

F- and F'-frames

In this chapter, we study F-frames and F'-frames. In the first section, we focus on F-frames and show that the class of F-frames contains the class of P-frames. We also note that the class of basically disconnected frames is also contained in the class of F-frames. We give ring-theoretic characterisations of F-frames in the first subsection. We show that L is an F-frame if and only if $\mathcal{R}L$ is a Bézout ring. In the second section, we show that the classes of P-frames, basically disconnected frames and F-frames are contained in the class of F'-frames. We also show that every weakly Lindelöf F'-frame is an F-frame and every zero-dimensional weakly Lindelöf F'-frame is a strongly zero-dimensional F-frame. Lastly, we give a few ring-theoretic characterisation of F'-frames.

5.1 F-frames

In this section, our focus is on F-frames as another generalisation of P-frames. F-frames were originally introduced by Ball and Walters-Wayland [16]. We start with a definition.

Definition 5.1.1. The frame L is said to be an *F-frame* if $\varphi : L \to\downarrow a$ is a C^*-quotient map for every $a \in CozL$.

We have the following as a characterisation of F-spaces, a space X is said to be an F-space if whenever A and B are disjoint cozero sets, then $\bar{A} \cap \bar{B} = \emptyset$. We can see that; A and B are completely separated. This was extended to frames by Ball and Walters-Wayland [16], a frame L is an F-frame if and only if whenever $a, b \in CozL$ such that $a \wedge b = 0$ there exist $c, d \in CozL$ such that $c \vee d = 1$ and $c \wedge a = 0 = d \wedge b$ (see [16, Proposition 8.4.10]).

Proposition 5.1.1. [16, Proposition 8.4.3] *Every basically disconnected frame is an F-frame.*

Proof. To show that L is an F-frame, let $a, b \in CozL$ be such that $a \wedge b = 0$. Basically disconnected frames are De Morgan frames, so $a \wedge b = 0$ implies that $(a \wedge b)^* = a^* \vee b^* = 1$. Hence L is an F-frame.

□

The following corollary follows immediately from Proposition 2.1.2 and Proposition 5.1.1.

Corollary 5.1.1. [16, Proposition 8.4.7] *Every P-frame is an F-frame.*

Dube [33] indicated that the following proposition can be shown easily with frame-theoretic terms without supplying proof. For the sake of completeness, we supply the proof.

Proposition 5.1.2. *A frame is basically disconnected if and only if it is a weakly cozero complemented F-frame.*

Proof. If L is a basically disconnected, then by Lemma 2.1.1 it is weakly cozero complemented. To show that it is an F-frame, let $a, b \in CozL$ be such that $a \wedge b = 0$. By Proposition 5.1.1, we are done.

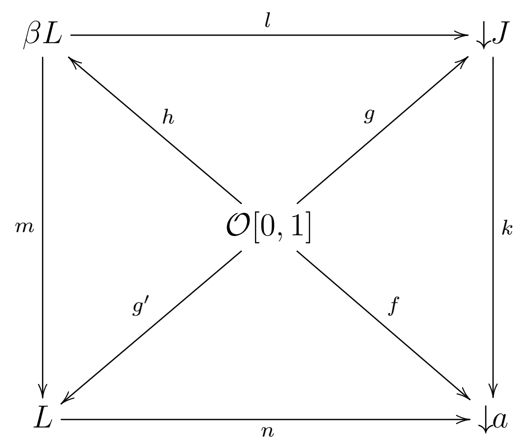
Conversely, let $a \in CozL$. By hypothesis L is weakly cozero complemented, so find $b \in CozL$ such that $a \wedge b = 0 = (a \vee b)^*$. The frame L is an F-frame, so there exist $c, d \in CozL$ such that $c \vee d = 1$ and $c \wedge b = 0 = d \wedge a$. Therefore $b \prec d$ and $a \prec c$ so that $b^* \vee d = 1$ and $a^* \vee c = 1$. Now $a^* \wedge b^* = 0$ implies that $a^* \leq b^{**}$ and $b^* \leq a^{**}$. Also $b \wedge c = 0 = d \wedge a$ implies that $c \leq b^*$ and $d \leq a^*$. Hence

$$1 = c \vee d \leq a^* \vee b^* \leq a^* \vee a^{**}.$$

Thus L is basically disconnected as expected. □

Theorem 5.1.1. [16, Proposition 8.4.11] *If L is an F-frame, then βL is an F-frame.*

Proof. Assume that L is an F-frame. Take $J \in Coz(\beta L)$, and put $a \equiv \bigvee J \in CozL$. Now take $g : \mathcal{O}[0,1] \to \downarrow J$ and consider the following diagram:



The maps in the diagram arise as follows: m is the canonical join map $I \to \bigvee I$, and k is its restriction to $\downarrow J$; f is kg; n is the open quotient map of a; g' is the extension of f over n whose existence is guaranteed by the fact that n is a C^*-quotient map; h is the result of factoring g' through m, which can be done because m is the co-reflection of L in compact regular frames; l is the open quotient map of J. We claim that h is an extension of g over l, i.e., that $lh = g$. To establish this claim first note that the outer square commutes, i.e., that $kl = nm$, since

$$\bigvee (J \wedge I) = \bigvee J \wedge \bigvee I = a \wedge \bigvee I.$$

Therefore

$$kg = f = ng' = nmh = klh,$$

and since k is monic in **regular frames** by virtue of being dense, it follows that $g = lh$. □

The following proposition is taken from [16, Corollary 8.4.12], our method of proof is different.

Proposition 5.1.3. [16, Corollary 8.4.12] *If L is an F-frame, then $\downarrow a$ is an F-frame for each $a \in CozL$.*

Proof. If $c, d \in Coz(\downarrow a)$ such that $c \wedge d = 0_{\downarrow a}$, then $c, d \in CozL$ such that $c \wedge d = 0_L$. L is an F-frame, so there exist $u, v \in CozL$ such that $u \vee v = 1_L$ and $c \wedge u = 0 = d \wedge v$. Now, $(a \wedge u) \vee (a \wedge v) = a \wedge (u \vee v) = a \wedge 1 = a$ and $a \wedge (c \wedge u) = 0 = a \wedge (d \wedge v)$. Thus $(a \wedge u), (a \wedge v) \in Coz(\downarrow a)$. Hence $\downarrow a$ is an F-frame. □

A frame homomorphism is said to satisfy property (β), if for every $a, b \in CozL$ with

$$h(a) \vee h(b) = 1,$$

then there exist $c, d \in CozL$ such that $c \vee d = 1$, $h(c) \leq h(a)$ and $h(d) \leq h(b)$.
The following proposition is taken from [32]. We omit the proof.

Proposition 5.1.4. [32, Proposition 2.6] *A quotient map $h : L \to M$ is a C^*-quotient map if and only if it is coz-onto and satisfies (β).*

The following result comes from [35].

Proposition 5.1.5. [35, Proposition 3.9] *The following are equivalent for a completely regular frame L.*

(1) *L is an F-frame.*

(2) *Every quotient map $h : L \to M$ satisfies (β).*

(3) *Every coz-onto homomorphism $h : L \to M$ is a C^*-quotient map.*

Proof. (1) $\Rightarrow$ (2): Let $a, b \in CozL$ such that $h(a) \vee h(b) = 1$. Since the join of cozeros is a cozero, it follows that $a \vee b \in CozL$ and L is an F-frame. The open quotient map $g : L \to \downarrow(a \vee b)$ is a C^*-quotient map. By proposition 5.1.4, g satisfies (β). Since $a, b \in Coz(\downarrow(a \vee b))$ with $g(a) \vee g(b) = 1_{\downarrow(a \vee b)}$, there exist $c, d \in CozL$ such that $c \vee d = 1$, $g(c) \leq g(a)$, and $g(d) \leq g(b)$.

Then $c \wedge (a \vee b) \leq a$ and $d \wedge (a \vee b) \leq b$, which implies $h(c) \wedge h(a \vee b) = h(c) \leq h(a)$ and $h(d) \wedge h(a \vee b) = h(d) \leq h(b)$. It follows that h satisfies (β).

(2) $\Rightarrow$ (3): By complete regularity, every *coz*-onto homomorphism $h : L \to M$ is onto. So from Proposition 5.1.4, it follows that h is a C^*-quotient map.

(3) $\Rightarrow$ (1): For any $a \in CozL$, the open quotient map $L \to \downarrow a$ is a *coz*-onto map by Lemma 1.3.1, and hence, a C^*-quotient map by hypothesis. Furthermore, L is an F-frame. □

Corollary 5.1.2. [77, Lemma 6.2.2] *If L is an F-frame and $h : L \to M$ is a quotient map, then M is an F-frame.*

A *weakly Lindelöf element* a is an element that is dense which is a join of countably many elements that are rather below (or, completely below) a.

Lemma 5.1.2. [55, Proposition 7] *Every cozero element of a weakly Lindelöf frame is weakly Lindelöf.*

Proposition 5.1.6. [23, Corollary 4] *The cozero elements of a Lindelöf frame are precisely the Lindelöf elements.*

The following lemma is found in [21] and [40].

Lemma 5.1.3. *Let $h : L \to M$ be a quotient map. If M is Lindelöf and L is completely regular, then h is coz-onto.*

Proof. Let $a \in CozM$ and $A = \{x \in CozL \mid x \leq h_*(a)\}$. Then, by completely regularity, we have $a = hh_*(a) = \bigvee h[A]$. Hence there is a countable $B \subseteq A$ such that $a = \bigvee h[B]$. By Proposition 5.1.6, thus $\bigvee B$ is a cozero element of L mapped to a by h. □

Ball and Walters-Wayland [16] showed that a quotient map $h : L \to M$ is a C^*-quotient map if and only if every two-element cover of M by cozero elements is the image of some two-element cover of L consisting of cozero elements. The following proposition is shown using this characterisation and the use of Proposition 1.3.3, and is the translation of a classical result from [75, Theorem 5.2] .

Proposition 5.1.7. [35] *Let L be an F-frame and $h : L \to M$ be a quotient map. If M is Lindelöf, then h is a C^*-quotient map.*

Proof. Let $a, b \in CozM$ such that $a \wedge b = 0_M$,

$$A = \{x \in CozL \mid x \leq h_*(a)\} \text{ and } B = \{y \in CozL \mid y \leq h_*(b)\}.$$

Since L is an F-frame, it is completely regular. Then

$$a = hh_*(a) = \bigvee h[A] \text{ and } b = hh_*(b) = \bigvee h[B].$$

Hence there are countable $C \subseteq A$ and $D \subseteq B$ such that $a = \bigvee h[C]$ and $b = \bigvee h[D]$. Since h is *coz*-onto and by Proposition 5.1.6, hence there exist $\bigvee C, \bigvee D \in CozL$ such that

$$\left(\bigvee C\right) \wedge \left(\bigvee D\right) = 0_L,$$

mapped to a and b by h. That is,

$$0_M = a \wedge b = \bigvee h[C] \wedge \bigvee h[D] = h\left[\bigvee C\right] \wedge h\left[\bigvee D\right] = h\left[\left(\bigvee C\right) \wedge \left(\bigvee D\right)\right] = h(0_L).$$

Let $n, m \in CozM$ such that $n \vee m = 1_M$. The frame L is an F-frame, hence for

$$(\bigvee C) \wedge (\bigvee D) = 0_L$$

implies that there exist $u, v \in CozL$ such that $u \vee v = 1_L$ and $(\bigvee C) \wedge u = 0 = (\bigvee D) \wedge v$, $n = h(u)$ and $m = h(v)$. Now

$$1_M = h(1_L) = h(u \vee v) = h(u) \vee h(v) = n \vee m.$$

□

Dube [35] noted that a Lindelöf quotient of an F-frame is a C^*-quotient without proving it. According to Dube, it follows from [16, Corollary 8.2.7], that if L is a Lindelöf F-frame, then βL is the only compactification of L which is an F-frame.

Lemma 5.1.4. *Let $h : L \to M$ be a quotient map. If M is weakly Lindelöf and L is completely regular, then h is coz-onto.*

Proof. Obviously for $1 \in M$, $1 \in CozM$. We have $1 \in 1_L, 1 \in CozL$ such that $h(1_L) = 1_M$. Now let $a \in CozM$ such that $a \neq 1$ and $A = \{x \in CozL \mid x \leq h_*(a)\}$. Then, by completely regular, we have $a = hh_*(a) = \bigvee h[A]$. Hence there is a countable $B \subseteq A$ such that $a = \bigvee h[B]$. By Lemma 5.1.2, thus $\bigvee B$ is a cozero element of L mapped to a by h. □

The following proposition is a point-free version of classical result in [29].

Proposition 5.1.8. *Let L be an F-frame and $h : L \to M$ be a quotient map. If M is weakly Lindelöf, then M is an F-frame.*

Proof. Let $a, b \in CozM$ such that $a \wedge b = 0_M$. By *coz*-onto (see Proposition 1.3.2), there exist $c, d \in CozL$ such that $c \wedge d = 0_L$, $a = h(c)$ and $b = h(d)$. Since L is an F frame, there exist $n, m \in CozL$ such that $n \vee m = 1_L$. Now

$$1_M = h(1_L) = h(n \vee m) = h(n) \vee h(m).$$

Thus $h(n), h(m) \in CozM$, since h preserves cozero elements. Therefore M is an F-frame. □

We end this section by strengthening Proposition 5.1.7 as follows, and the following proposition is a point-free version of classical result in [29].

Proposition 5.1.9. *Let L be an F-frame and $h : L \to M$ be a quotient map. If M is weakly Lindelöf, then h is a C^*-quotient map.*

Proof. Let $a, b \in CozM$ such that $a \wedge b = 0_M$. By *coz*-onto (see Proposition 1.3.2), there exist $c, d \in CozL$ such that $c \wedge d = 0_L$. The frame L is an F-frame, there exist $u, v \in CozL$ such that $u \vee v = 1_L$ and $u \wedge c = 0_L = v \wedge d$. Now

$$1_M = h(0_L) = h(u \vee v) = h(u) \wedge h(v)$$

and

$$h(u) \wedge a = h(u) \wedge h(c) = h(u \wedge c) = 0_M = h(v \wedge d) = h(v) \wedge h(d) = h(v) \wedge b.$$

Thus $h(u), h(d) \in CozM$, since h preserves cozero elements. Hence h is C^*-quotient map. $\square$

5.1.1 Ring-theoretic characterisations of F-frames

In this subsection, we show that a frame L is an F-frame if and only if the ring $\mathcal{R}L$ is a Bézout ring. We will make use of the following facts.

(i) For any space X, $C(X)$ and $\mathcal{R}(\mathcal{O}X)$ are isomorphic f-rings.

(ii) For any frame L, $\mathcal{R}(\beta L)$ and $\mathcal{R}^*(\beta L)$ are isomorphic as f-rings.

(iii) A space X is an F-space if and only if $\mathcal{O}X$ is an F-frame.

(iv) A frame L is an F-frame if and only if βL is an F-frame.

We shall use the properties of f-rings and characterisations of Bézout rings established in [70]. A homomorphic image of a Bézout ring is a Bézout ring. A ring is said to be *reduced* if it has no non-zero nilpotent elements. We omit the proof for the following lemma taken from [35].

Lemma 5.1.5. [35, Lemma 3.1] *Let R be a reduced commutative f-ring with unity, and let R^* denote its bounded part. Then the following holds.*

(1) *If a is a nonnegative invertible element of R, then $\frac{1}{a}$ is nonnegative.*

(2) *Suppose R has the bounded inversion property. Then R is Bézout if and only if R^* is Bézout.*

Proposition 5.1.10. [35, Proposition 3.2] *A completely regular frame L is an F-frame if and only if $\mathcal{R}L$ is a Bézout ring.*

Proof. The Lemma 5.1.5 and the facts preceding it show that L is an F-frame if and only if βL is an F-frame if and only if $\mathcal{R}L$ is Bézout if and only if $\mathcal{R}^*(L)$ is Bézout if and only if $\mathcal{R}L$ is Bézout. □

Thus, a spatial frame L is an F-frame if and only if $\mathcal{R}L$ is a Bézout ring.

Any completely regular frame L has the Hewitt realcompactification υL and the Lindelöfication λL. The join map $\upsilon L \to L$ and the join map $\lambda L \to L$ are both dense (so that each of the induced ring homomorphisms $\mathcal{R}(\upsilon L) \to \mathcal{R}L$ and $\mathcal{R}(\lambda L) \to \mathcal{R}L$ is one-to-one by [18, Lemma 1] and are C-quotient maps (so that each of the induced ring homomorphisms is onto). As a consequence we have:

Corollary 5.1.3. [17, 35] *The following are equivalent for a completely regular frame L.*

(1) *L is an F-frame.*

(2) *υL is an F-frame.*

(3) *λL is an F-frame.*

Next, we turn to the characterisation in terms of ideals. We start by recalling how the **M**- and **O**-ideals are defined. Let $h : \beta L \to M$ be a quotient map. Denote by $\mathbf{M}^h$ and $\mathbf{O}^h$ the ideals of $\mathcal{R}L$ by

$$\mathbf{M}^h = \{\varphi \in \mathcal{R}L \mid h(r(coz\varphi)) = 0\} \text{ and } \mathbf{O}^h = \{\varphi \in \mathcal{R}L \mid h(r(coz\varphi)^*) = 1\}.$$

In particular, if $I \in \beta L$ and $h : \beta L \to \uparrow I$ is the closed quotient map determined by I, we denote $\mathbf{M}^h$ and $\mathbf{O}^h$ by $\mathbf{M}^I$ and $\mathbf{O}^I$, respectively. Thus

$$\mathbf{M}^I = \{\varphi \in \mathcal{R}L \mid r(coz\varphi) \subseteq I\} \text{ and } \mathbf{O}^I = \{\varphi \in \mathcal{R}L \mid I \bigvee r((coz\varphi)^*) = 1\},$$

and hence, in light of properties of r,

$$\mathbf{O}^I = \{\varphi \in \mathcal{R}L \mid r(coz\varphi) \prec I\} = \{\varphi \in \mathcal{R}L \mid coz\varphi \in I\}.$$

We recall from [33] that an ideal I of a frame L is *coz-prime* whenever it contains the meet of two cozero elements, then it contains at least one of the elements. The following proposition is taken from [33], we omit the proof.

Proposition 5.1.11. [33, Proposition 4.9] *The following statements are equivalent for a frame L.*

(1) *L is an F-frame.*

(2) *$\mathbf{O}^I$ is prime for each maximal $I \in \beta L$.*

(3) *Each maximal $I \in \beta L$ is coz-prime.*

According to Dube [33], condition (4) in the following proposition is not straightforward, but clearly it holds (see [35, Proposition 3.4]).

Proposition 5.1.12. [33, Corollary 4.10] *The following are equivalent for a completely regular frame L.*

(1) *L is an F-frame.*

(2) *$\mathbf{O}^I$ is a prime ideal for each point I of βL.*

(3) *Minimal prime ideals of $\mathcal{R}L$ are precisely the ideals $\mathbf{O}^I$ for I a point of βL.*

(4) *Every maximal ideal of $\mathcal{R}L$ contains a unique minimal prime ideal.*

(5) *Every prime ideal of $\mathcal{R}L$ contains a unique minimal prime ideal.*

Proof. (1) $\Rightarrow$ (2): Let $\mu\nu \in \mathbf{O}^I$, and suppose $\mu \in \mathbf{O}^I$. We must show that $\nu \in \mathbf{O}^I$. Since $\mu\nu \in \mathbf{O}^I$, there exist $\tau \notin \mathbf{M}^I$ such that $\tau\mu\nu = 1$. Thus $coz\nu = coz(\tau\mu)$. Since L is an F-frame, there exist $\gamma, \delta \in \mathcal{R}L$ such that

$$coz\gamma \vee coz\delta = 1 \text{ and } coz\gamma \wedge coz\nu = 0 = coz\delta \wedge coz\tau\mu.$$

Since $\mathbf{M}^I$ is a proper ideal, it cannot contain both γ and δ, lest it contain the invertible element $r^2 + \delta^2$. Since $coz(\delta\tau) \wedge coz\mu = 0$ (so that $\delta\tau\mu = 0$), we have that $\delta\tau \in \mathbf{M}^I$, for otherwise we would have $\mu \in \mathbf{O}^I$. As $\tau \notin \mathbf{M}^I$, it follows by primeness that $\delta \in \mathbf{M}^I$. Consequently $\gamma \notin \mathbf{M}^I$. But $\gamma\nu = 0$, so we conclude that $\nu \in \mathbf{O}^I$.

(2) $\Rightarrow$ (3): By Proposition 5.1.11, $\mathbf{O}^I$ is prime for each I maximal in βL. But each element of $\mathbf{O}^I$ is annihilated by an element outside $\mathbf{O}^I$, so $\mathbf{O}^I$ is a minimal prime ideal. Now let P be a minimal prime ideal of $\mathcal{R}L$, and take a maximal $I \in \beta L$ such that $\mathbf{O}^I \subseteq P \subseteq \mathbf{M}^I$. The minimality condition on P therefore yields $P = \mathbf{O}^I$. We are done.

(3) $\Rightarrow$ (4): This is immediate since, by (3), $\mathbf{O}^I$ is the only minimal prime ideal contained in $\mathbf{M}^I$.

(4) $\Rightarrow$ (1): Let I be a maximal element of βL. We show that $\mathbf{O}^I$ is prime. The radical $\sqrt{\mathbf{O}^I}$ of $\mathbf{O}^I$ is the intersection of all prime ideals containing $\mathbf{O}^I$. Each prime ideal containing $\mathbf{O}^I$ contains a minimal prime ideal, which $\sqrt{\mathbf{O}^I}$ still contains $\mathbf{O}^I$. Thus, $\sqrt{\mathbf{O}^I}$ is the intersection of all minimal prime ideals containing $\mathbf{O}^I$. But now, any minimal prime ideal containing $\mathbf{O}^I$ is contained in $\mathbf{M}^I$. By (4) there is only one minimal prime ideal contained in $\mathbf{M}^I$, so $\sqrt{\mathbf{O}^I}$ is minimal prime. But $\mathbf{O}^I = \sqrt{\mathbf{O}^I}$, for if $\nu \in \sqrt{\mathbf{O}^I}$ and n is a nonnegative integer such that $\nu^n \in \mathbf{O}^I$, then $coz\nu = coz(\nu^n) \in I$, showing $\nu \in \mathbf{O}^I$. Therefore $\mathbf{O}^I$ is a prime ideal, and hence L is an F-frame.

(4) $\Rightarrow$ (5): It is trivial, since every maximal ideal is prime.

(5) $\Rightarrow$ (1): It is immediate from (4) $\Rightarrow$ (1). □

For the rest of this subsection, the results are based on the article by Acharyya *et al* [6] unless stated otherwise.

Proposition 5.1.13. *A frame L is an F-frame if and only if every finitely generated ideal of $\mathcal{R}L$ is flat.*

Proof. Suppose, L is an F-frame. Let I be a finitely generated ideal of $\mathcal{R}L$ by Proposition

5.1.10. Since L is an F-frame, $I = \langle f \rangle$ for some $f \in \mathcal{R}L$. Now it can be easily checked that

$$0 \to K \to \mathcal{R}L \xrightarrow{\varphi} I \to 0$$

is an exact sequence of $\mathcal{R}L$-modules, where $K = \{k \in \mathcal{R}L : kf = 0\}$ and $\varphi(g) = fg$, $g \in \mathcal{R}L$. Let J be another finitely generated ideal. Then J is also principal ideal and so $J = \langle r \rangle$ for some $r \in \mathcal{R}L$. We shall show that $K \cap J = kJ$. Firstly, $KJ \subseteq K \cap J$ always, so we must show that $K \cap J \subseteq KJ$. Let $gr \in K$ with $g \in \mathcal{R}L$. Then $grf = 0$ and so $coz(fg) \wedge cozr = 0$. Therefore $coz(fg)$ and $cozr$ are disjoint cozero elements of L, also since L is an F-frame, they are completely separated. Hence there exists $h \in \mathcal{R}L$ such that $coz(fg) \wedge cozh = 0$ and $cozr \wedge coz(1-h) = 0$. First equality ensures that, $gh \in K$ and second ensures that, $gr = ghr \in KJ$. Therefore by Lemma 2.2.19 and taking care of the fact that $\mathcal{R}L$ is flat as it is free and hence projection and every projective module is flat (see [82]), we can conclude that I is flat.

Conversely, suppose that every finitely generated ideal of $\mathcal{R}L$ is flat. To show that L is an F-frame, it is sufficient to show that disjoint cozero elements of L are completely separated. Let $cozf \wedge cozr = 0$ with $f, r \in \mathcal{R}L$, from which it follows that $fr = 0$. Consider the principal ideals $I = \langle f \rangle$, $J = \langle r \rangle$ and the exact sequence

$$0 \to K \to \mathcal{R}L \xrightarrow{\varphi} I \to 0$$

of $\mathcal{R}L$-modules with $K = \{k \in \mathcal{R}L : kf = 0\}$ and $\varphi(g) = fg$, $g \in \mathcal{R}L$. Then since I and $\mathcal{R}L$ are both flat we have $K \cap J = KJ$, by Lemma 2.2.19. Since $r \in K \cap J$ (as $rf = 0$), it follows that $r = kr$ for some $k \in K$. So $cozf \wedge cozk = 0$ and hence $cozf$ and $cozr$ are completely separated by k. □

The following corollary turns out to be the point-free extension of the classical result, which says that X is an F-space if and only if each ideal of $C(X)$ is flat, and this was established by Neville (see [76, Corollary 1.6]).

Corollary 5.1.4. *A frame L is an F-frame if and only if every ideal of $\mathcal{R}L$ is flat.*

Proof. Follows from [82, Proposition 3.48] and Proposition 5.1.13. □

The $pos(f)$ and $neg(f)$, denotes the positive part and the negative part of the function f, respectively.

Lemma 5.1.6. *Let $f \in \mathcal{R}L$. Then the ideal $\langle f, \mid f \mid \rangle$ is principal if and only if $pos(f)$ and $neg(f)$ are completely separated.*

A well-known characterisation of F-spaces says that; X is an F-space if and only if each ideal of the lattice ordered ring $C(X)$ is convex if and only if the positive and negative parts of each function in $C(X)$ are completely separated (see [52, Theorem 14.25]). We have the following proposition as a point-free version of this result.

Proposition 5.1.14. *For a frame L, $pos(f)$ and $neg(f)$ are completely separated, for each $f \in \mathcal{R}L$ if and only if it is an F-frame.*

Proof. First suppose that L is an F-frame and $f \in \mathcal{R}L$. Then

$$pos(f) \wedge neg(f) = coz(f^{+}) \wedge coz(f^{-}) = coz(f^{+}f^{-}) = 0$$

implies that $pos(f)$ and $neg(f)$ are completely separated.

Conversely, suppose that $pos(f)$ and $neg(f)$ are completely separated for each $f \in \mathcal{R}L$. To show L is an F-frame, it is sufficient to show in view of Corollary 5.1.3 and Proposition 5.1.12 that, $\mathbf{O}^{I} = \{g \in \mathcal{R}(\lambda L) : g \in I\}$ is a prime ideal of $\mathcal{R}(\lambda L)$, for each prime element I of $\beta(\lambda L)$, λL is the Lindelöf corefletion of L. Since $\mathbf{O}^{I}$ is a z-ideal of $\mathcal{R}(\lambda L)$, it is sufficient to show that, for any $f \in \mathcal{R}(\lambda L)$ there exists $g \in \mathbf{O}^{I}$ such that $pos(f) \leq cozg$ or $neg(f) \leq cozg$ (see [5, Lemma 4.8]). So let $f \in \mathcal{R}(\lambda L)$. Then by hypothesis $pos(\lambda L \circ f)$ and $neg(\lambda L \circ f)$ are completely separated and hence there exist $k, l \in \mathcal{R}L$ such that $pos(\lambda L \circ f) \wedge cozk = 0 = neg(\lambda L \circ f) \wedge cozl$ and $cozk \vee cozl = 1$, here $\lambda_L : \lambda L \to L$ is the coreflection map. Now, since L is a C-quotient of λL (see [16, Corollary 8.2.13]), we must have $\bar{k}, \bar{l} \in \mathcal{R}(\lambda L)$ such that $k = \lambda_L \circ \bar{k}$ and $l = \lambda_L \circ \bar{l}$. So $0 = pos(\lambda_L \circ f) \wedge cozk = \lambda_L(pos(f)) \wedge \lambda_L(coz\bar{k}) = \lambda_L(pos(f) \wedge coz\bar{k})$ implies

$pos(f) \wedge coz\bar{k} = 0$, as λ_L is dense. Similarly, $neg(f) \wedge coz\bar{l} = 0$ and $1 = cozk \vee cozl = \lambda_L(coz\bar{k} \vee coz\bar{l})$ implies $coz\bar{k} \vee coz\bar{l} = 1$, as λ_L is coz-codense (see [16, Theorem 8.2.12]). So $pos(f)$ and $neg(f)$ are completely separated in λL and hence $(pos(f))^* \vee (negf)^* = 1$. Therefore $r_{\lambda L}((pos(f))^* \vee (neg(f))^*) = r_{\lambda L}((pos(f))^*) \vee r_{\lambda L}((neg(f))^*) = \lambda L$ (see ([15, Lemma 3.1]) and hence $r_{\lambda L}((pos(f))^*) \nsubseteq I$ or $r_{\lambda L}((neg(f))^*) \nsubseteq I$, as I is prime. If $r_{\lambda L}((pos(f))^*) \nsubseteq I$, then $r_{\lambda L}((pos(f))^*) \vee I = \lambda L$, as I is a maximal element of $\beta(\lambda L)$ and so there exists $x \in \lambda L$ with $x \prec\prec (pos(f))^*$ and $y \in I$ such that $x \vee y = 1$. But $y \in I$ implies $y \leq cozg \in I$ with $g \in \mathcal{R}(\lambda L)$. So $x \vee cozg = 1$ and $pos(f) \leq (pos(f))^{**} \leq x^* \leq cozg$ with $g \in \mathbf{O}^I$. □

A space X is an F-space if and only if each ideal of $C(X)$ is convex. We have the following.

Theorem 5.1.7. *A frame L is an F-frame if and only if each ideal of $\mathcal{R}L$ is convex.*

Proof. Since $\mathcal{R}L$ is a semiprime f-ring with bounded inversion property (see [19, Proposition 11]), we have from [70, Theorem 1] that, every ideal of $\mathcal{R}L$ is convex if and only if it is Bézout ring. Therefore L is an F-frame by Proposition 5.1.10. Thus we get the required result. □

5.2 F'-frames

In this section, we investigate F'-frames which is part of generalisations of P-frames. F'-frames originally, were introduced by Ball and Walters-Wayland [16]. We study characterisations of F'-frames proposed by Dube [35] and other authors. We start with a definition culled from [16].

Definition 5.2.1. A frame L is said to be an *F'-frame* if for any two cozero elements in L which do not meet, then the join of their pseudocomplements is the top element.

Proposition 5.2.1. [16] *Every basically disconnected frame is an F'-frame.*

Proof. Let L be a basically disconnected frame and $a, b \in CozL$ such that $a \wedge b = 0$. Thus $a^* \vee a^{**} = 1 = b^* \vee b^{**}$. Since $b^{**} \leq a^*$ and $a^{**} \leq b^*$, thus

$$1 = (a^* \vee a^{**}) \vee (b^* \vee b^{**}) \leq (a^* \vee b^*) \vee (b^* \vee a^*) = a^* \vee b^*.$$

Therefore $a^* \vee b^* = 1$, thus L is an F'-frame. □

The following corollary is an immediate consequence of Proposition 2.1.2 and Proposition 5.2.1.

Corollary 5.2.1. [16] *Every P-frame is an F'-frame.*

The class of F'-frames contains the class of P-frames and the class of F-frames. It is only sufficient here to show that the class of F-frames is contained in the class of F'-frames. This statement is also mentioned by Ball and Walters-Wayland [16].

Lemma 5.2.1. [16] *Every F-frame is an F'-frame.*

Proof. Let L be an F-frame. We want to show that L is an F'-frame. Let $a, b \in CozL$, such that $a \wedge b = 0$. Then there exist $a^*, b^* \in CozL \Rightarrow a^*, b^* \in L$ such that $a \wedge a^* = 0 = b \wedge b^*$, but L is an F-frame, thus $a^* \vee b^* = 1$. Therefore L is an F'-frame. □

Theorem 5.2.2. [40, Corollary 4.7] *Every normal F'-frame is an F-frame.*

Proof. Let $a, b \in CozL$ be such that $a \wedge b = 0$. L is an F'-frame, so $a^* \vee b^* = 1$. Furthermore, L is normal so there exist $c, d \in CozL$ such that $c \leq a^*$ and $d \leq b^*$ with $c \vee d = 1$. Now $c \wedge a \leq a^* \wedge a = 0$ and $d \wedge b \leq b^* \wedge b = 0$. Thus L is an F-frame. □

The following corollaries are the consequence of Proposition 4.1.5, Corollary 4.1.3, and Theorem 5.2.2.

Corollary 5.2.2. *Every Lindelöf F'-frame is an F-frame.*

Corollary 5.2.3. *Every weakly Lindelöf almost P-frame which is also an F'-frame is an F-frame.*

The following remark is obtained in [72].

Remark : Let L be a frame and $a \in L$, for any $x \in \downarrow a$, let $x^{\odot}$ denote the pseudocomplement of x in $\downarrow a$. The $x^{\odot} = x^* \wedge a$.

Proof. The element $x^* \wedge a \in \downarrow a$ and $x \wedge (x^* \wedge a) = 0$. Therefore $x^* \wedge a \leq x^{\odot}$. Now let $z \in \downarrow a$ be such that $z \wedge x = 0_{\downarrow a}$. Then $z \leq x^*$. But $z \leq a$, so $z \leq x^* \wedge a$. Since $x^{\odot}$ is an element of $\downarrow a$ with $x \wedge x^{\odot} = 0_{\downarrow a} = 0$, it follows that $x^{\odot} \leq x^* \wedge a$. Hence $x^{\odot} = x^* \wedge a$. □

It is worthwhile to recall from Kohls [64] that every open subspace of an F'-space is an F'-space. This is captured in frames (see [35, Proposition 4.1]), we include it here for the sake of completeness and easy reference for the reader.

Proposition 5.2.2. [35, Proposition 4.1] *If L is an F'-frame, then $\downarrow a$ is an F'-frame for each $a \in L$.*

Proof. Let $c, d \in Coz(\downarrow a)$ be such that $c \wedge d = 0$. Since for any $x \in \downarrow a$, the pseudocomplement of x in $\downarrow a$ is given by $x^{\odot} = a \wedge x^*$, we must show that $(a \wedge c^*) \vee (a \wedge d^*) = a$. Let u be a cozero element of L with $u \leq a$. The map $\downarrow a \rightarrow \downarrow u$ given by $x \mapsto u \wedge x$ is a frame homomorphism. Therefore, $u \wedge c, u \wedge d \in Coz(\downarrow a)$ such that $(u \wedge c) \wedge (u \wedge d) = u \wedge (c \wedge d) = 0$. Since $\downarrow u$ is an F'-frame, it follows that

$$(u \wedge (u \wedge c)^*) \vee (u \wedge (u \wedge d)^* = u. \qquad (\ddagger\ddagger)$$

Now $(u \wedge (u \wedge c)^*) \wedge c = (u \wedge c) \wedge (u \wedge c)^* = 0$, which implies that $(u \wedge (u \wedge c)^*) \leq c^*$. Similarly, $(u \wedge (u \wedge d)^*) \leq d^*$. So it follows from $(\ddagger\ddagger)$ that $u \leq c^* \vee d^*$. Since, by complete regularity, a is the join of cozero elements of L below it, it follows that $a \leq c^* \vee d^*$. Furthermore $a = a \wedge (c^* \vee d^*)$, as required. □

Recall from [35] that a frame L is said to be *locally F'-frame* if for each $a \in L$ there exists $A \subseteq L$ such that $a = \bigvee A$, and $\downarrow w$ is an F'-frame for each $w \in A$. The following result also comes from [35].

Proposition 5.2.3. [35, Corollary 4.2] *A necessary and sufficient condition that a completely regular frame be an F'-frame is that it be locally an F'-frame.*

Proof. Let $a, b \in CozL$ such that $a \wedge b = 0$. There is a subset A of L such that $1_L = \bigvee A$ and $\downarrow w$ is an F'-frame for each $w \in A$. For any $t \in A$, we have that $(t \wedge a), (t \wedge b) \in Coz(\downarrow t)$ such that $(a \wedge t) \wedge (b \wedge t) = 0$. Thus, $(t \wedge a^*) \vee (t \wedge b^*) = t$, implying that $t \wedge (a^* \vee b^*) = t$, hence $t \leq (a^* \vee b^*)$. Consequently, $1 \leq a^* \vee b^*$. Thus $a^* \vee b^* = 1$, and so L is an F'-frame. □

The next result shows that F'-frames are preserved by *coz*-onto homomorphisms. The result comes from [40, Lemma 4.4].

Proposition 5.2.4. [40, Lemma 4.4] *Let $h : L \to M$ be a coz-onto homomorphism. If L is F'-frame, then so is M.*

Proof. Let $a, b \in CozM$ such that $a \wedge b = 0_M$, there exist $x, y \in CozL$ such that $h(x) = a$ and $h(y) = b$ (since h is *coz-onto*). Now $a \wedge b = h(x) \wedge h(y) = h(x \wedge y) = h(0_L) = 0_M$ (see Proposition 1.3.2). Hence $x \wedge y = 0_L$. But L is an F'-frame, thus $x^* \vee y^* = 1_L$. Furthermore $1_M = h(1_L) = h(x^* \vee y^*) = h(x^*) \vee h(y^*) \leq h(x)^* \vee h(y)^*$. Therefore M is an F'-frame. □

The following theorem is given in the context of classical topology, stated as each weakly Lindelöf F'-space is an F-space (see [29, Theorem 2.2]). In the context of point-free topology, it goes as follows.

Theorem 5.2.3. *Every weakly Lindelöf F'-frame is an F-frame.*

Proof. Suppose $a, b \in CozL$ such that $a \wedge b = 0$. The frame L is an F'-frame, so $a^* \vee b^* = 1$. So

$$(\bigvee\{x \in L \mid x \wedge a = 0\}) \vee (\bigvee\{y \in L \mid y \wedge b = 0\}) = 1, \bigvee\{c = x \vee y \mid x \wedge a = 0, y \wedge b = 0\}$$

is a cover of L. The frame L is weakly Lindelöf, so there is a countable subset $T = \{c_i\}_{i \in I}$ which is dense. That is

$$\left(\bigvee T\right)^* = \left(\bigvee_{i \in I}\{c_i = x_i \vee y_i \mid x_i \wedge a_i = 0, y_i \wedge b_i = 0\}\right)^* = 0.$$

Now

$$\left(\bigvee T\right)^* = \bigwedge_{i\in I}\{{c_i}^* = {x_i}^* \wedge {y_i}^* \mid x_i \wedge a_i = 0, y_i \wedge b_i = 0\}$$
$$= \left(\bigwedge_{i\in I}\{{x_i}^* \in L \mid x_i \wedge a_i = 0\}\right) \wedge \left(\bigwedge_{i\in I}\{{y_i}^* \in L \mid y_i \wedge b_i = 0\}\right)$$
$$= a^{**} \wedge b^{**}$$
$$= 0.$$

Thus L is an F-frame.

□

Proposition 5.2.5. [72] *Every frame with ccc is weakly Lindelöf.*

The following corollary is the extension of F'-space with *ccc* is an F-space to point-free setting (see [29]), and is a consequence of Theorem 5.2.3 and Proposition 5.2.5.

Corollary 5.2.4. *Every F'-frame with ccc is an F-frame.*

Recall from [16], that a frame L is said to be *zero-dimensional* frame if every element is a join of complemented elements. That is; if every element is a join of complemented elements below it. In [29, Theorem 2.1], it is shown that a zero-dimensional weakly Lindelöf F'-space is a strongly zero-dimensional F-space. We give the result in the point-free version, we use the characterisation of strongly zero-dimensional and recall that a frame L is strongly zero-dimensional if and only if $a \prec\prec b$ in L implies the existence of complemented element c in L such that $a \leq c \leq b$.

Theorem 5.2.4. *Every zero-dimensional weakly Lindelöf F'-frame L is a strongly zero-dimensional F-frame.*

Proof. Let $a, b \in CozL$ such that $a \prec\prec b$ in a zero-dimensional weakly Lindelöf F'-frame L. Thus every element is a join of complemented elements. Let $c \in CozL$ such that $a \prec\prec c \prec\prec b$. If c is complemented, then we are done. Suppose c is not complemented. Let $d \in CozL$ such that $a \wedge d = 0$ and $d \vee b = 1$. Since L is weakly Lindelöf F'-frame, thus $a^* \vee d^* = 1$, $a^{**} \wedge d^{**} = 0$ and c is not complemented. Thus d and d^* are complemented. Now $a \wedge d = 0$ implies that $a \leq d^*$ and $d \vee c = 1$; implies that $d^* \leq c$. Consequently $a \leq d^* \leq b$, and so L is strongly zero-dimensional F-frame. □

5.2.1 Ring-theoretic characterisations of F'-frames

In this subsection, we give characterisations of F'-frames in terms of ideals of $\mathcal{R}L$. The following useful proposition taken from [35] is needed in the sequel. We omit the proof.

Proposition 5.2.6. [35, Lemma 4.3] *A z-ideal Q is prime if and only if whenever $\alpha\beta = 0$, then $\alpha \in Q$ or $\beta \in Q$. Hence, a z-ideal is prime if and only if it contains a prime ideal.*

Proposition 5.2.7. [35, Proposition 4.5] *Let L be a completely regular frame. Consider the following statements.*

(1) *L is an F'-frame.*

(2) *$\mathbf{O}^I$ is a prime ideal for each point I of βL with $\bigvee I \neq 1$.*

(3) *The prime ideals of $\mathcal{R}L$ contained in any fixed maximal ideal form a chain.*

(4) *For any point I of βL with $\bigvee I \neq 1$, the prime ideals of $\mathcal{R}L$ that contains $\mathbf{O}^I$ form a chain.*

Then, (1) implies (2); (2), (3) and (4) are equivalent, and (4) implies (1) if L has enough points.

Proof. (1) $\Rightarrow$ (2): For any $J \in \beta L$ and $\varphi \in \mathcal{R}L$, if $\varphi \notin \mathbf{O}^J$, then $r((coz\varphi)^*) \vee J \neq 1_{\beta L}$, and hence, $r((coz\varphi)^*) \leq J$. Now let $\alpha, \beta \in \mathcal{R}L$ with $\alpha\beta = 0$. Suppose, by way of contradiction, that $\alpha \notin \mathbf{O}^I$ and $\beta \notin \mathbf{O}^I$. Write $a = coz\alpha$ and $b = coz\beta$. Then $a \wedge b = 0$, and so $a^* \vee b^* = 1$ (by (1)). Our assumption implies $r(a^*) \leq I$ and $r(b^*) \leq I$, therefore $r(a^*) \vee r(b^*) \leq I$. Taking joins yields $1 = a^* \vee b^* \leq I$, which is a contradiction. Since $\mathbf{O}^I$ is a z-ideal, quite clearly, it follows from Proposition 5.2.6 that it is a prime ideal.

(2) $\Rightarrow$ (3): The prime ideals that are contained in any fixed maximal ideal contain the ideal $\mathbf{O}^I$ for some point I of βL with $\bigvee I \neq 1$. Furthermore, if (2) holds, they form a chain.

(3) $\Rightarrow$ (4): The prime ideals that contain $\mathbf{O}^I$, where I is a point of βL with $\bigvee I \neq 1$, are contained in a fixed maximal ideal. Furthermore

(4) $\Rightarrow$ (2): Since $\mathbf{O}^I$ is clearly a radical ideal, it is the intersection of prime ideals containing it. Since the intersection of a chain of prime ideals is a prime ideal, (4) implies (2).

(4) $\Rightarrow$ (1): This is just one implication in the Mandelker result cited in [35]. $\square$

Chapter 6

P_F-frames

In this chapter, we introduce P_F-frames as another generalisation of P-frames. The reader must not be surprised as this is considered in the last chapter and not in the earlier chapters following the hierarchy of the generalisation of P-frames and we decided to consider P_F-frames at the end. The P_F-spaces were introduced in 2021 by Azarpanah *et al* [14]. We observe in the first section that the class of P-frames is contained in the class of P_F-frames in turn is contained in the class of F-frames, and P_F-frames and basically disconnected frames are incomparable. In the second section, we show that a frame L is a P_F-frame if and only if βL is a P_F-frame. Lastly, we introduce some of the ring-theoretic characterisations of P_F-frames.

6.1 Definition and examples

In this section, we introduce P_F-frames and give some of frame-theoretic and ring-theoretic characterisations. We observe that P_F-frames and basically disconnected frames are incomparable. Azarpanah *et al* [14] calls X a P_F-space if of any two zero-sets in X whose union is all of X at least one of them is open. Equivalently, of any two disjoint cozero-sets in X at least one of them is closed. This is extended to frames as follows.

Definition 6.1.1. A frame L is said to be a P_F-*frame* if whenever $a, b \in CozL$ such that

$a \wedge b = 0$, then at least one of them is complemented.

That is to say for $a^*, b^* \in L$, either $a \vee a^* = 1$ or $b \vee b^* = 1$ (or both). We can see that at least one of them is complemented (or both), thus either $a^* \in CozL$ or $b^* \in CozL$ (or both). Next, we show that the open cozero quotient of a P_F-frame is a P_F-frame.

Proposition 6.1.1. *If L is a P_F-frame, then $\downarrow a$ is a P_F-frame for each $a \in CozL$.*

Proof. Let $c, d \in Coz(\downarrow a)$ such that $c \wedge d = 0_{\downarrow a}$. Then $c, d \in CozL$ such that $c \wedge d = 0$. The frame L is a P_F-frame, so at least one of them is complemented, say c, that is, $c^* \wedge c = 0$ and $c^* \vee c = 1$. Since $c^{\odot} = (c^* \wedge a) \in \downarrow a$, we want to show that $c^{\odot} \in Coz(\downarrow a)$. Now,

$$(c \vee c^{\odot}) = c \vee (c^* \wedge a) = (c \vee c^*) \wedge (c \vee a) = 1 \wedge a = a = 1_{\downarrow a},$$

and

$$c \wedge c^{\odot} = c \wedge (c^* \wedge a) = (c \wedge c^*) \wedge a = 0 \wedge a = 0_{\downarrow a}.$$

Thus $c^{\odot} \in Coz(\downarrow a)$. Therefore c is complemented in $Coz(\downarrow a)$. □

The authors in [14] gave an example of a P_F-space which is not basically disconnected and an example of a basically disconnected space which is not a P_F-space. In the context of classical topology, P_F-space and basically disconnected spaces are incomparable. This will also hold in the larger terrain of point-free topology. Since in a P-frame every cozero element is complemented, the following is immediate.

Lemma 6.1.1. *Every P-frame is a P_F-frame.*

We observe that the converse is not true, that every P_F-frame is a P-frame.
Suppose L is a P_F-frame. We want to show that L is not a P-frame. Let $a, b \in CozL$. Since L is P_F-frame. Thus $a \wedge b = 0$ and either a is complemented or b is complemented or both are

complemented. Consider if only a is complemented, hence b is not complemented. Similarly, if only b is complemented, then a is not complemented. Thus L is not a P-frame.

Recall that a frame L is basically disconnected if for any $a \in CozL$, $a^* \vee a^{**} = 1$. P_F-frames and basically disconnected frames are incomparable, however, we have the following corollary as a consequence of Corollary 4.1.2 and Lemma 6.1.1.

Corollary 6.1.1. *Every basically disconnected (weakly cozero complemented) almost P-frame is a P_F-frame.*

The following corollary follows immediately from Corollary 4.1.1 and Lemma 6.1.1. Recall that a frame L is said to be an O_z-frame if every regular element is a cozero element. Because an O_z-frame is weakly cozero complemented, the following corollary is apparent.

Corollary 6.1.2. *Every frame that is both an O_z-frame and an almost P-frame is a P_F-frame.*

The following Corollary follows immediately as a consequence of Corollary 4.1.2 and Lemma 6.1.1. Recall that a frame L is said to satisfy *ccc* if every collection of pairwise disjoint elements of L is countable. Because a frame with *ccc* is weakly cozero complemented the following corollary is apparent.

Corollary 6.1.3. *Every frame that is an almost P-frame with ccc is a P_F-frame.*

We show below that the class of P_F-frames is contained in the class of F-frames. This containment is strict.

Proposition 6.1.2. *Every P_F-frame is an F-frame.*

Proof. Let $a, b \in CozL$ such that $a \wedge b = 0$. L is a P_F-frame, so at least one is complemented, say a. Then $a \vee a^* = 1$. Then a^* as a complemented element is a cozero element. By normality of $CozL$, there exist $c, d \in CozL$ such that $c \wedge d = 0$ and $c \vee a = 1 = d \vee a^*$. Now $c \vee a = 1 \Rightarrow a^* \prec c$. Also $c \vee d \geq a^* \vee d = 1$, and also $c \wedge d = 0$ and $c \vee a = 1 \Rightarrow d \prec a$. Then $d \wedge b \leq a \wedge b = 0$. Also $c \wedge d = 0$ and $d \vee a^* = 1 \Rightarrow c \prec a^*$, then $c \wedge a \leq a^* \wedge a = 0$. Therefore $c \wedge a = 0 = d \wedge b$. Thus L is an F-frame. ∎

The following corollary follows immediately as a consequence of Lemma 5.2.1 and Proposition 6.1.2.

Corollary 6.1.4. *Every P_F-frame is an F'-frame.*

Recall from [16], that a frame L is said to be a *quasi F-frame* if for every dense $a \in CozL$, then the open quotient map $h : L \to \downarrow a$ is a C^*-quotient map. We have the following characterisation of quasi F-frame which states that a frame L is said to be a *quasi F-frame* if and only if whenever $a, b \in CozL$ such that $a \wedge b = 0$ and $a \vee b$ is dense, then there exist $c, d \in CozL$ such that $c \vee d = 1$ and $c \vee a = 1 = d \vee b$ (see [16, Proposition 8.4.10]). The following theorem is well known.

Theorem 6.1.2. [16] *Every F-frame is a quasi F-frame.*

Proof. Let $a, b \in CozL$ such that $a \wedge b = 0$ and $a \vee b$ is dense. Since L is an F-frame, there exist $c, d \in CozL$ such that $c \vee d = 1$ and $c \wedge a = 0 = d \wedge b$. Thus L is a quasi F-frame. □

The following corollary is immediate from Proposition 6.1.2 and Theorem 6.1.2.

Corollary 6.1.5. *Every P_F-frame is a quasi F-frame.*

6.2 Transportations of P_F-frames

In this section, we show that P_F-frames are preserved under the transportation of *coz*-onto dense frame homomorphisms. On the other hand, P_F-frames are also reflected by nearly open frame homomorphisms which are *coz*-codense.

Proposition 6.2.1. *Let $h : L \to M$ be a coz-onto, dense frame homomorphism. If L is a P_F-frame, then so is M.*

Proof. Let $a, b \in CozM$ such that $a \wedge b = 0$. Since h is *coz-onto*, there exist $x, y \in CozL$ such that $h(x) = a$ and $h(y) = b$. Now $h(x \wedge y) = h(x) \wedge h(y) = a \wedge b = 0_M$. By density of h, $x \wedge y = 0_L$ (also by Proposition 1.3.2). But L is a P_F-frame, so at least one of them is complemented, say x. It is well known that frame homomorphisms preserve complemented elements. Thus M is a P_F-frame. □

Proposition 6.2.2. *Let $h : L \to M$ be a nearly open and coz-codense frame homomorphism. If M is a P_F-frame, then so is L.*

Proof. Let $a, b \in CozL$ such that $a \wedge b = 0$. Then since frame homomorphisms preserves cozero elements, it follows that $h(a)$ and $h(b)$ are cozero elements in M. Furthermore,

$$h(a) \wedge h(b) = h(a \wedge b) = h(0_L) = 0_M.$$

M is a P_F-frame, so at least one of $h(a)$ or $h(b)$ is complemented, say $h(a)$. That is,

$$h(a) \wedge h(a)^* = 0_M \quad \text{and } h(a) \vee h(a)^* = 1_M.$$

Now $h(a \wedge a^*) = h(a) \wedge h(a^*) \leq h(a) \wedge h(a)^* = 0_M$. By nearly openness of h, we have $h(a \vee a^*) = h(a) \vee h(a^*) = h(a) \vee h(a)^* = 1_M$. By *coz*-codense of h, $a \vee a^* = 1_L$. Hence a is a complemented element in L. Thus L is a P_F-frame. □

Next, we show that a frame is a P_F-frame precisely when its Stone-Čech compactification is a P_F-frame. We regard the Stone-Čech compactification βL of a frame L as a set of regular ideals. The map $h : \beta L \to L$ is given by join and its right adjoint is denoted by r.

Theorem 6.2.1. *A frame L is a P_F-frame if and only if βL is a P_F-frame.*

Proof. Assume that L is a P_F-frame. Take $I, J \in Coz\beta L$ such that $I \wedge J = 0_{\beta L}$. Put

$$a \equiv \bigvee I \in CozL \text{ and } b \equiv \bigvee J \in CozL.$$

Now the map $h : \beta L \to L$ is a frame homomorphism, so

$$a \wedge b = \bigvee I \wedge \bigvee J = \bigvee (I \wedge J) = h(I \wedge J) = 0_L.$$

Now L is a P_F-frame, so one of a or b is complemented, say a. That is, $a \vee a^* = 1_L$. That is

$$1_{\beta L} = r(a \vee a^*) = r(a) \vee r(a^*) \leq r(a) \vee r(a)^* = I \vee I^*.$$

Thus I is complemented in βL and hence βL is a P_F-frame.

Conversely, suppose that βL is a P_F-frame. Take $a, b \in CozL$ such that $a \wedge b = 0_L$. Put

$$a \equiv \bigvee I \in CozL \text{ and } b \equiv \bigvee J \in CozL.$$

Then by density of the join map it follows that

$$I \wedge J = 0_{\beta L}.$$

Now βL is a P_F-frame, so there exist $I^* \in Coz\beta L$ such that $I \vee I^* = 1_{\beta L}$. Thus

$$h(I \vee I^*) = \bigvee I \vee \bigvee I^* = (\bigvee I) \vee (\bigvee I)^* = a \vee a^* = 1_L.$$

Hence a is complemented and thus L is a P_F-frame. □

6.3 Ring-theoretic characterisations of P_F-frames.

In this section, we characterise P_F-frames in terms of the ring $\mathcal{R}L$. The following proposition is motivated by Lemma 3.1.1.

Proposition 6.3.1. *If L is a P_F-frame, then $\mathcal{R}L$ is a VN-local ring.*

Proof. Let $a, b \in CozL$ and $a \wedge b = 0$, thus $a \wedge b$ is complemented. There exists $c \in CozL$ such that $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c) = 1$, then $a \vee c = 1$ and $b \vee c = 1$. The frame L is a P_F-frame, say a is complemented. Therefore, $\mathcal{R}L$ is a VN-local ring. □

The following lemma is well-known in lattice theory.

Lemma 6.3.1. *If a and b are complemented, then the meet of a and b is complemented.*

Proof. If a and b are complemented, then $a \vee a^* = 1 = b \vee b^*$. Now since $a^* \vee b^* \leq (a \wedge b)^*$, we have

$$\begin{aligned}(a \wedge b) \vee (a \wedge b)^* &\geq (a \wedge b) \vee (a^* \vee b^*)\\ &= [(a \wedge b) \vee a^*)] \vee [(a \wedge b) \vee b^*]\\ &= (b \vee a^*) \vee (a \vee b^*)\\ &= (a^* \vee a) \vee (b \vee b^*)\\ &= 1 \vee 1 = 1.\end{aligned}$$

Thus $a \wedge b$ is complemented. □

We have the following characterisation; an ideal I in $\mathcal{R}L$ is said to be a *P-ideal* if and only if for each $f \in I$, then $cozf$ is complemented (see [83, Theorem 1.5]). The following theorem follows from the fact that $C(X)$ is a von Neumann regular ring if and only if all of its pure ideals are P-ideals (see [4, Theorem 2.2 and Theorem 2.3] and [7]).

Theorem 6.3.2. *An ideal I is a P-ideal in $\mathcal{R}L$ if and only if for every $f \in I$, $cozf$ is complemented in L.*

Proof. We want to prove that an ideal I of a distributive lattice L is a P-ideal if and only if for every $f \in I$, $cozf$ is complemented. If I is a P-ideal and $f \in I$, then there exists $g \in I$ such that $fg = 0$ and $g = 1$ on supp f. That is, $f \vee g = 1$. Hence $0 = coz(fg) = cozf \wedge cozg$ and $1 = coz(f \vee g) = cozf \vee cozg$. This shows that $cozf$ is complemented.

Conversely, if for every $f \in I$, $coz(f)$ is complemented in L, and $f, g \in L$ such that $f \wedge g \in I$, we need to show that f or g belongs to I. Let $h = (cozf \vee cozg)^*$. Since $cozf$ and $cozg$ are complements in L, we have:

$$cozf \wedge h = 0 \text{ and } cozg \wedge h = 0.$$

Thus, $cozf \leq h^*$ and $cozg \leq h^*$. It follows that $h^* \in I$ since I is an ideal. We have:

$$h \wedge (f \wedge g) = (cozf \vee cozg)^* \wedge (f \wedge g) = ((cozf)^* \wedge (f \wedge g)) \wedge ((cozg)^* \wedge (f \wedge g)) = 0.$$

This shows that h^* is an upper bound for $cozf$ and $cozg$. Since $cozf$ and $cozg$ are complements in L, it follows that h is a lower bound for f and g. Thus, we have:

$$f \vee g \leq h^* \in I$$

Therefore, I is a P-ideal. □

From Proposition 3.1.8, we can say that we are ready to give some algebraic characterisations of P_F-frames, and the following proposition is an extension of [14, Theorem 2.4].

Proposition 6.3.2. *For a frame L, the following statements are equivalent.*

(1) *A frame L is a P_F-frame.*

(2) *If $a, b \in CozL$ such that $a \wedge b$ is complemented, then at least one of them is complemented.*

(3) *Of any two ideals of $\mathcal{R}L$ whose product is a P-ideal, at least one is a P-ideal.*

(4) *Of any two principal ideals of $\mathcal{R}L$ whose product (intersection) is zero, at least one is semiprime.*

(5) *Of any two principal ideals of $\mathcal{R}L$ whose product (intersection) is semiprime, at least one is semiprime.*

(6) *L is an essential P-frame which is also an F-frame.*

Proof. (1) $\Rightarrow$ (2): Let $a, b \in CozL$ and $a \wedge b$ be complemented. If $a \wedge b = 0$, then we are done. Now assume without loss of generality that $a \wedge b \neq 0$. There exists $c \in CozL$ such that

$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c) = 1$ implies $a \vee c = 1$ and $b \vee c = 1$. Now, if $(a \wedge b) \wedge c = 0$ then $c \leq (a \wedge b)^*$. Put $c = a^* \vee b^*$. Now

$$\begin{aligned} b^* \vee b &= b^* \vee [(a^* \wedge b) \vee (a \wedge b)] \\ &= [b^* \vee (a^* \wedge b)] \vee (a \wedge b) \\ &= [(b^* \vee a^*) \wedge (b^* \vee b)] \vee (a \wedge b) \\ &= (b^* \vee a^*) \vee (a \wedge b) \\ &= c \vee (a \wedge b) \\ &= (c \vee a) \wedge (c \vee b) \\ &= 1. \end{aligned}$$

The fourth step holds because $a \wedge b \neq 0$. Hence b is complemented as required.

(2) $\Rightarrow$ (3): Let I and J be two ideals of $\mathcal{R}L$ whose product (intersection) is a P-ideal. Suppose J is not a P-ideal, then there is $j \in J$ such that $coz(j)$ is not complemented. Since $ij \in IJ$ for each $i \in I$, $coz(ij) = cozi \wedge cozj$ is complemented (IJ is a P-ideal). Now using (2), $cozi$ must be complemented for each $i \in I$, so I is a P-ideal and we are done.

For intersection of two two ideals I and J in $\mathcal{R}L$, where $I \bigcap J$ is a P-ideal. Then we have $I \bigcap J = IJ$.

(3) $\Rightarrow$ (4): If the product or intersection of two principal ideals $\langle f \rangle$ and $\langle g \rangle$ is zero, then $\langle f \rangle \langle g \rangle = \langle f \rangle \bigcap \langle g \rangle = \langle fg \rangle = \langle 0 \rangle$ is a P-ideal. Now using (3), one of the principal ideals $\langle f \rangle$ and $\langle g \rangle$ is a P-ideal whence it must be semiprime.

(4) $\Rightarrow$ (5): Let the product (intersection) of principal ideals $\langle f \rangle$ and $\langle g \rangle$ be semiprime. Then $\langle f \rangle \langle g \rangle = \langle f \rangle \bigcap \langle g \rangle = \langle fg \rangle$ and $coz(fg) = cozf \wedge cozg$ is complemented. Take complemented element $cozt = coz(fg)^*$ for some $t \in \mathcal{R}L$. Now $\langle tf \rangle \langle tg \rangle = \langle 0 \rangle$ implies that either $\langle tf \rangle$ or $\langle tg \rangle$ is semiprime (by (4)), say $\langle tf \rangle$ is semiprime. This implies that $coz(tf) = cozt \wedge cozf$ is complemented. On the other hand $cozt \vee cozf = 1$ implies that $cozf$ is complemented and

hence the principal ideal $\langle f \rangle$ will be semiprime.

(5) $\Rightarrow$ (1): Let $cozf \wedge cozg = 0$. Then $\langle f \rangle \bigcap \langle g \rangle = \langle 0 \rangle$ is semiprime. Hence by (5), at least one of the ideals $\langle f \rangle$ and $\langle g \rangle$ is semiprime. Hence either $cozf$ or $cozg$ is complemented. Therefore L is a P_F-frame.

(1) $\Rightarrow$ (6): We only need to show that L is an essential P-frame. Suppose on contrary that $a, b \in CozL$ which are not complemented with $a \neq b$. If $a \wedge b = 0$, then we are done. We assume without loss of generality that $a \wedge b \neq 0$. If $a \wedge b$ is complemented, then at least one of them is complemented, a contradiction. If $a \wedge b \in CozL$ is not complemented, then $(a \wedge b) \vee (a \wedge b)^* \neq 1$. The frame L is completely regular and hence $CozL$ generates L. There exist a cozero element $u \leq (a \wedge b)^*$. We claim that $u = b^* \vee a^*$ is a non complemented cozero element, then $u^* = (b^* \vee a^*)^* = b^{**} \wedge a^{**}$. Now

$$\begin{aligned} u \vee u^* &= (b^* \vee a^*) \vee (b^{**} \wedge a^{**}) \\ &= (b^* \vee a^* \vee b^{**}) \wedge (b^* \vee a^* \vee a^{**}) \\ &= (b^* \vee b^{**}) \wedge (a^* \vee a^{**}) \\ &\neq 1. \end{aligned}$$

The third step holding because $a^* \leq b^* \vee b^{**}$ and $b^* \leq a^* \vee a^{**}$. Thus u and $a \wedge b$ are disjoint cozero elements of L which are not complemented. Hence a contradiction since L is a P_F-frame. Hence L has at most one cozero element which is not complemented and we are done.

(6) $\Rightarrow$ (1): Suppose $a, b \in CozL$ such that $a \wedge b = 0$. Since L is an F-frame, there exist $c, d \in CozL$ such that $c \vee d = 1$ and $c \wedge a = 0 = d \wedge b$. Now $c \wedge a = 0$ implies $c \leq a^*$ and $d \wedge b = 0$ implies $d = b^*$. Now $a \prec d$ and $b \prec c$ implies $a^* \vee d = 1 = b^* \vee c$. Then $a^* \vee b^* = 1$. This shows that $a \wedge b$ is complemented. The frame L is an essential P-frame, so at least one of them is complemented.

□

Although the following corollaries are weaker than Proposition 3.1.4 and Corollary 3.1.4, they are worth to note.

Corollary 6.3.1. *A P_F-frame is strongly zero-dimensional.*

Corollary 6.3.2. *A P_F-space is strongly zero-dimensional.*

Theorem 6.3.3. *A normal frame L is a P_F-frame if and only if $\mathcal{R}L$ is a VN-local ring.*

The following proposition is an extension of [14, Proposition 2.5].

Proposition 6.3.3. *Every compact P_F-frame is finite. More generally, every pseudocompact P_F-frame is finite.*

Proof. Let L be a compact P_F-frame and suppose on contrary, that L is infinite. Then L is a one-point compactification of an indiscrete frame which implies that L is an F-frame. On the other hand L contains a copy of $\beta\mathbb{N}$, because by hypothesis is an infinite F-frame which is impossible. So L is finite and we are done. Whenever L is pseudocompact thus $\upsilon L = \beta L$ (Hewitt realcompactification is equals to Stone-Čech compactification), thus $\mathcal{R}L \cong \mathcal{R}(\upsilon L) = \mathcal{R}(\beta L)$ and implies that βL is a P_F-frame which must be finite as a consequence of the first part of the proof. □

The following proposition is an extension of [14, Proposition 2.7 (1)].

Proposition 6.3.4. *Every weakly cozero complemented P_F-frame is basically disconnected.*

Proof. We recall that a frame L is weakly cozero complemented if for each $a \in CozL$ there is $b \in CozL$ such that $a \wedge b = 0$ and $a \vee b$ is dense. Suppose that L is a weakly cozero complemented P_F-frame, we want to show that it is basically disconnected. Let $a \in CozL$. The frame L is weakly cozero complemented, so there is $b \in CozL$ such that $a \wedge b = 0$ and $a \vee b$ is dense. Again L is a P_F-frame, at least one of them is complemented, say a. Since, again, a is complemented hence, immediately, a^* is the complement of a and $a^{**} = a$. □

Recall from [47, 85], that a space X is an *almost P-space* if every dense G_δ-set of X has a non-empty interior. For the construction of the definition below, we first recall the definition 4.1.1. For the equivalence part, we recall from [72], that a point I of βL is an *almost P-point* if for any $\alpha \in \mathbf{M}^I$, $coz\alpha$ is not dense. This motivates us to note down the following definition in terms of ideals.

Definition 6.3.1. A frame L is *essential almost P-frame* if there is at most one cozero element which is not regular. Equivalently, if there is at most one $I \in \beta L$ such that $\alpha \in \mathbf{M}^I$, $coz\alpha$ is not dense.

Proposition 6.3.5. [14] *Every essential P-frame is an essential almost P-frame.*

Proof. It is immediate from definitions. □

The following proposition is an extension of [14, Proposition 2.7 (2)].

Proposition 6.3.6. *Every basically disconnected essential almost P-frame is a P_F-frame.*

Proof. Let $a, b \in CozL$ be such $a \wedge b = 0$. The frame L is basically disconnected, so

$$a^* \vee a^{**} = 1 = b^* \vee b^{**}.$$

Furthermore, L is almost P-frame, so at least of $a, b \in CozL$ is regular. Thus at least one of a and or b is complemented. Hence L is a P_F-frame. □

We close the section, with the following corollaries. Since every basically disconnected frame is weakly cozero complemented, by Lemma 2.1.1, Proposition 6.3.4 and Proposition 6.3.6, it follows that the following corollary (which is an extension of [14, Corollary 2.8]) is immediate.

Corollary 6.3.3. *A frame is a weakly cozero complemented P_F-frame if and only if it is a basically disconnected essential almost P-frame.*

By using Theorem 5.1.7, the following corollary (which is an extension of [14, Corollary 2.9]) is now an immediate consequence of Proposition 3.1.8.

Corollary 6.3.4. *A frame L is a P_F-frame if and only if of any two comaximal principal ideals of $\mathcal{R}L$, one is semiprime and the other is convex.*

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