ON YOSIDA FRAMES AND RELATED FRAMES

by

MATABANE MOGALATJANE EDWARD

(BSc, BSc Hons, PGCE)

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SUPERVISOR: Professor Hlengani J Siweya

DECLARATION

"I declare that the mini–dissertation hereby submitted to the University of Limpopo, for the degree of Master of Science has not previously been submitted by me for a degree at this or any other university; that it is my work in design and in execution, and that all material contained therein has been duly acknowledged."

Signature Date

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Abstract

Topological structures called *Yosida frames* and related *algebraic frames* are studied in the realm of Pointfree Topology. It is shown that in algebraic frames regular elements are those for which compact elements are rather below the regular elements, and algebraic frames are regular if and only if every compact element is rather below itself if and only if the frame has the Finite Intersection Property (*FIP*) and each prime element is minimal.

We also show that Yosida frames are those algebraic frames with the Finite Intersection Property and are finitely subfit; that these frames are also those semi-simple algebraic frames with *FIP* and a disjointification where $dim\ (L) \leq 1$; and we prove that in an algebraic frame with *FIP*, it holds that $dom\ (L) = dim\ (L)$. In relation to normality in Yosida frames, we show that in a coherent normal Yosida frame *L,* the frame is subfit if and only if it is regular if and only if it is zero- dimensional if and only if every compact element is complemented.

List of symbols

$Min(L)$	:	The collection of minimal primes of L
$Max(L)$	:	The set of all elements $x < e$ in L that are maximal
$L_{Max(L)}$	:	The set of all meet of maximal elements
$Max^*(x)$	:	$\Lambda \{m \in Max(L) \mid m \geq x\}$
$N(L)$	:	The collection of all nuclei on a frame L
$Fix(j)$	:	$\{x \in L \mid j(x) = x\}$
x^*	:	$V\{y \in L \mid x \wedge y = 0\}$.
$Spec(L)$	:	The collection of prime elements of L is denoted
$\mathrm{Reg}(L)$	:	The collection of all regular elements of L
dom (L)	:	The join of the lengths of dominance chains of L.
dim (L)	:	The maximum of the lengths of chains of primes.
ChFrm	:	The category of all coherent frames and coherent frame Homomorphisms.

Introduction

This dissertation is an exposition of the interplay of mathematical structures in General Topology and Algebra on the one hand, with those in Pointfree Topology on the other. It is based on three related research articles of Martinez and Zenk, namely:

- *Yosida frames*, Jour. of Pure and Appl. Alg. 204 (2006), 473-492.
- *When an algebraic frame is regular*, Algebra Univers. 50 (2003), 231-257.
- *Regularity in algebraic frames*, Jour. of Pure and Appl. Alg.211 (2007), 566 – 580.

In this study, we selected results whose proofs are sketchy but are related to Yosida or algebraic frames in general and organised them into three chapters.

In Chapter 1 (*Regularity in algebraic frames*), we study algebraic frames (those frames that are generated by compact elements). It must be recalled that frames are generalised complete lattices that are closed under finite meets and arbitrary joins in which the Generalised Distributive Law holds. We prove properties of compact and regular elements, the rather below relation and other frame-theoretic concepts in algebraic frames. We also study relative notions of regularity, namely, Reg(1), Reg(2), Reg(3) and Reg(4), in relation to pseudo-complements and complemented elements.

Chapter 2 deals with Yosida frames. These frames are not so well-known in the family of “Pointfree topologists” but enjoy very interesting properties.

For instance, these frames are precisely those algebraic frames with the Finite Intersection Property that are finitely subfit. In fact, it is proved that if L is a semi-simple algebraic frame with the Finite Intersection Property and disjointification with $dim(L) \leq 1$ is a Yosida frame. Thus there are new concepts such as semi-simple, disjointification, subfitness, Finite Intersection Property, the Compact Splitting Property and zero-dimensionality that we found enriching to study.

In the last Chapter, we bring a collection of results from one of the most familiar authorities in Pointfree Topology Bernhard Banaschewski alongside those of Martinez and Zenk. We succeeded in showing that compact normal frames are related to regular frames and that, importantly, in a normal coherent frame, the frame is subfit if and only if it is regular if and only if it is a zero-dimensional if and only if every compact element is complemented.

The approach, methods and techniques we used in this mini-dissertation are standard: many results are established from basic principles (i.e. definitions) and known techniques used in common (standard). There are no new results in the dissertation but some of the proofs constructed provide insight into the beauty of pointless thinking – and, indeed, as Peter Johnstone would say, there is a “point” in studying Pointless Topology.

Chapter 1

Regularity in algebraic frames

In this chapter we study the relationship between *regularity*, *compactness*, the *rather-below* relation, *denseness* and *pseudo-complements* in frames. We show how (pointless) regularity relates to *d-elements*, *pseudo-complements* and *compact elements* in an algebraic frame L. Some of the results we prove in this chapter are the following:

$i)$ Suppose that L is an algebraic frame. Then $x \in L$ is regular if and only if every compact element $c \leq x$ is rather below x (Proposition 1.2.4).

$ii)$ An algebraic frame L is regular if and only if for each $c \in \mathcal{C}(L)$, it holds that $c \vee c^* = e$ if and only if L has the Finite Intersection Property (FIP) and each prime of L is minimal (Theorem 1.2.6).

$iii)$ In algebraic frame in which the rather below relation interpolates, the collection of its regular elements is regular (Theorem 1.2.12).

$iv)$ In an algebraic frame L with *FIP*, the element x^{**} is regular if and only if it is complemented, for every compact $x \in L$ (Theorem 1.3.6).

$v)$ An algebraic frame L satisfies *Reg*(1) if and only if it has the Compact Splitting Property (Theorem 1.3.7).

$vi)$ In an algebraic frame L possessing a unit $u \in L$ and satisfying *Reg*(4), every complemented element is of the form a^{**} for some compact $x \in L$ (Proposition $1.3.13$).

1.1 Preliminary Concepts

We call a complete lattice L a *frame* if the following generalised *distributive law* holds:

$$b \wedge (\bigvee S) = \bigvee\{b \wedge s,\ s \in S\},$$

for each $b \in L$ and any $S \subseteq L$ .The *bottom* (respectively, *top)* element of L is denoted by 0 (respectively, e). A *frame homomorphism* $h : M \rightarrow L$ is a map between frames preserving finite meets (including e*)* and arbitrary joins (including 0). Frame homomorphisms are closed under composition, and therefore we have the category $\mathbb{F}rm$ of frames and frame homomorphisms (see Johnstone $[6]$).

Many of the concepts we use in this dissertation have their origin in "*pointful*" topology. Given a topological space (X, τ_X) and denoting by $O(X)$ the set τ_X of open subsets of X, we know that

$i)$ $\phi, X \in O(X)$

$ii)$ $U \cap V \in O(X)$, for all $U, V \in O(X)$ and $\left(\bigcap_{i \in I} U_i\right)^0 \in O(X)$ for each $\{U_i | i \in I\} \subseteq O(X)$.

$iii)$ $\bigcup_{i \in I} U_i \in O(X)$, for each $\{U_i | i \in I\} \subseteq O(X)$.

$iv)\ U \cap \bigcup_{i \in I} V_i = \bigcup_{i \in I} (U \cap V_i)$ for each $U \in O(X)$ and $\{V_i | i \in I\} \subseteq O(X)$.

Therefore with ϕ representing the bottom element and X the top element, $\cap$ and $\cup$ representing *meet* and *join*, respectively, the pair $(X, O(X))$ is an example of a frame.

Definition 1.1.1 (Martinez [12])

A *frame homomorphism* is a map $h: L \to M$ between frames satisfying the following properties:

$i)\ h(0) = 0;\ h(e) = e.$

$ii)\ h(x \wedge y) = h(x) \wedge h(y),$ for all $x, y \in L.$

$iii)\ h\left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} h(x_i),$ for all $\{x_i | i \in I\} \subseteq L.$

The preimage f^{-1} of any continuous function $f: (X, \tau_X) \to (Y, \tau_Y)$ between topological spaces satisfies (see Willard [16])

$i)\ f^{-1}(Y) = X$ and $f^{-1}(\phi) = \phi.$

$ii)\ f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V),$ for all $U, V \in \tau_Y.$

$iii)\ f^{-1}\left(\bigcup_{V_i \in \tau_Y} V_i\right) = \bigcup_{V_i \in \tau_Y} f^{-1}(V_i).$

Therefore, with $\cap$ and $\cup$ representing *meet* and *join*, respectively, and X and ϕ being the top and bottom elements, respectively, the function $O(f): O(Y) \to O(X)$ defined by $O(f)(U) = f^{-1}(U)$ for each $U \in \tau_Y$ is a *"perfect"* example of frame homomorphism as it preserves finite intersections and arbitrary unions.

Definition 1.1.2 (See e.g. Simmons [14])

Given a frame L, a *nucleus* on L is a function $j : L \to L$ satisfying:

$i)\ x \le j(x),\ \text{for all } x \in L.$

$ii)\ j(j(x)) = j(x),\ \text{for all } x \in L.$

$iii)\ j(\mathrm{x} \wedge \mathrm{y}) = j(x) \wedge j(y), \text{for all } x, y \in L.$

The collection of all nuclei on a frame L will be denoted by $N(L)$.

Proposition 1.1.3

For any subset $S \subseteq N(L)$, the function $\Lambda S : L \to L$ defined by

$$S(X) = \Lambda\{j(x), x \in L \mid j \in S\}$$

is a nucleus on L.

Proof:

$i)$ Let $j \in S$ and $x \in L$. Then since j is a nucleus on L, we have that

$$x \leq j(x)$$

$$\Rightarrow \mathrm{x} \leq \Lambda\{j(\mathrm{x}), x \in L \mid j \in S\}$$

$$\Rightarrow x \leq \Lambda S(x)$$

$ii)$ Take $x, y \in L$ and $j \in S$. Then since j is a nucleus on L, it follows that

$$j(x \wedge y) = j(x) \wedge j(y)$$

$$\Rightarrow \Lambda\{j(x \wedge y) \mid j \in S\} = \Lambda\{j(x) \wedge j(y) \mid j \in S\}$$

$$= \Lambda\{j(x) \mid j \in S\} \wedge \Lambda\{j(y) \mid j \in S\}$$

$$\Rightarrow \Lambda S(x \wedge y) \qquad = \Lambda S(x) \wedge \Lambda S(y).$$

$iii)$ Finally, we have that

$$\Lambda S[\Lambda S(x)] = \Lambda\{j(\mathit{\Lambda} S(x)) \mid j \in S\}$$

$$\leq \Lambda\{j(j(x)) \mid j \in S\}$$

$$= \Lambda\{j(x) \mid j \in S\}$$

$$= \Lambda S(x).$$

■

Recall (see e.g. Banaschewski [2]) that a frame homomorphism $h: M \to L$ is *dense* if whenever $h(x)=0$ implies $x=0$. Dually, h is said to be *codense* if whenever $h(x)=e$ then $x=e$. Denoting by jL the set

$$Fix(j) = jL = \{x \in L \mid j(x) = x\},$$

we now have the following relationship between a nucleus j on L and its denseness.

Observation 1.1.4 (Martinez [12])

The nucleus j is dense if and only if $0 \in jL = \{x \in L \mid j(x) = x\}$.

Proof:

$(\Rightarrow)$**:** Suppose that j is dense and $0 \notin jL$. Then $j(0) \neq 0$, so j is not dense, a contradiction. Thus $0 \in jL$.

$(\Leftarrow)$**:** On the other hand, if $0 \in jL$ and $j(x) = 0$, then $x \leq j(x) = 0$, proving that $x = 0$. ■

A *closure operator* on L is a map $j: L \to L$ which satisfies (i) and (ii) of Definition 1.1.2. Suppose $j_1, j_2 : L \to L$ are closure operators on L and that $j_1(x) \leq j_2(x)$ and take $s \in j_2L$. Then, by definition, we have that $s \leq j_1(s)$, for all $s \in L$. Now if $s \in j_2L$ then $s = j_2(s) \geq j_1(s)$ so that $j_1(s) = s$ which means that $s \in j_1L$. Hence $j_2L \subseteq j_1L$. These calculations prove that

Observation 1.1.5 (Martinez [12])

Closure operators on L are partially ordered by $j_1 \leq j_2$ if and only if $j_2 L \subseteq j_1 L$.

■

Following Martinez and Zenk $[11]$, we define the *pseudo-complement* of $x \in L$ to be

$$x^* = \bigvee \{y \in L \mid x \wedge y = 0\}.$$

We say that y is *rather below* x (and write $y \prec x$) if there exists an element $z \in L$ satisfying $x \wedge z = 0$ and $z \vee y = e$. It is immediate that $x \wedge x^* = 0$ because

$$x \wedge x^* = \bigvee \{x \wedge y \mid x \wedge y = 0\} = \bigvee 0 = 0.$$

However, $x \vee x^* \neq e$, in general. The frame L is *complemented* if $x \vee x^* = e$ for each $x \in L$.

The following equivalent notion of a pseudo-complement is used to prove Lemma 1.1.7.

Remark 1.1.6

$a)$ For each $x \in L$, if $z \wedge x = 0$, then $z \leq x^*$. This follows since then

$$z \in \{y \in L \mid y \wedge x = 0\}$$

so that $z \leq \bigvee \{y \in L \mid y \wedge x = 0\} = x^*$.

$b)$ $y \prec x$ if and only if $y^* \vee x = e$: If $y \prec x$, then $y \wedge z = 0$ and $z \vee x = e$ for some

$z \in L$. By $a)$, $z \leq y^*$ so that $e = z \vee x \leq y^* \vee x$. Thus $y^* \vee x = e$.

On the other hand, if $y^* \vee x = e$, then $y \wedge y^* = 0$ and $y^* \vee x = e$ gives $y \prec x$.

Lemma 1.1.7

For any elements x, y in a frame L, the following hold:

$i)$ $x \leq x^{**}$.

$ii)$ If $x < y$, then $y^* \leq x^*$.

$iii)$ $x^{***} = x^*$.

$iv)$ If $x \prec y$ then $x^{**} \leq y$.

$v)$ If $x \prec y$ in L, then $x \leq y$.

$vi)$ If $a \leq x \prec y \leq b$, then $a \prec b$.

$vii)$ If $a \prec c$ and $b \prec c$, then $a \vee b \prec c$.

Proof:

$i)$ Since $x^* \wedge x = 0$ and $x^* \wedge x^{**} = 0$, it follows from Remark 1.1.6. that $x \leq x^{**}$.

$ii)$ Since $x \wedge y \leq^* y \wedge y^* = 0$, it follows that $x \wedge y^* = 0$. But $x \wedge x^* = 0$,

so from Remark 1.1.6 we find that $y^* \leq x^*$.

$iii)$ Since $x \leq x^{**}$, then $x^{***} \leq x^*$. Again $x^* \leq x^{***}$ by $i)$, thus $x^* = x^{***}$.

$iv)$ We only need to show that $x^{**} = x^{**} \wedge y$. We proceed as follows:

$$x^{**} = x^{**} \wedge e$$

$$= x^{**} \wedge \left(x^{*} \vee y\right) \qquad \left(\text{since } x \prec y\right)$$

$$= \left(x^{**} \wedge x^{*}\right) \vee \left(x^{**} \wedge y^{*}\right)$$

$$= 0 \vee \left(x^{**} \wedge y\right)$$

$$= \left(x^{**} \wedge y\right)$$

$v)$ Suppose that $x \prec y$ in L. Then there exists $z \in L$ such that $x \wedge z = 0$ and $y \vee z = e$. So we have

$$x = x \wedge \left(y \vee z\right)$$

$$= \left(x \wedge y\right) \vee \left(x \wedge z\right)$$

$$= \left(x \wedge y\right) \vee 0$$

$$= \left(x \wedge y\right)$$

$$\leq y.$$

$vi)$ Suppose that $a \leq x \prec y \leq b$ in L. Find $z \in L$ such that

$$x \wedge z = 0 \text{ and } y \vee z = e.$$

Then $a \leq x$ implies

$$a \wedge z \leq x \wedge z = 0 \text{, so that } a \wedge z = 0.$$

On the other hand, $y \leq b$ implies that $b \vee z \geq y \vee z = e$ and hence

$b \vee z = e$. Thus $a \prec b$.

$vii)$ Given $a \prec c$ and $b \prec c$. Find w,t such that $w \wedge a = 0, \mathrm{t} \vee \mathrm{c} = \mathrm{e}$.

Also, there exist $p, q \in L$ such that $p \wedge b = 0$, $q \vee c = e.$ Now we have that

$$(w \wedge p) \wedge (a \vee b)$$

$$= (w \wedge p \wedge a) \vee (w \wedge p \wedge b)$$

$$= 0 \vee 0$$

$$= 0$$

and

$$(t \vee q) \vee z$$

$$= (t \vee z) \vee (q \vee z)$$

$$= e \vee e$$

$$= e.$$

On the basis of these equations, we conclude that $a \vee b \prec c$. ■

1.2 Algebraic frames

Let Y be a subset of a topological space X; then a *cover* of Y is a collection of subsets of X whose union contains Y. Classically, a topological space X is *compact* if each open cover of X has a finite subcover. (See, for example, Willard [16].) That is, for every arbitrary collection $\{U_i\}_{i\in A}$ of open subsets of X such that $X = \bigcup_{i\in A} U_i$, there is a finite subset B of A such that $X = \bigcup_{i\in B} U_i$. We now have

Definition 1.2.1 (Martinez [10])

Let L be a frame.

$i)$ An element $x \in L$ is said to be *compact* if whenever $x \leq \bigvee S$ for any $S \subseteq L$ it holds that $x \leq \bigvee F$ for some finite subset F of S. A frame L is said to be *compact* if its top element e is compact. The set of all compact elements of L will be denoted by $\mathcal{C}(L)$. In addition, a frame L is *algebraic* if every element of L is a join of compact elements.

$ii)$ An element x of a frame L is said to be *regular* if

$$x = \bigvee \{y \in L \mid y \prec x\}.$$

A frame L is said to be *regular* if each $x \in L$ is regular.

Pseudo-complementation and complementation are related as follows:

Proposition 1.2.2 (See also Birkhoff [4])

In a frame L, the pseudo-complement is complemented if and only if for any $x, y \in L$, it holds that $x^* \vee y^* = (x \wedge y)^*$.

Proof:

$(\Rightarrow)$: Since $x \wedge y \leq x$ and $x \wedge y \leq y$, it follows from Lemma 1.1.7 that $x^* \vee y^* \leq (x \wedge y)^*$. It remains to show that $(x \wedge y)^* \leq x^* \vee y^*$.

By assumption, the element $(x \wedge y)^*$ is complemented and so

$$(x \wedge y)^* \vee (x \wedge y)^{**} = e.$$

Therefore, we need only show that

$$(x^* \vee y^*) \vee (x \wedge y)^{**} = e$$

to prove that $(x \wedge y)^* = x^* \vee y^*$. To this end we have

$$\begin{aligned}(x \wedge y)^{**} \vee (x^* \vee y^*) &\geq (x \wedge y) \vee (x^* \vee y^*) \\ &= [(x \wedge y) \vee x^* \vee y^*] \\ &= [(x \vee x^*) \wedge (y \vee x^*)] \vee y^* \\ &= [e \wedge (y \vee x^*)] \vee y^* \\ &= y \vee x^* \vee y^* \\ &= e \vee x^* \\ &= e\end{aligned}$$

$(\Leftarrow)$: Suppose $x^* \vee y^* = (x \wedge y)^* = (x \wedge y)^*$, for any $x, y \in L$. We need only show that $x^* \vee x^{**} = e$. In the equation $x^* \vee y^* = (x \wedge y)^*$, we set $y = x^*$ and find that

$$\begin{aligned} x^* \vee x^{**} &= x^* \vee (x^*)^* \\ &= x^* \vee y^* \\ &= (x \wedge y)^* \quad (\text{By assumption}) \\ &= (x \wedge x^*)^* \\ &= 0^* \\ &= e. \end{aligned}$$

and so the pseudo-complement x^* is complemented. ■

We need the following result to prove one of our main results, namely Theorem 1.2.6.

Lemma 1.2.3

If $y \leq z$ and $z \prec x$ in L, then $y \prec x$.

Proof:

Suppose that $y \leq z$ and $z \prec x$. Then $e = z^* \vee x \leq y^* \vee x$, whence $y^* \vee x = e$ so that $y \prec x$. ■

The following result gives a necessary and sufficient condition for the regularity of an element in an algebraic frame.

Proposition 1.2.4 (Martinez [12])

An element x in an algebraic frame L is regular if and only if every compact element $c \leq x$ satisfies $c \prec x$.

Proof:

$(\Rightarrow)$: Suppose that $x \in L$ is regular and that $c \leq x$ is compact. We must show that $c \prec x$. By regularity, we have

$$x = \bigvee \{a \in L \mid a \prec x\}.$$

But $c \leq x$ is compact with $c \leq \bigvee \{a \in L \mid a \prec x\}$, so compactness implies that

$$c \leq \bigvee \{a_i \in L \mid a_i \prec x, \quad i = 1,2,\ldots,n\}.$$

From this observation, it follows that $a_1 \vee a_2 \vee \ldots \vee a_n \prec x$ and so Lemma 1.2.3 ensures that $c \prec x$.

$(\Leftarrow)$**:** Conversely, suppose that any compact $c \leq x$ satisfies $c \prec x$. Since L is regular, we must have that

$$x = \bigvee \{ c \in L \mid c \in \mathcal{C}(L) \}.$$

But each $c \in \mathcal{C}(L)$ with $c \leq x$ satisfies $c \prec x$ by hypothesis, so

$$x = \bigvee \{ c \in L \mid c \prec x, c \in \mathcal{C}(L) \},$$

making x regular as was to be proved. ■

Definition 1.2.5 (Martinez [12])

$i)$ A frame L is said to have the finite Intersection Property (FIP) if for $a,b \in \mathcal{C}(L)$ it holds that $a \wedge b \in \mathcal{C}(L)$.

$ii)$ An element p in L is said to be prime if whenever $x \wedge y \leq p$ and $p < e$ implies that $x \leq p$ and $y \leq p$. The collection of prime elements of L is denoted by $Spec(L)$ and is called the spectrum of the frame. A typical Zorn's Lemma argument guarantees that when primes exist then so do minimal primes. We denote the collection of minimal primes of L by $Min(L)$.

$iii)$ An element $a \in L$ is said to be a d-element if it is expressible in the form

$$a = \bigvee\{ c^{**} \mid c \leq a,\ c \in \mathcal{C}(L) \}.$$

$iv)$ In an algebraic frame L with the FIP, we denote by $Max(L)$ the set of all elements $x < e$ in L that are maximal and by $L_{Max(L)}$ the set of all meet of maximal elements. Moreover,

$$Max^*(x) = \Lambda\{m \in Max(L) \mid m \geq x\}$$

$v)$ Let L be an algebraic frame. Then L is said to have the *Compact Splitting Property* (CSP) if each compact element of L is complemented.

Theorem 1.2.6 (**Characterization of Algebraic Regular Frames**)

In an algebraic frame L, the following are equivalent.

$i)$ *L* is regular.

$ii)$ Each compact element $x \in \mathcal{C}(L)$ satisfies $x \vee x^* = e$, that is each compact element is rather below itself.

$iii)$ *L* has the *FIP* and each prime element is minimal.

Proof:

$i) \Rightarrow ii)$: Assume that L is regular and take a compact $x \in \mathcal{C}(L)$. To see

that $x \vee x^* = e$, we need only show that $x \prec x$. To this end, it follows from regularity that (since L is algebraic)

$$x = \bigvee \{ x_i \in L \mid x_i \prec x, x_i \in \mathcal{C}(L) \}.$$

Since x is compact, we must have that

$$x = \bigvee \{ x_i \in \mathcal{C}(L) \mid x_i \prec x, \quad i = 1, 2, \ldots, n \}.$$

Since each $x_i \prec x$ for $i = 1, 2, \ldots, n$, the observation in first part of the proof of Proposition 1.2.4 ensures

$$x = a_1 \vee a_2 \vee \ldots \vee a_n \prec x$$

which means that

$$x \vee (a_1 \vee a_2 \vee \ldots \vee a)^* = x \vee x^* = e.$$

$ii) \Rightarrow iii)$: Assume that $x \prec x$ and take $x, y \in \mathcal{C}(L)$.

We will show that $x \wedge y \in \mathcal{C}(L)$. To this end, we assume (without loss of generality, (since L is algebraic)) that

$$x \wedge y \leq \bigvee x_i \ \ (x_i \, \mathcal{C}(L)).$$

Since $x = x \vee (x \wedge y)$, we have that

$$x \leq x \bigvee (\bigvee \ x_i \ \ (x_i \in \ \mathcal{C}(L)).$$

But $x \in \mathcal{C}(L)$, so (by re-arrangement if necessary) we find that

$$x \leq x \bigvee (\bigvee \{x_i \mid i = 1,2,\ldots,n\}).$$

And since $x \wedge y \prec x$ we must have that $x \wedge y \leq \bigvee \{x_i \mid i = 1,2,\ldots,n\}$, which proves that $x \wedge y \in \mathcal{C}(L)$.

$iii) \Rightarrow i)$ Suppose that *L* has the *FIP* and that each prime element is minimal. This is equivalent to the Compact Splitting Property (Martinez and Zenk [10, Definition & Remarks 2.1 (i)]), so for each compact element x we will have $x^{**} = x \prec x$. Now given $x \in L$, we have that (since L is algebraic)

$$x \leq \bigvee x_i \, (x_i \leq x, x_i \in \mathcal{C}(L)).$$

It follows then that $x_i \prec x_i \leq x$, hence $x_i \prec x$ (Lemma 1.2.3), showing that x is regular. Consequently, L is regular and $i)$ follows. ■

Definition 1.2.7 (Martinez [12])

Let L be an algebraic frame. We say that L has a disjointification (or simply, That L is a frame with disjointification) if for each pair of compact elements $a,b \in L$ there exists disjoint $c,d \in \mathcal{C}(L)$ such that

$i)$ $c \leq a$ and $d \leq b$, and

$ii)$ $a \vee b = a \vee d = c \vee b$

Definition 1.2.8 (Banaschewski [1])

A subset $F \subseteq L$ with $0 \notin F$ is called a filter if the following conditions hold:

$i)$ $e \in F$.

$ii)$ $a \wedge b \in F$, whenever $a,b \in F$.

$iii)$ $a \in F$ for any $a \geq b$ where $b \in F$.

Definition 1.2.9 (Martinez [12])

Suppose that L is an algebraic frame with the FIP. For $p \in Spec(L)$, define

$$O(p) = \bigvee \{ a^* \mid a \in \mathcal{C}(L),\ a \nleq p \}.$$

Theorem 1.2.10 (Martinez [12])

Suppose that L is an algebraic frame with the FIP. Let $p \in Spec(L)$.

$a)$ If $p \leq q$, then $O(q) \leq O(p)$

$b)$ $O(p)$ is a $d-$element.

$c)$ $q \in Min(L)$ and $q \leq p$ imply that $O(p) \leq q$.

$d)$ If $O(p) \leq q$ and q is a minimal element over $O(p)$ then $q \leq p$.

Proof:

$a)$ Suppose $p \leq q$. Then

$$O(q) = \bigvee \{ a^* \mid a \in \mathcal{C}(L),\ a \nleq q \}$$

$$\leq \bigvee \{ a^* \mid a \in \mathcal{C}(L),\ a \nleq p \}$$

$$= O(p).$$

$b)$ $O(p)$ is a d - element since in the definition above a^* could be replaced by $(a^*)^{**}$ (by Lemma 1.1.7). In its new form, then, $O(p)$ is a d - element.

$c)$ Assume that $q \in Min(L)$ and $q \leq p$. Note that in the definition of $O(p)$ the join in question is over an upward directed set. Thus if $O(p) \nleq q$, there exist disjoint compact elements a and b such that $a \nleq p$ and $b \nleq q$. But then $a \nleq q$ as well because $a \leq q$ would imply $a \leq p$, a contradiction, so $O(p) \leq q$.

$d)$ Suppose $O(p) \leq q$ and q is minimal over $O(p)$. Suppose also that there is a compact element $c \leq q$ such that $c \nleq p$. For each pair of compact elements $a \nleq q$ and $b \nleq p$, we have that $0 < a \wedge b$. This implies that such a and b generate a filter F of compact elements, which is contained in the ultra-filter U (say). If we set

$$m = \bigvee \{a^* \mid a \in U\},$$

we find a minimal prime m satisfying $O(p) \leq m \leq q$. By assumption, since $c \in F \subseteq U$, we therefore conclude that $q \leq p$, as required. ■

Returning to regular elements in L, we denote by $\mathrm{Reg}(L)$ the collection of all regular elements of L and note that there is a natural inclusion $l : \mathrm{Reg}(L) \to L$ such that $l(r) = r$, for each $r \in \mathrm{Reg}(L)$ and $\mathrm{Reg}(L)$ is a sbframe of L. There is then a right adjoint $l_* : L \to \mathrm{Reg}(L)$ of l defined by

$$l_*(x) = \bigvee \{y \in \mathrm{Reg}(L) \mid y \leq x\}.$$

Proposition 1.2.11

For an algebraic frame L with the *FIP* it holds that $l_*(x) = q$, for each $x \in L$ where

$$q = \bigvee \{\, y \in L \mid y \prec x,\ y \in \mathcal{C}(L) \}.$$

Proof:

By definition, each y in the definition of $l_*(x)$ is a regular element. But then Proposition 1.2.4 implies that every compact element $y \leq x$ satisfies $y \prec x$, hence $l_*(x) \leq q$.

For the reverse inequality, suppose that $z \in \mathcal{C}(L)$ and $z \leq q$. Then there are finitely many $y_1, y_2, \ldots, y_n \in \mathcal{C}(L)$ such that $y_i \prec x$ for $i = 1,2,\ldots,n$ and

$x \leq y_1 \vee y_2 \vee \ldots \vee y \prec x$ so that $z \prec x$ by Proposition 1.2.4. This means that $z \leq x$ is a compact element satisfying $z \prec x$, so (by the same result) z must be regular, hence $z \in$ *Reg*(L) and so $l_*(x) \geq q$. ■

Theorem 1.2.12

In an algebraic frame L, if $\prec$ interpolates then the subframe *Reg*(L) is regular.

Proof:

Suppose that $x \in$ *Reg*(L). We will show that there exists an element $y \in L$ satisfying $y \prec x$. Since L is an algebraic frame, we may (and do) choose $z \in \mathcal{C}(L)$ such that $z \leq x$. By Proposition 1.2.4, we know that $z \prec x$.

But $\prec$ interpolates, so some element $y \in L$ exists such that $z \prec y \prec x$. In addition, we find that

$$z \prec l_*(y) \leq y \prec x$$

showing that

$$x = \bigvee \{ l_*(y) \mid y \prec x \text{ interpolates} \}.$$

Hence *Reg*(L) is regular. ■

In relation to regular elements in L, we have

Theorem 1.2.13

An element $x \in L$ is regular if and only if whenever $x \leq p$ then $x \leq O(p)$.

Proof:

$(\Rightarrow)$: Suppose that $x \in L$ is regular such that $x \leq p$, where $p \in Spec(L)$. Suppose, for a contradiction, that $x \nleq O(p)$. Then regularity of x ensures the existence of some compact $y \in L$ such that $y \prec x$ (compactness stems from the fact that L is algebraic) and $x \nleq O(p)$. By definition, we also have $x \vee y^* = e$ whereas $y \nleq O(p)$ implies that $y^* \leq O(p)$, a contradiction to

$y \nleq O(p)$. Hence $x \leq O(p)$.

$(\Leftarrow)$: Conversely, suppose that whenever $x \leq p$ then $x \leq O(p)$. Since L is algebraic, we pick some $y \in \mathcal{C}(L)$ such that $y \leq x$. We claim that $y \prec x$ i.e., $x \vee y^* = e$. For, if $x \vee y^* < e$ were true, then there would be a prime element $p \in L$ such that $x \vee y^* < p$ and so $y \nleq O(p)$, a contradiction to $y \leq x \leq O(p)$. We also have that $x \leq p$ (as $x \vee y^* \leq p$) so that $x \leq O(p)$, which does not make sense either. Therefore, we must have $x \vee y^* = e$ or $y \prec x$, as desired. ■

1.3 Relative notions of regularity

Definition 1.3.1

Given an algebraic frame L in which the FIP holds, we define the following concepts relating to regularity:

$i)$ satisfies $\mathrm{Reg}(1)$ if L is regular.

$ii)$ satisfies $\mathrm{Reg}(2)$ if each $d-$element is regular.

$iii)$ L satisfies $\mathrm{Reg}(3)$ if each pseudo-complement in L is regular.

$iv)$ L satisfies $\mathrm{Reg}(4)$ if, for each compact x, the pseudo-complement x^* is regular.

Remark 1.3.2

Since every regular element is a $d-$element and each pseudo-complement is regular, it easily follows that $\mathrm{Reg}(1)\Rightarrow \mathrm{Reg}(2)\Rightarrow \mathrm{Reg}(3)\Rightarrow \mathrm{Reg}(4)$.

In the following result, we characterize the equivalent conditions $\mathrm{Re}\,g(2)$ and $\mathrm{Re}\,g(3)$.

Observation 1.3.3

$\mathrm{Reg}(2)\Leftrightarrow \mathrm{Reg}(3)$

Proof:

We need only show that *Reg(3)* $\Rightarrow$ *Reg(2)*. Suppose then that *Reg(3)* holds and take a d-element $y\in L$, say

$y = \bigvee \{ x^{**} \mid x \leq y, x \in \mathcal{C}(L) \}$.

By Reg(3), the pseudo-complement x^*, and in particular x^{**}, is regular; so there exists a $z \in L$ such that that $x \prec x^{**}$. But each x^{**} satisfies $x^{**} \leq y$, so Lemma 1.2.3 ensures that $x \prec y$; thus y is regular and *Reg*(2) follows. ■

Proposition 1.3.4

Suppose that L is algebraic with the *FIP*. Then L satisfies *Reg*(2) if and only if every x^{**} is regular for every compact $x \in L$.

Proof:

Let L be algebraic with FIP.

$(\Rightarrow)$: Suppose that L satisfies *Reg*(2) and *let* $x \in \mathcal{C}(L)$. We must show that $x^{**} \vee x^{***} = e$. By Proposition 1.3.4, if $x \in \mathcal{C}(L)$ then x^{**} is regular. Thus

$$x \leq x^{**} = \bigvee \{ y \in L \mid y \prec x \}.$$

Since x is compact, $x \leq y_1 \vee y_2 \vee \ldots \vee y_n$ with $y_i \prec x^{**}$ some $i = 1,2,\ldots,n$. Since $y_i \prec x^{**}, x \leq \bigvee_{i=1}^{n} y_i \prec x^{**}$ so that $x \prec x^{**}$. Hence $x^* \vee x^{**} = e$.

$(\Leftarrow)$: Conversely, suppose that every x^{**} is regular for a compact $x \in L$ and let y be a *d*-element so that

$$y = y = \bigvee \{ x^{**} \mid z \leq y,\ z \in \mathcal{C}(L) \}.$$

By assumption, each z^{**} in the equation is regular (as each z is compact). Therefore, the *d*-element y is a join of regular elements, so (by definition) y must be regular. ■

Lemma 1.3.5

Suppose that L is an algebraic frame with FIP. Then L satisfies *Reg*(2) if and only if every x^{**} is complemented for each compact $x \in L$.

Proof:

Let L be an algebraic frame with FIP.

$(\Rightarrow)$: Suppose that L satisfies *Reg*(2) and let $x \in \mathcal{C}(L)$. We must $x^{**} \vee x^{***} = e$.
By Proposition 1.3.4, if $x \in \mathcal{C}(L)$, then x^{**} is regular. Thus

$$x \leq x^{**} = \bigvee \{y \in L \mid y \prec x^{**}\}.$$

Since x is compact, $x \leq y_1 \vee y_2 \vee \ldots \vee y_n$ with $y_i \prec x^{**}$ for some $i = 1,2,\ldots,n.$ Since $y_i \prec x^{**}$, $x \leq \bigvee_{i=1}^{n} y_i \prec x^{**}$ so that $x \prec x^{**}$. Hence $x \vee x^{**} = e.$

$(\Leftarrow)$: Take a *d*-element $x \in L$ and assume that the condition is satisfied. Then we have that

$$x = \bigvee\{ y^{**} \mid y \leq x, \ \ y \in \mathcal{C}(L) \},$$

with each y^{**} complemented. Then $y^{*} \vee y^{**} = e,$ so that
$y \prec y^{**} \leq x$ and thus $y \prec x$. Hence $x = \bigvee\{y \in L \mid y \prec x\},$ which shows that, x is regular. ■

Combining the above two propositions, we have the following characterisation.

Theorem 1.3.6

In an algebraic frame L with FIP, the element x^{**} is regular if and only if it is complemented, for every compact $x \in L$ ■

Theorem 1.3.7

Let L be an algebraic frame. Then L satisfies *Reg*(1) if and only if it has the Compact Splitting Property.

Proof:

$(\Rightarrow)$: If L is regular, then each element of L is a *d*-element. This means that $c = c^{**}$, for each $c \in \mathcal{C}(L)$. Moreover, *Reg*(2) holds and, therefore, Lemma 1.3.5. Then it is clear that each compact element is complemented. Hence L has the *CSP*.

$(\Leftarrow)$ Follows from Theorem 1.2.6. ■

We will now state without proof the following result:

Lemma 1.3.8 (Martinez and Zenk [10, Lemma 2.2])

Suppose that L is an algebraic frame possessing the FIP. Then $p \in Spec(L)$ is minimal if and only if

$$p = \bigvee \{\ c^{*} \mid c \in \mathcal{C}(L), \quad c \not\leq p\ \}.$$

Theorem 1.3.9 (Knox and McGovern [8, Lemma 3.1])

Suppose L is an algebraic frame.

$i)$ The frame L satisfies *Reg* (4) if and only if for any disjoint $a, b \in \mathcal{C}(L)$,

$a^* \vee b^* = e$.

$ii)$ If L has the FIP and satisfies *Reg*(4) then $p \vee q = e$ for all distinct $p, q \in Min(L)$.

Proof:

$i)$ Suppose *Reg*(4) holds and let a and b be compact elements with $a \wedge b = 0$. Then since a^* is regular, we have from Proposition 1.2.4 that $b \prec a^*$, hence $a^* \vee b^* = e$. Conversely, suppose the condition holds and let $x \in \mathcal{C}(L)$. We need to show that x^* is regular. To this end, take $y \leq x^*$. In view of Proposition 1.2.4, we need only show that $y \prec x^*$, that is, that $x^* \vee y^* = e$.

But this follows from the fact that
Since $y \in \mathcal{C}(L)$ with $y \leq x^*, y \wedge x \leq x \wedge x^* = 0$. Thus $y \wedge x = 0$. Since $x, y \in \mathcal{C}(L)$ are disjoint, by the hypothesis $y^* \vee x^* = e$. Thus $y \prec x^*$ so that by Proposition 1.2.4, x is regular.

$ii)$ Suppose that L has the FIP and satisfies *Reg*(4), and consider distinct minimal primes p and q. By Lemma 1.3.8,

$$p = \bigvee \{ c^* \mid c \in \mathcal{C}(L), c \not\leq p \} \text{ and } q = \bigvee \{ c^* \mid c \in \mathcal{C}(L), c \not\leq q \}.$$

Thus there exist disjoint compact elements a and b such that $b^* \leq q$ and $a^* \leq p$, so that $e = a^* \vee b^* \leq p \vee q$, hence $p \vee q = e$, as claimed. ■

Definition 1.3.10 (Knox and McGovern [8])

$a)$ An element $x \in L$ is said to *zero-dimensional* if it is a join of complemented elements, that is,

$$x = \bigvee c \;\; (c \text{ is complemented}).$$

$b)$ A frame L is said to be *zero-dimensional* when every element is zero dimensional. Equivalently, L is a zero dimensional if every compact element is complemented.

Theorem 1.3.11

A compact algebraic frame L is regular if and only if it is zero-dimensional.

Proof:

This follows from the equivalence of each of these statements to the CSP. See Martinez and Zenk [10, Theorem 2.4]. ■

Definition 1.3.12 (Knox and McGovern [8])

If $x \in \mathcal{C}(L)$ has the property that $x^* = 0$, then x is called a *unit* and the frame L is said to be *possessing a unit*.

Proposition 1.3.13

Suppose that *L* is an algebraic frame possessing a unit, say *u* ∈ *L*. If *L* satisfies *Reg(4)*, then every complemented element is of the form *a*** for some *a* ∈ $\mathcal{C}$(L).

Proof:

Suppose $x \in L$ is a complemented element and let $y = x^*$. Now,

$$u = (x \wedge u) \vee (y \vee u)$$

and since L is algebraic we can write each of the components of u as a join of compact elements. Since u is a unit, it is compact and thus we can write $u = s \vee t$ where $s \leq x, t \leq y,$ and $s,t \in \mathcal{C}(L)$. We claim that $s^{**} = x$. Clearly, $s^{**} \leq x$. Since $s \wedge t = 0$ it follows that $t \leq s^*$, thus $t^{**} \leq s^*$. Again,

$$\begin{aligned}(s^* \wedge t^*) \wedge u &= (s^* \wedge t^*) \wedge (s \vee t) \\ &= (s^* \wedge t^* \wedge s) \vee (s^* \wedge t^* \wedge t) \\ &= 0 \wedge 0 \\ &= 0\end{aligned}$$

from which it follows that $s^* \wedge t = 0$, thus $s^* \leq t^{**}$. We therefore conclude that $s^* = t^{**}$. By assumption, L satisfies *Reg*(4) and since both s and t are compact it follows that $s^* \vee t^* = e$, thus s^{**} and t^{**} is a complementary pair. Since $x \leq x^* < t^* = s^{**} \leq x,\ s^{**} = x$. ■

Chapter 2

Yosida frames

In this chapter, we study the relationship between *complementation, completeness, Boolean Algebra,* and *subfitness* in frames. We introduce Yosida frames and study how they relate to finite subfitness. Some of the results we prove are:

$i)$ Yosida frames are precisely those algebraic frames with the Finite Intersection Property that are finitely subfit (Theorem 2.9).

$ii)$ A semi-simple algebraic frame L with the Finite Intersection Property and disjointification with $\dim(L) \leq 1$ is a Yosida frame (Theorem 2.11).

$iii)$ Suppose that L is an algebraic complete lattice and j is a closure operator. Then $\mathcal{C}(jL) = j\,\mathcal{C}(L)$ (Theorem 2.6).

$iv)$ In an algebraic frame L with the Finite Intersection Property, it holds that $dom(L) = \dim(L)$ (Theorem 2.13).

Definition 2.1 (Martinez and Zenk [11])

An algebraic frame L is a *Yosida frame* if every $x \in \mathcal{C}(L)$ is a meet of maximal elements, thus, L is a Yosida frame if $L = L_{\max(L)}$.

Notation: An element $y \in L$ is a *complement* of an element $x \in L$ if $x \wedge y = 0$ and $x \vee y = e$ in which case y is denoted by $\sim x$. We recall that a frame in which every element has a complement is called a *complemented frame*.

In addition, a *Boolean algebra* is a distributive lattice in which every element is complemented and a *complete Boolean algebra* is a Boolean algebra which is *complete* as a partially ordered set. See Johnstone $[6]$.

Note that one of the important properties of complementation on a Boolean algebra L is the fact that

$$x \wedge y \leq z \Leftrightarrow x \leq \sim y \vee z \qquad \text{for any } x, y, z \in L.$$

Proposition 2.2

Each complete Boolean algebra is a frame.

Proof:

Suppose that L is a complete Boolean algebra. Since L is a complete lattice, we need only show that

$$y \wedge \bigvee X = \bigvee \{ y \wedge x \mid x \in X \}$$

for any $y \in L$ and $X \subseteq L$. Since $y \wedge x \leq y \wedge \bigvee X$ for any $y \in L$ and $x \in X$, we must have that

$$\bigvee \{ y \wedge x \mid x \in X \} \leq y \wedge \bigvee X .$$

On the other hand, for each $y \in L$ we find that

$$x \wedge y \leq \bigvee (y \wedge x)$$

$$\Rightarrow x \leq \sim y \vee [\ \bigvee \{y \wedge x \ \mid x \in X\}]$$

$$\Rightarrow \bigvee x \leq \sim y \vee [\ \bigvee \{y \wedge x \ \mid x \in X\}]$$

So that

$$y \wedge \bigvee X \leq y \wedge \left(\sim y \vee \left[\bigvee \{y \wedge x \mid x \in X\} \right] \right)$$

$$= (y \wedge \sim y) \vee (\ y \wedge [\bigvee y \wedge x \mid x \in X\ ])$$

$$= 0 \vee (\ y \wedge [\bigvee \{y \wedge x \mid x \in X\}])$$

$$= y \wedge [\bigvee \{y \wedge x \mid x \in X\}]$$

$$\leq \bigvee \{y \wedge x \mid x \in X\}.$$

Thus

$$y \wedge \bigvee X \leq \bigvee \{y \wedge x \mid x \in X\} \leq y \wedge \bigvee X$$

So that

$$y \wedge \bigvee X = \bigvee \{y \wedge x \mid x \in X\}.$$

■

Proposition 2.3

The map $j: L \to L$ on a frame L defined by $j(x) = x^{**}$ is a nucleus.

Proof:

$i)$ Since $x \leq x^{**}$ (Lemma 1.1.7), it follows that $x \leq j(x)$.

$ii)$ Note that

$$(x \wedge y) \leq x \Rightarrow x^{*} \leq (x \wedge y)^{*} \qquad \text{so that} \quad (x \wedge y)^{**} \leq x^{**}$$

and, similarly, $(x \wedge y)^{**} \leq y^{**}$ hence $(x \wedge y)^{**} \leq x^{**} \wedge y^{**}$. It remains to show that $(x \wedge y)^{**} \geq x^{**} \wedge y^{**}$. But this follows from the calculations:

$$\begin{aligned}(x \wedge y)^{**} \wedge (x \wedge y)^{*} &= \left(x^{**} \wedge y^{**}\right) \wedge \left(x^{*} \vee y^{*}\right) \\ &= \left[x^{**} \wedge \left(x^{*} \vee y^{*}\right)\right] \wedge \left[y^{**} \wedge \left(x^{*} \vee y^{*}\right)\right] \\ &= \left[\left(x^{**} \wedge x^{*}\right) \vee \left(x^{**} \wedge y^{*}\right)\right] \wedge \left[\left(y^{**} \wedge x^{*}\right) \vee \left(y^{**} \wedge y^{*}\right)\right] \\ &= \left[0 \vee \left(x^{**} \wedge y^{*}\right)\right] \wedge \left[\left(y^{**} \wedge x^{*}\right) \vee 0\right] \\ &= \left(x^{**} \wedge y^{*}\right) \wedge \left(y^{**} \wedge x^{*}\right) \\ &= 0.\end{aligned}$$

Thus $x^{**} \wedge y^{**} \leq (x \wedge y)^{**}$ so that $x^{**} \wedge y^{**} = (x \wedge y)^{**}$. Therefore,

$$j(x \wedge y) = (x \wedge y)^{**} = x^{**} \wedge y^{**} = j(x) \wedge j(y).$$

$iii)$ To prove that $(j(x)) = j(x)$, we have

$$j(j(x)) = j(x^{**})$$

$$= x^{****}$$

$$= x^{**} \qquad (\text{Lemma } 1.1.7)$$

$$= j(x).$$

Hence j is a nucleus. ∎

Remark 2.4 (See Vickers [17])

For any x, y in a frame L, we define

$$x \to y = \bigvee \{a \in L \mid x \wedge a \leq y\}.$$

It is immediate that

$x \longrightarrow 0 =$ $x \to 0 = \bigvee \{a \in L \mid x \wedge a \leq 0\} = x^*$

and

$$x \wedge (x \to y) = x \wedge \bigvee \{a \mid x \wedge a \leq y\}$$

$$= \bigvee \{x \wedge \mathrm{a} \mid x \wedge a \leq y\}$$

$$\leq y.$$

Observation: From this definition, we note that

$$(x \to y) \wedge (x \to z) = \bigvee\{p \in L \mid x \wedge p \leq y\} \wedge \left(\bigvee\{t \in L \mid t \wedge x \leq z\}\right)$$

$$= \bigvee\{s \in L \mid x \wedge s \leq y \wedge z\}$$

$$= x \to (y \wedge z),$$

which helps in the proof of the following

Proposition 2.5 (Martinez and Zenk [9])

If the operator $j : L \to L$ is a nucleus and $y \in jL = Fix(j)$ then $x \to y \in jL$.

Proof:

Suppose that $j : L \to L$ is a nucleus and take $y \in jL$. Then $j(y) = y$. We must prove that $j(x \to y) = x \to y$. Since $x \to y \leq j(x \to y)$, we only need to show that $x \to y \geq j(x \to y)$. But this follows from

$$y = j(y)$$

$$\Rightarrow y \wedge j(y) = j(y)$$

$$\Rightarrow x \to j(y) = x \to (y \wedge j(y))$$

$$= (x \to y) \wedge (x \to j(y))$$

$$\Rightarrow x \to j(y) \leq x \to y.$$

■

Theorem 2.6

Suppose that L is an algebraic complete lattice and j is a closure operator. Then $\mathcal{C}(jL) = j\,\mathcal{C}(L)$.

Proof:

$i)$ To show that $\mathcal{C}(jL) \subseteq j\,\mathcal{C}(L)$, take $x \in \mathcal{C}(jL)$. Then x is compact and $x \in j(L)$, thus $j(x) = x$. We have $j(x) = \bigvee j(a)$, where $a \in \mathcal{C}(L)$ and $a \leq x$, so that $x \in j\,\mathcal{C}(L)$.

$ii)$ On the other hand we will show that $j\,\mathcal{C}(L) \subseteq \mathcal{C}(jL)$. Take $x \in j\,\mathcal{C}(L)$. Then $x = j(y)$ with $y \in \mathcal{C}(L)$. To see that $x \in jL$, we note that

$$
\begin{aligned}
j(x) &= j(j(y)) \\
&= j(y) \qquad (j \text{ is closure operator}) \\
&= x.
\end{aligned}
$$

To see that x is compact with this property, suppose that $x \leq \bigvee_{i \in I} x_i$. *Since* $x = j(y)$ and $y \leq j(y)$, we must have that $y \leq \bigvee_{i \in I} x_i$. But $y \in \mathcal{C}(L)$, we must have $y \leq \bigvee_{i \in F} x_i$ *for some finite* $F \subseteq I$. We then take $\{x \wedge x_i \mid i \in F\}$ and note that $x \leq \bigvee \{x \wedge x_i \mid i \in F\}$; thus x is compact. ■

Prime elements in L are related to $Fix(j) = jL = \{x \wedge L \mid j(x) = x\}$ as follows:

Proposition 2.7

For a frame L and a nucleus $j : L \to L$ it holds that

$$Spec(Fix(j)) = Fix(j) \cap Spec(L).$$

Proof:

Let $p \in Spec(Fix(j))$. Then p is prime in L.To see this, suppose that $x, y \in L$ and $x \wedge y \le p$. Then $j(x) \wedge j(y) = j(x \wedge y) \le j(p) = p$. Since $j(j(x)) = j(x)$ and $j(j(y)) = j(y)$, both $j(x)$ and $j(y)$ are elements of $Fix(j)$. Since p is prime in $Fix(j)$ and $j(x) \wedge j(y) \le p$, we have $x \le j(x) \le p$ and $y \le j(y) \le p$ so that $x \le p$ and $y \le p$. Hence $p \in Spec(L)$. Thus $p \in Fix(j) \cap Spec(L)$ so that

$$Spec(Fix(j)) \subseteq Fix(j) \cap Spec(L).$$

On the otherhand, if $p \in Fix(j) \cap Spec(L)$, then $j(p) = p$ and p is prime in L. Now let $x, y \in Fix(j)$ and $x \wedge y \le p$. Since p is prime in L we immediately have that $x \le p$ and $y < p$. Consequently, $p \in Spec(Fix(j))$. Thus $Fix(j) \cap Spec(L) \subseteq Spec(Fix(j))$. ■

Definition 2.8

Following Martinez and Zenk [11], we say that a frame L is *subfit* if whenever $x < y$ in L there exists a $x \in L$ satisfying $x \vee z < y < z = e$ An algebraic frame L is said to be *finitely subfit* if whenever $x < y$ in L with both $x, y \in \mathcal{C}(L)$ there exists a $z \in L$ satisfying $x \vee z < y \vee z = e$.

In relation to finite subfitness, we have the following characterization

Theorem 2.9 (Yosida Characterization Theorem)

An algebraic frame L with the Finite Intersection Property is a Yosida frame if and only if it is finitely subfit.

Proof:

$(\Rightarrow)$:Suppose that L is Yosida frame and take $x, y \in \mathcal{C}(L)$ such that $x < y$. By definition, we have $x = \bigwedge\{z \in L : z \in Max(L)\}$ and so there exists some $m \in Max(L)$ for which $x \leq m$ and $y \vee m = e$ which shows that such m satisfies finite subfitness.

$(\Leftarrow)$:Conversely, suppose that L is finitely subfit and take $x, y \in \mathcal{C}(L)$ with $x < y$. Then by finite subfitness, we find $z \in L$ satisfying $x \vee z = e$ and $y \vee z = e$. We also have that

$$y \vee (x \vee z) = x \vee (y \vee z) = x \vee e = e$$

But then $y \not< x \vee y$, so by compactness of $y \in \mathcal{C}(L)$ and appealing to Zorn's Lemma we infer that $x \vee z$ is maximal with respect to the conditions

$$x \leq x \vee z < e \text{ and } y \vee (x \vee z) = e.$$

For, suppose that $k \in L$ satisfies the condition that $x \leq k < e$ and $y \leq k < e$. With $x \vee z \leq k$, we claim that $k = x \vee z$. This follows from the calculations:

$$k = k \wedge e$$

$$= k \wedge [y \vee (x \vee z)]$$

$$= (k \wedge y) \vee [k \wedge (x \vee z)]$$

$$= (k \wedge y) \vee (x \vee z) \qquad (\text{since } x \vee k \leq k)$$

$$= [k \vee (x \vee z)] \wedge [y \vee (x \vee z)]$$

$$= (x \vee z) \wedge e$$

$$= x \vee z.$$

This argument is enough to conclude that for each compact element $x \in \mathcal{C}(|L)$ it holds that $x =$ *Max*(x)* and, therefore, L is a Yosida frame. ■

To prove another characterization of Yosida frames, we need to define the *dimension* of an algebraic frame *L* which is given in (Martinez [10]).

A chain of primes $p_0 < p_1 < p_2 < \ldots < p_k$ is said to be of *length k*, and the *dimension dim*(L) of L is the maximum of the lengths of chains of primes. In addition, we mention without proof the following (Martinez [11]).

Theorem 2.10 ("Prime-free" Criterion for *dim*$(L) \leq k$*)*

In a Yosida Frame L*, dim*$(L) \leq k$ if and only if for each chain $p_0 < p_1 < p_2 < \ldots < p_{k+1}$ of nonzero compact elements of L there exists $q_1, q_2, q_3, \ldots, q_{k+1} \in \mathcal{C}(L)$ such that

$$p_i \vee q_{i+1} = p_{i+1} \quad \text{for each} \quad i = 0,1,2,\ldots,k, \quad \text{and} \quad p_0 \wedge q_1 \wedge q_{k+1} = 0 .$$

Recalling Martinez [11], we say a frame L is *semi-simple* if $\bigwedge \max(L) = 0$ which we need in the following

Theorem 2.11

If L is semi-simple algebraic frame with the FIP and disjointification with $\dim(L) \leq 1$, then L is a Yosida frame.

Proof:

In view of the Characterization Theorem 2.9, we need only prove that L is finitely subfit. Assume then that $x, y \in \mathcal{C}(L)$ with $x < y$.

Note that if $y = e$, then L is finitely subfit and the result follows.

We therefore assume without loss of generality that $y < e$. Then, by the "Prime–free" Criterion Theorem, we find compact elements $z,t \in \mathcal{C}(L)$ satisfying:

$$x \wedge z \wedge t = 0, \; x \vee z = y \text{ and } y \vee t = e .$$

If $x \vee t < e$, then L is finitely subfit and the result follows. Therefore, we assume that $x \vee t = e$. Then (easily from the above equations)

$$x \vee z \wedge t = (x \vee z) \wedge (x \vee t)$$

$$= (x \vee z) \wedge e$$

$$= x \vee z$$

$$= y.$$

Thus $x \wedge (z \wedge t) = 0$ and $x \vee (z \wedge t) = y$ making $z \wedge t$ a complement of x in $\downarrow y$.

On the other hand, we also have that

$$x \vee (y \wedge t) = (x \vee y) \wedge (x \vee t)$$

$$= (x \vee y) \wedge e$$

$$= x \vee y$$

$$= y$$

and, similarly,

$$x \wedge (y \wedge t) = (x \wedge y) \wedge (x \wedge t)$$

$$= (x \wedge y) \wedge 0$$

$$= 0.$$

Since complements are unique (if they exist), it follows that $z \wedge t = y \wedge t$. Since $x \vee t = e$ (by assumption) and $x \wedge t = 0$, it follows that x is complemented. Since $\uparrow (x) = \{y \in L \mid \geq x\}$, it follows that

$$\Lambda (Max[\uparrow (x)]) = x,$$

and so $\uparrow (x)$ is also semi-simple, hence x is the meet of maximal elements. Since $x < y$, we conclude that maximal elements $m \in Max[\uparrow (x)]$

exist with $m \geq x$ such that $y \not\leq m$, giving rise to $y \vee m = e$. Thus, L is finitely subfit. ■

Definition 2.12 (Martinez [10])

$a)$ Suppose that L is an *algebraic frame* and $a_0 < a_1 < a_2 < \ldots < a_k$ is a chain of compact elements of . We say that it is a dominance chain of length if *there is a prime element of such that, in* $\uparrow p$,

$$p < a_0 \vee p < \ldots < a_k \vee p < \ldots$$

The *dominance* of L, *denoted dom*(L), is the join of the lengths of dominance chains of L.

$b)$ A chain $a_0 < a_1 < a_2 < \ldots < a_n < \ldots$ is an *ascending dominance chain* if there is a prime element p such that

$$p < a_0 \vee p < \ldots < a_k \vee p$$

If $p_0 < p_1 < p_2 < \ldots < p_k$ is a chain of primes, we may find, for each $i = 0,1,2,\ldots,k$, *a compact element* a$_i$ *such that* $a_i < p_{i+1}$ for each $i = 0,1,2,\ldots,k-1$, *and* $a_i \not< p_i$ for each $i = 0,1,2,\ldots,k$. Without loss of generality we may assume that $a_0 < a_1 < a_2 < \ldots < a_k$. It is easy to see that

$$p_0 < a_0 \vee p_0 < \ldots < a_k \vee p_0$$

and so

$$a_0 < a_1 < a_2 < \ldots < a_k$$

is a *dominance chain.* Thus, $\dim(L) \leq dom(L)$.

We now provide a condition for the reverse inequality to hold.

Theorem 2.13

If L is an algebraic frame with disjointification, then $\dim(L) = dom(L)$.

Proof:

We only need to show that $dom(L) \leq \dim(L)$. To this end suppose that $a_0 < a_1 < a_2 < \ldots < a_k$ is a dominance chain. *As* $p < a_i \vee p$, we may select a prime $p_i \geq p$ which is maximal with respect to $a_{i-1} \vee p \leq p_i$ and $a_i \vee p \not\leq p$ (for each $i = 1,2,\ldots,k$) and $a_0 \vee p \not\leq p_0$. Since $\uparrow p$ is a chain, we have

$$p \leq p_0 \leq a_0 \vee p \leq p_1 < \ldots \leq p_k < a_k \vee p.$$

In particular, $p_0 < p_1 < p_2 < \ldots < p_k$, which proves that $dom(L) \leq \dim(L)$, *as* claimed. ■

Chapter 3

Coherent normal Yosida frames

This chapter is concerned with the relationship between *normality*, *codenseness* and *coherent* frames. We also show how *normal subfitness* relates to regularity. Some of the results we prove in this chapter are the following:

$i)$ A normal subfit frame is regular (Proposition 3.1.6)

$ii)$ The following statements are equivalent for a normal coherent frame L

Theorem 3.1.11)

$a)$ $L = SL$.

$b)$ L is subfit.

$c)$ L is regular.

$d)$ L is a zero- dimensional.

$e)$ Every compact element of L is complemented.

3.1 Normality in Yosida Frames

Motivating Example:

Recall that a topological space (X,τ) is normal if whenever A,B are disjoint closed subsets of X with $A \bigcup B = X$ then there exists disjoint open subsets U,V such that $A \subseteq U$ and $B \subseteq V$. It follows then that $X - A$, X-B are open subsets of X satisfying

$$B \subseteq X - A,\ A \subseteq X\text{-}B \text{ and } (X - A) \bigcup (X - B) = X.$$

In particular, it holds (easily) that for the open subsets $X - A$, X-B the disjoint open subsets U,V satisfy

$$U \bigcup (X - A) = X = V \bigcup (X - B)$$

which paves way for the following Pointfree version of normality.

Definition 3.1.1 (Banaschewski [3])

$a)$ A frame L is *normal* if whenever $x \vee y = e$ in L then there exist $s,t \in L$ such that $s \vee x = e = y \vee t$ and $s \wedge t = 0$.

$b)$ Given a distributive lattice B, with top e and bottom 0, we call a subset $j \subseteq B$ an *ideal* of B if it satisfies the following two conditions (see Johnstone$[6]$:

$i)$ $x \wedge y \in j$ for all $x, y \in j$.

$ii)$ If $x \leq y$ and $y \in j$ then $x \in j$.

For an algebraic frame L with the FIP, normality and $O(p)$ for $p \in Spec(L)$ are related as follows:

Theorem 3.1.2

Suppose that L is normal and let $m,n. \in Max(L)$.

$a)$ If m,n are distinct, then $O(m) \vee O(n) = e$.

$b)$ $O(m)$ is regular.

$a)$ By normality, we have $a \leq m$ and $b \leq n$ with $a \wedge b = 0$ and $a \vee n = b \vee m = e$.

Clearly, $a \nleq n$, which means that $b \leq O(n)$ so that $e = b \vee m \leq O(n) \vee m$. Since e is the top element, we must have $m \vee O(n) = e$. Now, if $O(m) \vee O(n) \leq e$, there is a maximal $q \geq O(m) \vee O(n)$. Then $q = m$, and similarly, $q = n$, making $m = n$,

a contradiction to the fact that m,n are distinct, hence $O(m) \vee O(n) = e$.

$a)$ If $p \in Spec(L)$ and $O(m) \leq p$, then $p \leq n$, for a suitable maximal n. By $a)$ it follows that $n = m$, and thus, $O(m) \leq O(p)$, which proves that $O(m)$ is regular. ■

Definition 3.1.3 (See also Banaschewski [3])

A frame L is said to be coherent if and only if it is an algebraic frame and satisfies the finite Intersection Property.

There is one- one relationship between coherent frames and distributive lattices in the sense that

Theorem 3.1.4 (See Siweya [15])

$i)$ If L is a coherent frame, then $\mathcal{C}(L)$ is a distributive lattice with top e And bottom 0.

$ii)$ Given a distributive lattice B with top e and bottom 0, then the lattice $J(B)$ of all ideals of B is a coherent frame.

In fact, we can say more and do better about this relationship as follows: Now, denote by ***D*** the category of all distributive lattices with top e and bottom

0 and by $\mathcal{C}\mathbf{h}\mathbb{F}\mathbf{rm}$ the category of all coherent frames and coherent frame homomorphisms.

Theorem 3.1.5

There is a categorical equivalence

$$\mathcal{D} \underset{\ell}{\overset{J}{\rightleftarrows}} \mathcal{C}\mathbf{h}\mathbb{F}\mathbf{rm} \text{ between categories } \mathcal{D} \text{ and } \mathcal{C}\mathbf{h}\mathbb{F}\mathbf{rm}.$$

Proof:

For $J\colon\ \mathcal{D} \to \mathcal{C}\mathbf{h}\mathbb{F}\mathbf{rm}$, if $B \in Ob(\mathcal{D})$ then $J(B)$ is the lattice of all ideals of B which is known to be coherent by the Theorem 3.1.4. For a $\mathcal{D}$-morphism $B \xrightarrow{f} D$ we have that

$$J\left(B \xrightarrow{f} D\right) = J(B) \xrightarrow{J(f)} J(D,)$$

a frame morphism between coherent frames, where

$$J(f)\left[\left\langle b_i \mid i \in I\right\rangle\right] = \left\langle J[f(b_i)] \mid i \in I\right\rangle$$

Here $\left\langle b_i \mid i \in I\right\rangle$ denotes the ideal of B that is generated by $\left\langle b_i \mid i \in I\right\rangle \subseteq B$ Since $f(b_i) \in D$, it follows that $\left\langle f(b_i) \mid i \in I\right\rangle$ is the ideal of D that is generated by $\left\{f(b_i) \mid i \in I\right\} \subseteq D$.

On the other hand, given a $\mathcal{C}\mathbf{h}\mathbb{F}\mathbf{rm}$-object of L we define $\mathcal{C}(L)$ to be the distributive lattice (with top e and bottom 0) consisting of all compact elements of L. If $L \xrightarrow{f} M$ is a $\mathcal{C}\mathbf{h}\mathbb{F}\mathbf{rm}$-morphism, then $\mathcal{C}\left(L \xrightarrow{f} M\right) = \mathcal{C}(L) \xrightarrow{C(f)} \mathcal{C}(M)$ where for each $x \in \mathcal{C}(L)$ we have

$$(\mathcal{C}(f))(x) = \mathcal{C}[f(x)]$$

which is compact. To complete the proof, given a $\mathcal{D}$-*morphism* $B \xrightarrow{f} D$ we find that

$$C[J(B \xrightarrow{f} D)] = C[J(B)] \xrightarrow{J(f)} C[J(D,)]$$
$$= C \circ J(f)$$

so that $\mathcal{C}oJ = I_{\mathcal{D}}$, the identity functor on the category $\mathcal{D}$. On the other hand, if $L \xrightarrow{f} M$ is a $\mathcal{C}\mathbf{h}\mathbb{F}\mathbf{rm}$-morphism, then

$$J[\mathcal{C}(L) \xrightarrow{C(f)} \mathcal{C}(M)] = Jo\mathcal{C}(L) \xrightarrow{JoC(f)} [Jo\mathcal{C}](M)$$

$$= [Jo\mathcal{C}](f)$$

giving rise to $Jo\mathcal{C} = I_{\mathcal{C}\mathbf{h}\mathbb{F}\mathbf{rm}}$, the identity functor on $\mathcal{C}\mathbf{h}\mathbb{F}\mathbf{rm}$. Then the equivalence follows. ■

In the following result, we provide a "weak partial converse" to the result of the well known teacher J. Isbell [5, Theorem 2.3]).

Proposition 3.1.6 (Banaschewski [3, Lemma 1.1])

Any normal subfit frame is regular.

Proof:

Let L be a normal subfit frame and take $x \in L$. Suppose, for a contradiction, that

$$z = \bigvee y \in L \mid y \in x$$

Now, if $z < x$ then by subfitness there is some $w \in L$ for which $z \vee w < e = x \vee w$. Since L is normal, there exists a disjoint pair $s, t \in L$ such that

$$x \vee s = e = w \vee t .$$

But then $s \wedge t = 0$ and $x \vee s = e$ is the same as $t \prec x$ and so $t \leq z$ (by Lemma $1.1.7(v)$), giving rise to

$$e = t \vee w \leq z \vee w$$

Hence $z \vee w = e$, a contradiction. ■

In the following result we show that normality is preserved by condense homomorphisms.

Proposition 3.1.7 (Banaschewski [3])

Any codense image of normal frame is normal.

Proof:

Suppose that $L : M \to L$ is a codense onto frame homomorphism and that M is a normal frame. We will show that L is also normal.

To this end, take $x, y \in L$ such that $x \vee y = e$. Since M is onto, we pick $m, n \in M$ such that $f\ (m) = x, f(n) = y$. By $x \vee y = e$ we have that

$$f(m \vee n) = f(m) \vee f(n) = x \vee y = e ,$$

so codenseness of f ensures that $m \vee n = e$ in M. By normality of M, there exist disjoint $s, t \in M$ such that:

$$m \vee s = e = n \vee t .$$

In particular, we also find that

$$f(s)\wedge f(t)= f(s\wedge t)= f(0)0,$$

and

$$x\vee f(s)= f(m)\vee f(s)$$

$$= f(m\vee s)$$

$$= f(e)$$

$$= e$$

$$= f(n\vee t)$$

$$= y\vee f(t).$$

Thus, $f(s)$ and $f(t)\in L$ satisfy the normality condition and L is therefore normal. ■

Definition 3.1.8 (Banaschewski [3])

In a compact frame L, an element $x\in L$ is said to be *a-small* for any $a\in L$ if whenever $x\vee y=e$, then $a\vee y=e$ for all $y\in L$.

Remark 3.1.9

We observe that the set $J=\{x\in L\,|\,x$ is a-small$\}$ is an ideal and $\bigvee J$ is also *a*-small containing the element $a\in L$. In fact, it is the largest *a*-small element of L, which is denoted by $S(a)=\bigvee\{x\in L\,|\,x$ is a-small$\}$. The resulting map $s:L\to L$, given by $s(a)=S(a)$, is a codense nucleus such that, for

$SL = Fix(s)$, $s : L \to SL$ is the unique smallest codense quotient of L (Banaschewski [3]).

Proof:

We show that $J = \{x \in L \mid x \text{ is a small}\}$ is an ideal for each $a \in L$. Note that

$$e = (x \wedge y) \vee z$$

$$= (x \vee y) \wedge (y \vee z)$$

$$\leq (x \vee z).$$

Thus $x \vee z = e$. Since x is a-small, $a \vee z = e$. Thus $x \wedge y$ is a-small.

$b)$ Suppose that $x \leq y$ and $y \in J$. We claim that x is also a-small. To this end, suppose that $x \vee z = e$. We will show that $a \vee z = e$. But this follows from the fact that

$$e = x \vee z \leq y \vee z \Rightarrow y \vee z = e$$

and since y is a-small, we must have $a \vee y = e$. On the basis of $a)$ and $b)$, we conclude that J is an ideal. ■

Proposition 3.1.10

A compact frame L is normal if and only if SL is regular.

Proof:

$(\Rightarrow)$: Suppose that L is compact and normal. Then SL is subfit (Banaschewski [3, Lemma 1.2]) and since $s: L \to SL$ is a codense quotient of L, it follows from Proposition 3.1.7 that SL is normal. But a normal subfit frame is regular, so Proposition 3.1.6 ensures that SL is regular.

$(\Leftarrow)$: Conversely, suppose that L is compact and SL is regular. We must show that L is normal. To this end, pick $x, y \in L$ such that $x, y = e$ in L. Applying $s: L \to L$, we have $s(x) \vee s(y) = e$ in SL.

Note that SL is compact since s is codense, so there exist $p \prec s(x)$ and $q \prec s(y)$ in SL, by regularity, such that $p \vee q = e$ in SL which implies that

$$s(p \vee q) = s(p) \vee s(q) = e.$$

But *s* is codense, so we must have $p \vee q = e$ in L. Now, if $u, t \in SL$ are such that

$$p \wedge u = 0,\ s(x) \vee u = e,\ q \wedge t = 0 \text{ and } s(y) \vee t = e \ in \ SL$$

from which it follows that

$$s(x \vee u) = s(x) \vee s(u)$$

$$= s(x) \vee u$$

$$= e$$

$$= s(y) \vee t$$

$$= s(y) \vee t$$

$$= s(y) \vee s(t)$$

$$= s(y \vee t)$$

and, by codenseness of s, we arrive at

$$x \vee u = e = y \vee t$$

On the other hand, we also have

$$u \wedge t = e \wedge (u \wedge t)$$

$$= (p \vee q) \vee (u \wedge t)$$

$$= (p \wedge u \wedge t) \vee (q \wedge u \wedge t)$$

$$= 0 \vee 0$$

$$= 0,$$

proving that L is normal as desired. ■

Our main result is

Theorem 3.1.11

The following statements are equivalent for a normal coherent frame L

$i)$ $L = S\mathrm{L}$.

$ii)$ L is subfit.

$iii)$ L is regular.

$iv)$ L is a zero-dimensional.

$v)$ Every compact element of L is complemented.

Proof:

$i) \Rightarrow ii)$: Since SL is known to be subfit (Banaschewski [3, Lemma 1.2]), it follows that if $L = SL$ then L is subfit as well.

$ii) \Rightarrow iii)$: Suppose that L is subfit. Then regularity of L is immediate since L is normal (Proposition 3.1.7)

$iii) \Rightarrow iv)$: Suppose that L is regular. By coherence, every element of L is a joint of compact elements. Given a compact element $x \in L$, we have that (by regularity)

$$x = V\{y \in L \mid y \prec x\}$$

since x is compact, $x = \overset{n}{\underset{i=1}{V}}\{y_i \in L \mid y_i \prec x\}$.

However, $y_i \prec x$ for each $i = 1,2,\ldots,n$ gives $x = \overset{n}{\underset{i=1}{V}}\{y_i \in L \mid y_i \prec x\} \prec x$ so that $x \prec x$.

Hence $x \vee x^* = e$ which shows that x is complemented and hence L is zero dimensional.

$iv) \Rightarrow v)$: Suppose L is zero-dimensional and take a compact element $x \in L$. By coherence and zero- dimensionality of L, the compact element $x \in L$ is a join of finitely many complemented elements, so it is complemented.

$v) \Rightarrow i)$: Suppose every compact element of L is complemented and let $x < y$ in L. Since L is coherent, there is a compact $z \leq y$ such that $z \neq a.$ Now, we have (since z is complemented)

$$y \vee z^* \geq z \vee z^* = e \Rightarrow y \vee z^* = e.$$

But $x \vee z^* < y \vee z^* = e$, so we must have $s(x) = x$, establishing the result that $L = SL$. ■

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