

ZERO DIVISORS IN BANACH ALGEBRAS

by

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Introduction

Numerous characterizations of semi-Fredholm operators exist in the classical theory of operators on a Hilbert space. Zero divisors in the "Calkin" algebra were used to define the classes of Fredholm-type operators. We refer the reader to [5] where the characterizations for classical semi-Fredholm operators on a Hilbert space in terms of the set of divisors of zero in the Calkin algebra were obtained by Fillmore, Stampfli and Williams. Such characterizations for operators on Banach spaces also exist in [12] that under certain conditions on a Banach space the left and right Fredholm operators were characterized respectively in terms of the left and right algebraic divisors of zero in the Calkin algebra.

It was shown by Lebow and Schechter [11] with additional conditions on the Banach space that the left and the right topological divisors of zero coincide with the left and right algebraic divisors of zero respectively. Stroh and Swart in [17] proved similar results hold if a Fredholm theory is considered in a semi-finite von Neumann algebra with non-large center relative to the ideal of compact operators (the

norm closure of the ideal generated by the finite projections), then the left and right Fredholm operators can be characterised in terms of the left and right algebraic divisors of zero in the Calkin algebra and left (right) algebraic zero divisors coincide with the left (right) topological zero divisors in the Calkin algebra.

Some of these characterisations are used in the study of the structure of the essential spectrum. [Several of these results hold when a Fredholm theory is studied relative to any von Neumann algebra.]

Operational quantities such as minimum modulus and essential minimum modulus have attracted interest over the last three decades [6], [9], [14] and [20].

In particular the minimum modulus and its connections with Fredholm perturbation theory and best approximation have been extensively studied [2,3,6,10,11]. In [3] in the case of Banach algebra, it was shown that the essential lower bound $m_e(\mathcal{T})$ of an operator \mathcal{T} measures the distance of \mathcal{T} to the complement of the set of left Fredholm operators. In particular, it is shown that if

$$m_e(\mathcal{T}) = m_e(\mathcal{T}^*) = 0,$$

then \mathcal{T} is in the boundary of the group of the invertible operators. Some of these ideas were generalised to a von Neumann algebra setting in [7, 8, 9]. Characterisation of topological zero divisor in the quotient algebra in a von Neumann setting is studied in terms of the essential lower bounds in [6].

The purpose of this dissertation is to provide a self contained summary of these results and to illustrate the importance of the algebraic and topological zero divisors in Fredholm and spectral theory.

Chapter 1

Preliminary

1.1 Banach Algebras

Definition 1.1 (Banach Algebra) \mathcal{B} is called a Banach algebra

(with unit) if:

(i) \mathcal{B} is a Banach space;

(ii) there is a multiplication $\mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ that has the following

properties:

$$(xy)z = x(yz), (x + y)z = xz + yz,$$

$$x(y + z) = xy + xz, c(xy) = (cx)y = x(cy)$$

for all $x, y, z \in \mathcal{B}$, $c \in \mathbb{C}$.

Moreover, there is a unit element e :

$$ex = xe = e \text{ for all } x \in \mathcal{B};$$

$$(iii) \quad \|e\| = 1;$$

$$(iv) \quad \|xy\| \leq \|x\|\|y\| \quad \text{for all } x, y \in \mathcal{B}.$$

In this dissertation we will consider only Banach algebras with unit.

In fact all von Neumann algebras (which is of particular importance) do have units.

An element $x \in \mathcal{B}$ is invertible (regular) in \mathcal{B} if there exists an element $x^{-1} \in \mathcal{B}$ such that $x^{-1}x = e = xx^{-1}$.

Every invertible element in a Banach Algebra has a unique inverse.

We call an element of \mathcal{B} **singular** if it is not invertible in \mathcal{B} .

If $x \in \mathcal{B}$ satisfies $x \neq 0$, $x \neq e$ and $x^2 = x$, then x is called a **non-trivial idempotent**.

Definition 1.2 (Involution) *Let \mathcal{B} be a Banach algebra.*

A mapping $x \rightarrow x^$ from \mathcal{B} into itself is called an **involution** on \mathcal{B} if it satisfies the following conditions for all $x, y \in \mathcal{B}$ and $\alpha \in \mathbb{C}$:*

$$(i) \quad (x + y)^* = x^* + y^*,$$

$$(ii) \quad (\alpha x)^* = \bar{\alpha}x^*,$$

(iii) $(xy)^* = y^*x^*$ and

(iv) $(x^*)^* = x$.

Definition 1.3 (Banach *-algebra) A Banach algebra with an involution is sometimes called a **Banach *-algebra**.

Definition 1.4 (Hermitian or self-adjoint) If \mathcal{B} is a Banach *-algebra then any $x \in \mathcal{B}$, such that $x = x^*$ is called **hermitian** or **self-adjoint**.

Definition 1.5 (Normal) If \mathcal{B} is a Banach *-algebra then any $x \in \mathcal{B}$, such that $xx^* = x^*x$ is called **normal**.

Definition 1.6 (Unitary) If \mathcal{B} is a Banach *-algebra then for any $x \in \mathcal{B}$, such that $xx^* = x^*x = e$ is called **unitary**.

The set of all invertible elements in a Banach algebra \mathcal{B} is denoted by $\mathcal{B}^{-1} = \mathcal{G}$, which forms a group under multiplication.

The complement of \mathcal{G} denoted by \mathcal{S} , is called the **set of singular elements**.

Theorem 1.7 ([16],p306) Every element x in a Banach algebra \mathcal{B} for which $\|x - 1\| < 1$ is regular, and the inverse of such an

element is given by the formula

$$x^{-1} = 1 + \sum_{n=1}^{\infty} (1-x)^n$$

Proof.

If $r = \|x - 1\|$ such that $r < 1$, then

$$\|(1-x)^n\| \leq \|1-x\|^n = r^n$$

shows that the partial sums of the series

$$\sum_{n=1}^{\infty} (1-x)^n$$

form a Cauchy sequence in \mathcal{B} .

Since \mathcal{B} is complete, these partial sums converge to an element \mathcal{B}

which is denoted by

$$\sum_{n=1}^{\infty} (1-x)^n.$$

If we define y by

$$y = 1 + \sum_{n=1}^{\infty} (1-x)^n,$$

then the joint continuity of multiplication in \mathcal{B} implies that

$$y - xy = (1-x)y = (1-x) + \sum_{n=2}^{\infty} (1-x)^n = \sum_{n=1}^{\infty} (1-x)^n = y - 1.$$

So, $xy = 1$.

Similarly, we can prove that $yx = 1$. ■

Theorem 1.8 ([16],p306) \mathcal{G} is an open set, and therefore \mathcal{S} is a closed set.

Proof.

Let x_0 be an element of \mathcal{G} , and x be another element in \mathcal{B} such that

$$\|x - x_0\| < \frac{1}{\|x_0^{-1}\|}.$$

It is clear that

$$\|x_0^{-1}x - 1\| = \|x_0^{-1}(x - x_0)\| \leq \|x_0^{-1}\|\|x - x_0\| < 1,$$

This shows that $x_0^{-1}x$ is in \mathcal{G} .

Since $x = x_0(x_0^{-1}x)$,

it follows that x is also in \mathcal{G} , so \mathcal{G} is open. ■

Theorem 1.9 ([16],p306) The mapping $x \rightarrow x^{-1}$ of \mathcal{G} into \mathcal{G} is continuous and is therefore a homeomorphism of \mathcal{G} onto itself.

Proof.

Let x_0 be element of \mathcal{G} , and x be another element of \mathcal{G} such that

$$\|x - x_0\| < \frac{1}{2\|x_0^{-1}\|}.$$

Since $\|x_0^{-1}x - 1\| = \|x_0^{-1}(x - x_0)\| \leq \|x_0^{-1}\|\|x - x_0\| < \frac{1}{2}$

We know that $x_0^{-1}x$ is in \mathcal{G} and

$$x^{-1}x_0 = (x_0^{-1}x)^{-1} = 1 + \sum_{n=1}^{\infty} (1 - x_0^{-1}x)^n.$$

Conclusion now follows from

$$\begin{aligned} \|x^{-1} - x_0^{-1}\| &= \|(x^{-1}x_0 - 1)x^{-1}\| \leq \|x_0^{-1}\| \|x^{-1}x_0 - 1\| = \|x_0^{-1}\| \\ &\| \sum_{n=1}^{\infty} (1 - x_0^{-1}x)^n \| \leq \|x_0^{-1}\| \sum_{n=1}^{\infty} \|1 - x_0^{-1}x\|^n \\ &= \|x_0^{-1}\| \|1 - x_0^{-1}x\| \sum_{n=0}^{\infty} \|1 - x_0^{-1}x\|^n \\ &= \frac{\|x_0^{-1}\| \|1 - x_0^{-1}x\|}{1 - \|1 - x_0^{-1}x\|} \\ &< 2\|1 - x_0^{-1}\| \|1 - x_0^{-1}x\| \leq 2\|x_0^{-1}\|^2 \|x - x_0\| \end{aligned}$$

■

Definition 1.10 A *left ideal* of a Banach algebra \mathcal{B} is a subalgebra \mathcal{I} of \mathcal{B} such that $ax \in \mathcal{I}$ whenever $a \in \mathcal{B}$, $x \in \mathcal{I}$. A *right ideal* of \mathcal{B} is a subalgebra \mathcal{I} such that $xa \in \mathcal{I}$ whenever $a \in \mathcal{B}$, $x \in \mathcal{I}$. A *two-sided ideal* is a subalgebra of \mathcal{B} that is both left and right ideals. With respect to the above definitions, \mathcal{I} is said to be **proper** if $\mathcal{I} \neq \mathcal{B}$ and **maximal** if \mathcal{I} is proper and not contained in any larger proper ideal.

We know that proper ideals cannot contain invertible elements and the closure of an ideal is also an ideal of the same kind. It has been proved that every proper ideal of \mathcal{B} is contained in a maximal ideal of the same kind.

Definition 1.11 (Radical) *The radical of \mathcal{B} denoted by $\text{Rad } \mathcal{B}$ is the intersection of all its maximal left (right) ideals.*

Theorem 1.12 (*[1],p.34*) *In a Banach algebra \mathcal{B} the following sets are identical:*

- (i) *the intersection of all maximal left ideals of \mathcal{B} ,*
- (ii) *the intersection of all maximal right ideals of \mathcal{B} ,*
- (iii) *$\{x \in \mathcal{B} : e - zx \text{ is invertible in } \mathcal{B}, \text{ for all } z \in \mathcal{B}\},$*
- (iv) *$\{x \in \mathcal{B} : e - xz \text{ is invertible in } \mathcal{B}, \text{ for all } z \in \mathcal{B}\}.$*

■

It is clear from the above theorem that the $\text{Rad } \mathcal{B}$ is a two sided ideal.

If $\text{Rad } \mathcal{B} = \{0\}$, we say that \mathcal{B} is **semi-simple**.

Maximal ideals are closed sets and hence $\text{Rad } \mathcal{B}$ is also closed.

Through out this dissertation, whenever we talk of an ideal we mean a proper closed two-sided ideal.

If \mathcal{I} is a closed two-sided ideal of a Banach algebra \mathcal{B} , then \mathcal{B}/\mathcal{I} is a Banach space with norm

$$\|x + \mathcal{I}\| = \inf_{a \in \mathcal{I}} \|x - a\| \quad (x \in \mathcal{B})$$

\mathcal{B}/\mathcal{I} becomes a Banach algebra if coset multiplication is defined by

$$(x + \mathcal{I})(y + \mathcal{I}) = (xy + \mathcal{I}) \quad (x, y \in \mathcal{B}).$$

The **quotient algebra** \mathcal{B}/\mathcal{I} is also called \mathcal{B} modulo \mathcal{I} . ([**Aup**],p.33)

1.2 C^* -algebra

Definition 1.13 (C^* -algebra) *A Banach \star -algebra \mathcal{A} with the property that $\|xx^*\| = \|x\|^2$ for $x \in \mathcal{A}$ is called a C^* -algebra.*

1.3 Operational quantities

Operational quantities such as minimum modulus and essential minimum modulus have attracted interest over the last three decades, in particular, the minimum modulus and its connection in relation to topological zero divisors have been studied extensively [6], [9], [14] and [20].

For an example, the characterisation of topological zero divisors in the quotient algebra \mathcal{A}/\mathcal{I} in a von Neumann algebra setting has been established in terms of the essential lower bounds [6].

Definition 1.14 (Minimum modulus) *The minimum modulus $m(T)$ of an operator T on a Banach space \mathcal{B} is defined by,*

$$m(T) = \inf\{\|Tx\| : x \in \mathcal{B}, \|x\| = 1\}.$$

Definition 1.15 (Spectrum) *If \mathcal{B} is a Banach algebra and $x \in \mathcal{B}$, then the **spectrum** of x is defined to be the set $\sigma(x) = \{\lambda \in \mathbb{C} : \lambda e - x \notin \mathcal{B}^{-1}\}$.*

The complement of $\sigma(x)$ is called the **resolvent set**, $\{\sigma(x)\}^c$.

If $\lambda \in \{\sigma(x)\}^c$ then $\lambda e - x$ is invertible.

The spectral radius of x is the number $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$.

The minimum modulus can also be determined spectrally in terms of the absolute value $|T|$, where $m(T) = \inf\{\lambda : \lambda \in \sigma(|T|)\}$.

Theorem 1.16 (*[15], p235*) *If \mathcal{B} is a Banach algebra and $x \in \mathcal{B}$, then*

(i) $\sigma(x)$ is compact and non-empty,

(ii) $r(x)$ satisfies $r(x) = \lim_n \|x^n\|^{\frac{1}{n}}$.

■

The above theorem was used to prove the following Gelfand-Mazur theorem.

Theorem 1.17 ([15], p237) *If \mathcal{B} is a Banach algebra in which every non-zero element is invertible, then \mathcal{B} is isometrically isomorphic to \mathbb{C} .*

Proof.

We shall show that $\mathcal{B} = \{\zeta e \mid \zeta \in \mathbb{C}\}$, and hence the mapping $\zeta e \rightarrow \zeta$ clearly defines a mapping from \mathcal{B} onto \mathbb{C} with the desired properties.

As before, we assume that $\|e\| = 1$. To prove the above, suppose $x \in \mathcal{B}$ and $x - \zeta e \neq 0$, $\zeta \in \mathbb{C}$. Since \mathcal{B} is a division algebra, it follows that $(x - \zeta e)^{-1}$ exists for each $\zeta \in \mathbb{C}$ and, in particular, that $x \neq 0$ and x^{-1} exists. Let x^* be a continuous linear functional on \mathcal{B} such that $x^*(x^{-1}) = 1$. Such a functional exists as $x^{-1} \neq 0$. Now define the function $g : \mathbb{C} \rightarrow \mathbb{C}$ by $g(\zeta) = x^*[(x - \zeta e)^{-1}]$, $\zeta \in \mathbb{C}$. We claim

that g is a bounded entire function.

Before we prove this we need to note one preliminary fact; namely,

if $(x - \zeta e)^{-1}$ and $(x - \zeta_0 e)^{-1}$ exists, then

$$(x - \zeta e)^{-1}(x - \zeta_0 e)^{-1} = (x - \zeta_0 e)^{-1}(x - \zeta e)^{-1}.$$

This, however, clearly holds if and only if

$$(x - \zeta_0 e)(x - \zeta e) = (x - \zeta e)(x - \zeta_0 e),$$

as inverses are unique. But elementary calculations reveal that both sides of the last equation are equal to

$$x^2 - \zeta_0 x - \zeta x + \zeta_0 \zeta e,$$

thereby establishing the desired assertion. Using this fact, we see at

once that, for any $\zeta, \zeta_0 \in \mathbb{C}$,

$$(x - \zeta e)^{-1} - (x - \zeta_0 e)^{-1} = (x - \zeta e)^{-1}(x - \zeta_0 e)^{-1}[(x - \zeta_0 e) - (x - \zeta e)]$$

$$= (\zeta - \zeta_0)(x - \zeta e)^{-1}(x - \zeta_0 e)^{-1}.$$

Consequently, given $\zeta_0 \in \mathbb{C}$, we see that

$$\begin{aligned} \frac{g(\zeta) - g(\zeta_0)}{\zeta - \zeta_0} &= \frac{x^*[(x - \zeta e)^{-1}] - x^*[(x - \zeta_0 e)^{-1}]}{\zeta - \zeta_0} \\ &= \frac{x^*[(\zeta - \zeta_0)(x - \zeta e)^{-1}(x - \zeta_0 e)^{-1}]}{\zeta - \zeta_0} \end{aligned}$$

$$= x^*[(\zeta - \zeta_0)(x - \zeta e)^{-1}(x - \zeta_0 e)^{-1}] \quad (\zeta \in \mathbb{C}; \quad \zeta \neq \zeta_0).$$

However, since the mapping $\zeta \rightarrow x - \zeta e$, $\zeta \in \mathbb{C}$, is clearly continuous from \mathbb{C} to \mathcal{B} and inversion in \mathcal{B} is continuous, we conclude that

$$\lim_{\zeta \rightarrow \zeta_0} \frac{g(\zeta) - g(\zeta_0)}{\zeta - \zeta_0} = x^*[(x - \zeta_0 e)^{-1}(x - \zeta_0 e)^{-1}];$$

that is, the derivative of g at ζ_0 exists.

Since $\zeta_0 \in \mathbb{C}$ was arbitrary, we see that g is an entire function.

To see that the entire function g is bounded it suffices to show that $\lim_{|\zeta| \rightarrow \infty} g(\zeta) = 0$. But if $\zeta \neq 0$, it is evident that

$$(x - \zeta_0 e)^{-1} = (x/\zeta_0 - e)^{-1}/\zeta,$$

whence from the continuity of inversion in \mathcal{B} we deduce that

$$\begin{aligned} \lim_{|\zeta| \rightarrow \infty} (x - \zeta e)^{-1} &= \lim_{|\zeta| \rightarrow \infty} \frac{(x/\zeta - e)^{-1}}{\zeta} \\ &= 0. \end{aligned}$$

Thus, since x^* is continuous, we obtain

$$\begin{aligned} \lim_{|\zeta| \rightarrow \infty} g(\zeta) &= \lim_{|\zeta| \rightarrow \infty} x^*[(x - \zeta e)^{-1}] \\ &= x^*(0) \\ &= 0. \end{aligned}$$

Hence g is a bounded entire function, and so, we see that g is a

constant.

Since $\lim_{|\zeta| \rightarrow \infty} g(\zeta) = 0$, we see, moreover, that $g(\zeta) = 0$, $\zeta \in \mathbb{C}$.

However, this contradicts the fact that $g(0) = x^*(x^{-1}) = 1$.

Therefore for each $x \in \mathcal{B}$ there exists some $\zeta \in \mathbb{C}$ such that $x = \zeta e$.

■

Proposition 1.18 *If x and y are elements of a Banach algebra \mathcal{B} , then $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$.*

Proof.

Let $x, y \in \mathcal{B}$ be arbitrary. By construction 0 belongs to both sets, hence let $\alpha \neq 0$, $\alpha \in \sigma(xy)$. Assume $\alpha e - xy$ is not invertible. Then we can find a normalised sequence (z_n) with $\|z_n\| = 1$, all n , such that $(\alpha e - xy)z_n \rightarrow 0$. Thus, also

$$y(\alpha e - xy)z_n \rightarrow 0 \text{ and hence } (\alpha e - xy)yz_n \rightarrow 0.$$

If $yz_n \rightarrow 0$, then

$$\begin{aligned} |\alpha| - \|x\|\|yz_n\| &= |\alpha|\|z_n\| - \|x\|\|yz_n\| \leq \|\alpha z_n\| - \|xyz_n\| \\ &\leq \|(\alpha e - xy)z_n\| \end{aligned}$$

we obtain $|\alpha| \leq 0$, which is a contradiction. Therefore, $yz_n \not\rightarrow 0$ so that for some $\varepsilon > 0$ there is a subsequence (yz_{n_k}) such that $\|yz_{n_k}\| \geq \varepsilon$, for all k .

It follows that

$$\frac{\|(\alpha e - yx)yz_{n_k}\|}{\|yz_{n_k}\|} \leq \frac{\|(\alpha e - yx)yz_{n_k}\|}{\varepsilon} \rightarrow 0$$

and hence $\alpha \in \sigma(yx)$. If we assume that $z_n(\alpha e - xy) \rightarrow 0$, then one can easily show by similar argument that $\alpha \in \sigma(yx)$.

Hence

$$\sigma(xy) \cup \{0\} \subseteq \sigma(yx) \cup \{0\}.$$

By symmetry we also have

$$\sigma(yx) \cup \{0\} \subseteq \sigma(xy) \cup \{0\}$$

and hence the result. ■

Definition 1.19 (Essential minimum modulus) *Let*

$\pi : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{I}$ *be the canonical quotient map.*

*For $x \in \mathcal{B}$, the $m_{\mathcal{I}}(T) = \inf\{\lambda | \lambda \in \sigma(\pi(|x|))\}$ is called the **essential minimum modulus** of T .*

Definition 1.20 (Reduced minimum modulus) *For a general Banach algebra \mathcal{B} and an element $x \in \mathcal{B}$, we define the reduced*

minimum modulus to be the quantity $\gamma(x) = \inf\{\sigma(|x|) \setminus \{0\}\}$ where we set $\gamma(x) = \infty$ if $\sigma(|x|) = \{0\}$.

Definition 1.21 (Left singular spectrum) For a Banach algebra \mathcal{B} we define the **left singular spectrum** of $x \in \mathcal{B}$ to be the set $\tau^l(x) = \{\alpha \in \mathbb{C} : \alpha I - x \in Z_l\}$.

The **right singular spectrum** of $x \in \mathcal{B}$ is the set

$$\tau^r(x) = \{\alpha \in \mathbb{C} : \alpha I - x \in Z_r\}.$$

The **singular spectrum**, $\tau(x)$, of $x \in \mathcal{B}$ is the union $\tau^l(x) \cup \tau^r(x)$.

That is $\tau(x) = \{\alpha \in \mathbb{C} : \alpha I - x \in Z\}$.

If \mathcal{B} is a finite dimensional Banach algebra then

$$\tau(x) = \sigma(x) \text{ for all } x \in \mathcal{B}.$$

This is also true in case where the Banach algebra is a \mathcal{C}^* -algebra.

$$\tau(x) = \sigma(x) \text{ for all } x \in \mathcal{B}.$$

Definition 1.22 (Modulus of integrity) ([13], p20) For any x in a Banach algebra \mathcal{B} , we define the quantities λ and ρ , as

$$\lambda(x) = \inf_{y \neq 0} \frac{\|xy\|}{\|y\|}$$

and

$$\rho(x) = \inf_{y \neq 0} \frac{\|yx\|}{\|y\|}$$

$\lambda(x)$ and $\rho(x)$ are called the left and right modulus of integrity respectively.

The above definition can also be written as

$$\lambda(x) = \inf_{\|y\|=1} \|xy\|$$

and

$$\rho(x) = \inf_{\|y\|=1} \|yx\|$$

We now give some basic properties of the above functions, λ and ρ , including continuity and a boundedness property.

Theorem 1.23 (*[13], p20*) *The functions λ and ρ have the properties:*

$$1. |\lambda(x) - \lambda(y)| \leq \|x - y\| \text{ and } |\rho(x) - \rho(y)| \leq \|x - y\|$$

for all $x, y \in \mathcal{B}$.

$$2. \lambda(x)\lambda(y) \leq \lambda(xy) \leq \|x\| \lambda(y) \text{ and } \rho(x)\rho(y) \leq \rho(xy) \leq \|y\| \rho(x)$$

for all $x, y \in \mathcal{B}$.

Proof.

It will be sufficient to consider only λ .

For all $z \in \mathcal{B}$,

$$\lambda(x) \leq \frac{\|xz\|}{\|z\|} = \frac{\|(x - y)z + yz\|}{\|z\|} \leq \|x - y\| + \frac{\|yz\|}{\|z\|}.$$

Therefore $\lambda(x) \leq \|x - y\| + \lambda(y)$, so that $\lambda(x) - \lambda(y) \leq \|x - y\|$.

By symmetry, $\lambda(y) - \lambda(x) \leq \|x - y\|$ and hence (i) follows.

Next we have

$$\begin{aligned}\inf_z \frac{\|xz\|}{\|z\|} \inf_z \frac{\|yz\|}{\|z\|} &\leq \inf_z \frac{\|xyz\|}{\|yz\|} \inf_z \frac{\|yz\|}{\|z\|} \\ &\leq \inf_z \frac{\|xyz\|}{\|z\|} \leq \|x\| \inf_z \frac{\|yz\|}{\|z\|}.\end{aligned}$$

In other words $\lambda(x)\lambda(y) \leq \lambda(xy) \leq \|x\|\lambda(y)$, which proves (ii). ■

Chapter 2

Zero divisors

Definition 2.1 (Zero divisor) *An element $x \neq 0$ in a Banach algebra \mathcal{B} is said to be a **left zero divisor** if there exists an element $y \neq 0$ such that $xy = 0$. Similarly, an element $x \neq 0$ in a Banach algebra \mathcal{B} is said to be a **right zero divisor** if there exists an element $y \neq 0$ such that $yx = 0$.*

An element in a Banach algebra that is both a left and a right zero divisor is simply called the **zero divisor**.

A non-trivial idempotent is always a divisor of zero.

Definition 2.2 (Topological zero divisor) *If \mathcal{B} a Banach algebra then $x \in \mathcal{B}$ is called a **left (right) topological zero divisor** if*

there exists a sequence (z_n) in \mathcal{B} such that $\|z_n\| = 1$, $n \in \mathbb{N}$ and $\|xz_n\| \rightarrow 0$ ($\|z_nx\| \rightarrow 0$).

The sets of left and of right topological divisors of zero are denoted by Z_l and Z_r respectively.

If an element in a Banach algebra belongs to both Z_l and Z_r , then it is called a **topological divisor of zero**, denoted by $Z_{l,r}$.

Let $Z = Z_l \cup Z_r$ and $Z_{l,r} = Z_l \cap Z_r$.

If \mathcal{B} is a Banach algebra and \mathcal{O} is a closed subalgebra of \mathcal{B} containing the unit e , then it might happen that $x \in \mathcal{O}$ is not invertible in \mathcal{O} , though it is invertible in \mathcal{B} . However, if x is not a topological divisor of zero in \mathcal{O} then x cannot be the topological zero divisor in \mathcal{B} since $xz_n \rightarrow 0$, $\|z_n\| = 1$, $z_n \in \mathcal{B}$

also applies to \mathcal{B} . This property possessed by topological divisor of zero is called **permanent singularity**.

It is clear that every zero divisor in a Banach algebra is a topological zero divisor and a topological zero divisor in a Banach algebra with identity need not be zero divisor that can not be invertible. We shall give examples of Banach algebras which make the above statements clear that these two concepts are indeed not equivalent. It can easily

be proved that zero divisor is a topological zero divisor. The example below shows that the converse need not be true.

Example 1.

If \mathcal{B} is any finite dimensional Banach algebra, let $x \in Z_l(Z_r)$. Then there is a sequence (z_n) with $\|z_n\| = 1$ such that

$$xz_n \rightarrow 0 \quad (z_n x \rightarrow 0).$$

Since the closed unit ball is compact, (z_n) has a convergent subsequence $z_{n_k} \rightarrow z$.

Continuity of multiplication now implies that

$$xz_{n_k} \rightarrow xz = 0 \quad (z_{n_k} x \rightarrow zx = 0).$$

i.e. x is a zero divisor.

Now we give an example to illustrate that a topological zero divisor is not necessarily a zero divisor.

Example 2.

If \mathcal{B} is the Banach algebra $\mathcal{C}([0, 1])$ then

$x(t) = t$ is a topological divisor of zero.

Let

$$z_n(t) = \begin{cases} -nt + 1, & t \in [0, \frac{1}{n}] \\ 0, & t \in (\frac{1}{n}, 1] \end{cases}$$

but clearly $x(t)$ is not a divisor of zero.

2.1 Topological zero divisors in Banach Algebras

It is clear that no topological zero divisor is invertible. This statement can be proved by using the norm inequality together with the uniqueness of limits.

We give the characterisation of these elements in terms of the moduli of integrity.

Theorem 2.3 ([4],p22) $\lambda(x) = 0$ if and only if $x \in Z_\ell$ and

$\rho(x) = 0$ if and only if $x \in Z_r$.

Proof.

If $\lambda(x) = 0$, then by definition $\inf_{y \neq 0} \frac{\|xy\|}{\|y\|} = 0$. This implies that either there is $y_0 \neq 0$ such that $\frac{\|xy_0\|}{\|y_0\|} = 0$ (in which case x is a left divisor of zero) or there is a sequence (y_n) , ($y_n \neq 0$ such that $\frac{\|xy_n\|}{\|y_n\|} \rightarrow 0$ which means that $x \in Z_l$. Suppose on the other hand that $x \in Z_l$. By definition there is a sequence (z_n) with $\|z_n\| = 1$ such that $\|xz_n\| \rightarrow 0$.

Since $\{\frac{\|xz_n\|}{\|z_n\|} : n \in \mathbb{N}\} \subseteq \{\frac{\|xy\|}{\|y\|} : y \neq 0, y \in A\}$

we have

$$\inf_{y \neq 0} \frac{\|xy\|}{\|y\|} \leq \inf_n \frac{\|xz_n\|}{\|z_n\|} = 0$$

and thus $\lambda(x) = 0$.

Similarly we have prove $\rho(x) = 0$ if and only if $x \in Z_r$. ■

Theorem 2.4 ([4],p17) *In a Banach algebra the sets Z_l , Z_r , $Z_{l,r}$ and Z are closed.*

Proof.

We only prove that Z_l is closed; the proof for Z_r is similar and then obviously Z and $Z_{l,r}$ are closed. Suppose (x_n) is a sequence in Z_l such that $x_n \rightarrow x$.

Then, to each x_n there corresponds a sequence $(y_k^{(n)})$, $k \in \mathbb{N}$, such that $x_n y_k^{(n)} \rightarrow 0$ where $\|y_k^{(n)}\| = 1$, $k \in \mathbb{N}$ ($k \rightarrow \infty$).

It follows that there exists a sequence $(x_n y_{k_n}^{(n)})$ such that

$$x_n y_{k_n}^{(n)} \rightarrow 0 \quad (n \rightarrow \infty).$$

We now have that

$$\|x_n y_{k_n}^{(n)}\| = \|x_n y_{k_n}^{(n)} - x_n y_{k_n}^{(n)} + x_n y_{k_n}^{(n)}\| \leq \|x - x_n\| + \|x_n y_{k_n}^{(n)}\| \rightarrow 0$$

$(n \rightarrow \infty)$.

Thus $x \in Z_l$ so that Z_l is closed.

Similarly Z_r , $Z_{l,r}$ and Z are also closed. ■

It is clear that topological zero divisors in Banach algebras with identity cannot be invertible. Arens ([10],P48) showed that $x \in \mathcal{B}$ is a topological divisor of zero if and only if x is permanently singular.

The boundary of the group of invertible elements corresponds to topological zero divisor.

Theorem 2.5 ([18],p397) *If $\partial\mathcal{B}^{-1}$ denotes the topological boundary of the group of invertible elements of \mathcal{B} then $\partial\mathcal{B}^{-1} \subseteq Z_{l,r}$.*

Proof.

If $x \in \partial\mathcal{B}^{-1}$, then there is a sequence (x_n) in \mathcal{B}^{-1} such that $x_n \rightarrow x$ by the definition of $\partial\mathcal{B}^{-1}$.

Since \mathcal{B}^{-1} is open we have that $x \notin \mathcal{B}^{-1}$.

Thus, (x_n) is a sequence of invertible elements converging to a non-invertible element, so that $\|x_n^{-1}\| \rightarrow \infty$.

It follows that

$$x \frac{x_n^{-1}}{\|x_n^{-1}\|} = (x - x_n) \frac{x_n^{-1}}{\|x_n^{-1}\|} + \frac{e}{\|x_n^{-1}\|} \rightarrow 0$$

since

$$x_n \text{ and } \frac{1}{\|x_n^{-1}\|} \rightarrow 0$$

Similarly we may prove that

$$\frac{x_n^{-1}}{\|x_n^{-1}\|}x \rightarrow 0$$

and hence $x \in Z_{l,r}$.

■

Theorem 2.6 ([10],p48) *Let \mathcal{B} be a commutative Banach algebra and let $x \in \mathcal{B}$. Then the following are equivalent:*

- (i) *x is a topological divisor of zero in \mathcal{B} ,*
- (ii) *x is singular in every superalgebra Y of \mathcal{B} .*

Proof.

If x is a topological zero divisor in \mathcal{B} , then x is clearly a two-sided topological zero divisor in every superalgebra Y of \mathcal{B} , whence, x is singular in Y . Thus part (i) implies part (ii).

Conversely, suppose x is not a topological zero divisor in \mathcal{B} . Then we shall construct a superalgebra Y of \mathcal{B} in which x is regular.

First, we note that since x is not a topological zero divisor, we have

$$\xi(x) > 0.$$

Let

$$\rho > \frac{1}{\xi(x)}$$

and consider the commutative algebra Y_1 consisting of all formal power series in t ,

$$y(t) = \sum_{k=0}^{\infty} y_k t^k,$$

$$y_k \in \mathcal{B}, \quad k = 0, 1, 2, \dots,$$

such that $\|y(t)\| = \sum_{k=0}^{\infty} \|y_k\| \rho^k$ is finite.

For example, if $y \in \mathcal{B}$ is such that

$$\|y\| < \frac{1}{\rho},$$

then

$$y(t) = \sum_{k=0}^{\infty} y^k t^k$$

belongs to Y_1 , where, of course, $y^0 = e$. The algebra operations in Y_1 are the usual formal operations of addition, multiplication and scalar multiplication applied to power series. Moreover, it is not difficult to verify that $\|\cdot\|$ defined above is a norm on Y_1 under which Y_1 is a commutative normed algebra. The completion of Y_1 , denoted by \overline{Y}_1 , is a commutative Banach algebra.

Let \mathcal{I} be the closed in \overline{Y}_1 generated by the element $e - xt$; that is, \mathcal{I} is the closure in \overline{Y}_1 of the ideal $\{(e - xt)w | w \in \overline{Y}_1\}$. Then

Y is defined to be the quotient algebra $Y = \overline{Y}_1/\mathcal{I}$ with the usual quotient norm $\|w + \mathcal{I}\| = \inf_{v \in \mathcal{I}} \|w + v\|$. Y is a commutative Banach algebra. We claim that Y is a superalgebra of \mathcal{B} .

Indeed, it is evident that the mapping $\varphi : \mathcal{B} \rightarrow Y$, is defined by

$$\varphi(z) = \sum_{k=0}^{\infty} \varphi(z)_k t^k + \mathcal{I}$$

($z \in \mathcal{B}$), where $\varphi(z)_0 = z$, $\varphi(z)_k = 0$, $k = 1, 2, 3, \dots$, is a homomorphism of \mathcal{B} into Y .

Furthermore, given $z \in \mathcal{B}$ and $(e - xt)y(t) \in \mathcal{I}$, where

$$y(t) = \sum_{k=0}^{\infty} y_k t^k,$$

we see that

$$\begin{aligned} \left\| \sum_{k=0}^{\infty} \varphi(z)_k t^k + (e - xt)y(t) \right\| &= \left\| z + y_0 + \sum_{k=1}^{\infty} (y_k - xy_{k-1})t^k \right\| \\ &= \|z + y_0\| + \sum_{k=1}^{\infty} \|y_k - xy_{k-1}\| \rho^k \\ &\geq \|z\| - \|y_0\| + \sum_{k=1}^{\infty} (\|xy_{k-1}\| - \|y_k\|) \rho^k \\ &= \|z\| - \sum_{k=0}^{\infty} \|y_k\| \rho^k + \rho \sum_{k=0}^{\infty} \|xy_k\| \rho^k \\ &\geq \|z\| + \sum_{k=0}^{\infty} [\xi(x)\rho - 1] \|y_k\| \rho^k \\ &= \|z\| + [\xi(x)\rho - 1] \|y(t)\| \\ &\geq \|z\| \end{aligned}$$

The final inequality is valid since

$$\rho > \frac{1}{\xi(x)},$$

and the penultimate inequality utilizes the fact that $\|xy\| \geq \xi(x)\|y\|$, $y \in \mathcal{B}$. It is then apparent from the previous inequality and the fact that $\{(e - xt)y(t) | y(t) \in Y_1\}$ is dense in \mathcal{I} that

$$\|\|\varphi(x)\|\| = \|\|\sum_{k=0}^{\infty} \varphi(z)_k t^k + \mathcal{I}\|\| \geq \|z\|$$

($z \in \mathcal{B}$). The inequality in the opposite direction is trivial, so we conclude that $\|\|\varphi(z)\|\| = \|z\|$, $z \in \mathcal{B}$; that is, φ is an isometric algebra isomorphism of \mathcal{B} into Y . Furthermore, an elementary argument reveals that $\varphi(e) = e + \mathcal{I}$ is an identity for Y . Thus Y is a commutative superalgebra of \mathcal{B} .

Finally, we claim that x is regular in Y ; that is, $\varphi(x) = x + \mathcal{I}$ is regular in Y . Indeed, since $e - xt \in \mathcal{I}$, we have

$$(x + \mathcal{I})(et + \mathcal{I}) = xt + \mathcal{I} = e + \mathcal{I}, \text{ that is, } (x + \mathcal{I})^{-1} = et + \mathcal{I}.$$

Therefore part (ii) of the theorem implies part (i), and the proof is complete. ■

2.2 Topological zero divisors in von Neumann Algebras

Recall that an element a in a von Neumann algebra \mathcal{A}/\mathcal{I} is called a left (resp. right) topological zero divisor relative to an ideal \mathcal{I} if there exists a normalized sequence $\{\pi(b_n)\}$ in \mathcal{A}/\mathcal{I} such that

$$\lim_{n \rightarrow \infty} \|\pi(a) \pi(b_n)\| = 0 \quad (2.1)$$

respectively,

$$\lim_{n \rightarrow \infty} \|\pi(b_n) \pi(a)\| = 0 \quad (2.2)$$

We denote the set of these elements $Z_l(\mathcal{A}, \mathcal{I})$

(resp. $Z_r(\mathcal{A}, \mathcal{I})$).

Put $Z(\mathcal{A}, \mathcal{I}) = Z_l(\mathcal{A}, \mathcal{I}) \cap Z_r(\mathcal{A}, \mathcal{I})$.

Let $\Phi_l(\mathcal{A})$ (resp. $\Phi_r(\mathcal{A})$) denote the set of all elements a in \mathcal{A} such that $\pi(a)$ is left (resp. right) invertible in \mathcal{A}/\mathcal{I} .

Put $\Phi(\mathcal{A}) = \Phi_l(\mathcal{A}) \cap \Phi_r(\mathcal{A})$, $\Phi_l^c(\mathcal{A}) = \mathcal{A} \setminus \Phi_l(\mathcal{A})$,

$\Phi_r^c(\mathcal{A}) = \mathcal{A} \setminus \Phi_r(\mathcal{A})$ and $\Phi^c(\mathcal{A}) = \mathcal{A} \setminus \Phi(\mathcal{A}) = \Phi_l^c \cup \Phi_r^c$.

For any norm closed two-sided ideal \mathcal{I} in \mathcal{A} let $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{I}$ denote the canonical the canonical quotient map.

Definition 2.7 (Fredholm elements) *An element $a \in \mathcal{A}$ is called **left Fredholm relative to \mathcal{I}** if $\pi_{\mathcal{I}}(a)$ is left invertible in \mathcal{A}/\mathcal{I} . We denote this semigroup of elements by $\Phi_l(\mathcal{A}, \mathcal{I})$. The class $\Phi_r(\mathcal{A}, \mathcal{I})$ of right Fredholm elements is defined in the similar way.*

Let $N_l(\mathcal{A}, \mathcal{I})$ denote the class of elements $a \in \mathcal{A}$ for which $\pi(a)$ is a left algebraic divisor of zero in \mathcal{A}/\mathcal{I} ; i.e., $a \in N_l(\mathcal{A}, \mathcal{I})$ if and only if there exists a $b \notin \mathcal{I}$ with $ab \in \mathcal{I}$. A similar definition holds for $N_r(\mathcal{A}, \mathcal{I})$. These classes possess a number of algebraic and topological properties.

The Fredholm elements relative to \mathcal{I} are those elements contained in $\Phi(\mathcal{A}, \mathcal{I}) := \Phi_l(\mathcal{A}, \mathcal{I}) \cap \Phi_r(\mathcal{A}, \mathcal{I})$.

Proposition 2.8 ([19], p23) *Let $a \in \mathcal{A}$.*

(a) *$a \in \Phi_r(\mathcal{A}, \mathcal{I})$ if and only if $a^* \in \Phi_l(\mathcal{A}, \mathcal{I})$*

(b) *$a \in Z_r(\mathcal{A}, \mathcal{I})$ if and only if $a^* \in Z_l(\mathcal{A}, \mathcal{I})$*

(c) *$a \in N_r(\mathcal{A}, \mathcal{I})$ if and only if $a^* \in N_l(\mathcal{A}, \mathcal{I})$*

where a^ denotes the adjoint operator of a .*

Proof.

(a) $a \in \Phi_r(\mathcal{A}, \mathcal{I})$ implies there exists an $b \in \mathcal{A}$ such that $ba - e \in \mathcal{I}$, so that $b^*a^* - e \in \mathcal{I}$.

Hence $a^* \in \Phi_l(\mathcal{A}, \mathcal{I})$.

Similarly one can prove the converse.

(b) Suppose $a \in Z_r(\mathcal{A}, \mathcal{I})$ and let $(\pi(b_n))$ be a normalised sequence in \mathcal{A}/\mathcal{I} such that $\lim_{n \rightarrow \infty} \|\pi(a)\pi(b_n)\| = 0$.

Then $(\pi(b_n^*))$ is also normalised sequence in \mathcal{A}/\mathcal{I} and

$$\lim_{n \rightarrow \infty} \|\pi(a^*)\pi(b_n^*)\| = 0.$$

Hence $a^* \in Z_l(\mathcal{A}, \mathcal{I})$.

Similarly the converse can be proved.

c) Suppose $a \in N_r(\mathcal{A}, \mathcal{I})$ and let $b \notin \mathcal{I}$ with $ba \in \mathcal{I}$. Then $b^* \notin \mathcal{I}$ and $a^*b^* \in \mathcal{I}$, so that $a^* \in N_l(\mathcal{A}, \mathcal{I})$.

Similarly one can prove the converse.

■

Theorem 2.9 ([19], p24) Let \mathcal{A} be a von Neumann algebra and \mathcal{I} any norm closed two-sided ideal in \mathcal{A} . Then $a \in Z_l(\mathcal{A}, \mathcal{I})$ if and only if $m_{\mathcal{I}}(a) = 0$.

Proof.

Suppose $a \in \Phi_l(\mathcal{A}, \mathcal{I})$, then by definition there exists a normalised sequence $(\pi(a_n))$ such that $\lim_{n \rightarrow \infty} \|\pi(a_n)\| = 0$. This implies that $m_{\mathcal{I}}(a) = 0$.

For the converse inclusion, assume $m_{\mathcal{I}}(a) = 0$.

(Using [19], p13, prop 1.4) We have

$$\inf\{\|\pi(aP)\| : P \in \mathcal{A}^p \text{ and } P\mathcal{I}\} = 0.$$

So we can find a sequence (P_n) of projections not in \mathcal{I} such that

$$\|\pi(aP_n)\| < \frac{1}{n} \text{ for every } n.$$

It is easy to see that $\|\pi(P_n)\| = 1$ for every n and hence $a \in \Phi_l(\mathcal{A}, \mathcal{I})$.

By a similar argument we can show that $a \in \Phi_r(\mathcal{A}, \mathcal{I})$ if and only if

$m_{\mathcal{I}}(a^*) = 0$. Hence the following corollary holds. ■

Corollary 2.10 ([19], p25) $a \in Z_r(\mathcal{A}, \mathcal{I})$ if and only if

$$m_{\mathcal{I}}(a^*) = 0.$$
■

Theorem 2.11 ([19], p25) Let \mathcal{A} be a von Neumann algebra and \mathcal{I}

any norm closed two-sided ideal in \mathcal{A} .

Then

$$\Phi_l(\mathcal{A}, \mathcal{I}) = (Z_l(\mathcal{A}, \mathcal{I}))^c \text{ and } \Phi_r(\mathcal{A}, \mathcal{I}) = (Z_r(\mathcal{A}, \mathcal{I}))^c.$$

■

Note: Xue extended some of the above results to a \mathcal{C}^* -algebra settings and gave the characterisation of topological zero divisor in terms of essential minimum modulus.

Theorem 2.12 ([20], p274) *Let \mathcal{A} be a \mathcal{C}^* -algebra and \mathcal{I} any norm closed two-sided ideal in \mathcal{A} . Then $a \in Z_l(\mathcal{A}, \mathcal{I})$ if and only if $m_{\mathcal{I}}(a) = 0$.*

■

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