ON CLOSED AND QUOTIENT MAPS OF LOCALES

by

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DECLARATION

"I declare that the (mini)-dissertation hereby submitted to the University of Limpopo, for the degree of Master of Science has not previously been submitted by me for a degree at this or any other university; that it is my work in design and in execution, and that all material contained therein has been duly acknowledged."

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Date

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ABSTRACT

The category $\mathbb{L}\mathbf{oc}$ of locales and continuous maps is dual to the category $\mathbb{F}\mathbf{rm}$ of frames and frame homomorphisms. Regular subobjects of a locale A are elements of the form

$$A_j = \{j : A \to A \mid j(a) = a\}.$$

The subobjects of this form are called *sublocales* of A. They arise from the lattice $\mathcal{O}X$ of open sets of a topological space X in a natural way. The right adjoint of a frame homomorphism maps closed (dually, open) sublocales to closed (dually, open) sublocales.

Simple coverings and separated frames are studied and conditions under which they are closed (or open) are those that are related to coequalizers are shown. Under suitable conditions, simple coverings are regular epimorphisms.

Extremal epimorphisms and strong epimorphisms in the setting of locales are studied and it is shown that strong epimorphisms compose. In the category $\mathbb{L}\mathbf{oc}$ of locales and continuous maps, closed surjections are regular epimorphisms at least for those surjections with subfit domains.

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SUMMARY

This mini-dissertation is based on the paper "Quotient Maps of Locales" of Till Plewe where it was shown, among other results, that closed surjections with subfit domains are regular epimorphisms. In the said paper, most of the proofs are not detailed. The aim of this dissertation is to use Plewe's paper as a starting point to study closed and quotient maps of locales. We provide detailed proofs of selected results in Plewe's paper and a few other papers. All references are fully acknowledged.

In Chapter 1 (**Closed Morphisms and Complemented Sublocales**), we look at closedness and complementation in the setting of frames. A condition under which a frame homomorphism is closed is given and some properties of complemented sublocales are studied.

The existence of separated frames and coequalizers is studied in Chapter 2 (**Simple coverings in locales**) in which their relationship is presented. We also show that simple coverings, under suitable conditions, are regular epimorphisms.

The centre of attraction in this mini-dissertation is Chapter 3 (**Quotient Maps of Locales**) wherein we study the concept of extremal and strong epimorphisms and show that strong epimorphisms compose (Section 3.1) and, in Section 3.2 we study the characterization of quotient maps in locales and show that closed surjections whose domains are subfit are regular epimorphisms.

Chapter 1

Closed Morphisms and Complemented Sublocales

This chapter is concerned with the relationship between the category $\mathbb{T}\mathbf{op}$ of (all) topological spaces and continuous functions, the category $\mathbb{F}\mathbf{rm}$ of (all) frames and frame homomorphisms, and the category $\mathbb{L}\mathbf{oc}$ of (all) locales (as generalized spaces) and locale maps.. We use a familiar result in the theory of locales to understand why a closed frame homomorphism is defined in terms of its adjoint. Properties related to (complemented) sublocales are also studied.

1.1 Introduction to Pointless Topology

If X is a topological space, the partially ordered set of open subsets of X is a complete lattice, in which the **infinite distributive law**

$$U \wedge \bigvee \mathfrak{B} = \bigvee \{U \wedge S \mid S \in \mathfrak{B}\}$$

holds for all open subsets U of X and collections $\mathfrak{B} \subseteq \mathcal{P}(X)$ of open subsets of X. A *frame* is an abstract lattice with this property. In a frame L the bottom element is denoted by 0 and the top element by e.

Given a topological space (X, τ_X), we consider the collection $\mathcal{O}X$ of all open subsets of X. Then the following properties in $\mathcal{O}X$ are well known:

(i) $U, V \in \mathcal{O}X \Rightarrow U \cap V \in \mathcal{O}X$;

(ii) $\varnothing, X \in \mathcal{O}X$;

(iii) If $V \in \mathcal{O}X$ and $\mathcal{U}$ is a subcollection from $\mathcal{O}X$, then

$$V \cap (\bigcup \mathcal{U}) = \bigcup_{U \in \mathcal{U}} (V \cap U).$$

Recall that for any topological space X, the family $\mathcal{O}X$ of open subsets of X uniquely determine the topology on X. In fact, for the family $\mathcal{O}X$ to be a topology on X it should contain the empty set $\emptyset$ and the whole set X as well as be closed under finite intersections, say $\bigcap\{A_k \subseteq X \mid 1 \leq k \leq n\}$ and arbitrary unions, say $\bigcup_{k \in I} \{A_k \subseteq X\}$. *What this means is that the collection*

$(X, \mathcal{O}X, \subseteq, \cap, \cup, \emptyset)$ *is a typical example of a frame.*

Definition 1.1.1

A *frame homomorphism* $h : L \to M$ between frames satisfies the following properties:

(h_1) $h(0) = 0$; $h(e) = e$.

(h_2) $h(x \wedge y) = h(x) \wedge h(y)$, $\forall\ x, y \in L$.

(h_3) $h(\bigvee_{i \in I} x_i) = \bigvee_{i \in L} h(x_i)$.

Now given a continuous function $f : (X, \tau_X) \to (Y, \tau_Y)$ between topological spaces, consider the function $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ defined by $(\mathcal{O}f)(V) = f^{-1}(V)$ for each $V \in \mathcal{O}Y$. Then we easily prove that

(i) $\mathcal{O}f\,(U \cap W) = \mathcal{O}f\,(U) \cap \mathcal{O}f\,(V)$;

(ii) $\mathcal{O}f(\varnothing) = \varnothing$ and $\mathcal{O}f(Y) = X$;

(iii) $(\mathcal{O}f)\left(\bigcup_{i \in I} V_i\right) = \bigcup_{i \in I} (\mathcal{O}f)\,(V_i)$.

By (i) and (iii), it follows that for any $V, V_i \in \mathcal{O}Y$ we have

$$(\mathcal{O}f)\left(V \cap \bigcup_{i \in I} V_i\right) = \bigcup_{i \in I} [(\mathcal{O}f)\,(V) \cap (\mathcal{O}f)\,(V_i)],$$

which shows that $\mathcal{O}f$ is a frame homomorphism.

Note that frame maps are closed under composition and hence we have the category $\mathbb{F}\mathbf{rm}$ of (all) frames and frame homomorphisms. The dual category $\mathbb{F}\mathbf{rm}^{op}$ to $\mathbb{F}\mathbf{rm}$ is the category $\mathbb{L}\mathbf{oc}$ whose objects are frame objects but whose morphisms are defined in the opposite direction to those of $\mathbb{F}\mathbf{rm}$.

Remark 1.1.2

(a) A topological space, in a classical sense, is often described in terms of its points and sometimes in terms of a system of open sets. These open sets form a (complete) lattice with the operations of intersection and union. The study of lattices like these is what *pointless or pointfree topology* is all about. There are many lattices whose origin is not associated with open sets derived from a topological space.

(b) The category of pointless topological spaces (which are called *locales*) is an extension of the category $\mathbb{T}\mathbf{op}$ of ordinary topological spaces and continuous functions.

(c) $\mathbb{F}\mathbf{rm}$-morphisms are functions $f : M \to L$ that preserves finite meets (including the top element e) and arbitrary joins (including the bottom element 0). It follows immediately that frame homomorphisms are *monotone* in the sense that if $x \leq y$ in M then $f(x) \leq f(y)$ in L: for, we have that

$$\begin{aligned} x &\leq y \\ \Rightarrow \quad & x = x \wedge y \\ \Rightarrow \quad f(x) &= f(x \wedge y) = f(x) \wedge f(y) \\ \Rightarrow \quad f(x) &\leq f(y). \end{aligned}$$

(d) The $\mathbb{L}\mathbf{oc}$-morphisms are opposite (in terms of domain and codomain) to $\mathbb{F}\mathbf{rm}-$morphisms. In a sense $\mathbb{L}\mathbf{oc}$-morphisms correspond to continuous functions in $\mathbb{T}\mathbf{op}$ as follows: for any function $f : X \to Y$, the inverse map $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ between the corresponding power sets is a homomorphism of complete Boolean algebras. Therefore, if $(X, \tau_X), (Y, \tau_Y)$ are topological spaces with lattices $\tau_X = \mathcal{O}X$, $\tau_Y = OY$ in $\mathcal{P}(X)$, $\mathcal{P}(Y)$ respectively and $f : X \to Y$ is continuous, then $f^{-1} : \tau_Y = \mathcal{O}Y \to \mathcal{O}X = \tau_X$ preserves finite meets and arbitrary joins making $\mathcal{O}(f) = f^{-1}$ a frame homomorphism whenever f is a continuous function. This exposition shows that there is a functor $\mathcal{O} : \mathbb{T}\mathbf{op} \to \mathbb{L}\mathbf{oc}$ for which $\mathcal{O}(X \xrightarrow{f} Y) = \mathcal{O}X \xrightarrow{\mathcal{O}(f)} \mathcal{O}Y$. The opposite functor is one that takes a continuous function $X \xrightarrow{f} Y$ in $\mathbb{T}\mathbf{op}$ to a $\mathbb{F}\mathbf{rm}$-morphism $\mathcal{O}Y \xrightarrow{f^{-1}} \mathcal{O}X$. In fact, $\mathcal{O} : \mathbb{F}\mathbf{rm} \to \mathbb{T}\mathbf{op}$ is a contravariant functor which shows that topological properties can also be conveniently analysed through frame theoretical techniques (Simmons [15]).

Definition 1.1.3

(a) An element $y \in L$ is a *complement* of an element $x \in L$ if $x \wedge y = 0$ and $x \vee y = e$ in which case y is denoted by $\neg x$.

(b) A *complemented frame* is one in which every element has a complement.

(c) A *Boolean algebra* is a distributive lattice in which every element is complemented and a *complete Boolean algebra* is a Boolean algebra which is *complete* as a partially ordered set.

Note that one of the important properties of complementation on a Boolean algebra L is the fact that

$$x \wedge y \leq z \Longleftrightarrow x \leq \neg y \vee z$$

for any $x, y, z \in L$. We use this characterization to prove that a complete Boolean algebra is a frame.

Proposition 1.1.4 (Simmons [15])

Each complete Boolean Algebra is a frame.

Proof:

Suppose that L is a complete Boolean algebra. Since L is a complete lattice, we need only show that

$$y \wedge \bigvee X = \bigvee \{y \wedge x \mid x \in X\}$$

for any $y \in L$ and $X \subseteq L$. Since $y \wedge x \leq y \wedge \bigvee X$ for each $y \in L$ and $x \in X$, we must have that

$$\bigvee \{y \wedge x \mid x \in X\} \leq y \wedge \bigvee X.$$

On the other hand, for each $y \in L$ we find that

$$
\begin{aligned}
x \wedge y &\leq \bigvee (y \wedge x) \\
\Rightarrow x &\leq \neg y \vee \left[\bigvee_{x \in L} (y \wedge x) \right] \\
\Rightarrow \bigvee_{x \in L} x &\leq \neg y \vee \left[\bigvee_{x \in L} (y \wedge x) \right] \\
\Rightarrow y \wedge \bigvee_{x \in L} x &\leq \bigvee_{x \in L} (y \wedge x),
\end{aligned}
$$

completing the reverse inclusion. □

Definition 1.1.5 (See e.g. Chen [3])

Given a frame homomorphism $h : L \to M$, its *right adjoint* is the map $h_* : M \to L$ defined by

$$h_*(x) = \bigvee \{y \in L \mid h(y) \leq x\}$$

for each $x \in M$.

Remark 1.1.6

(a) It follows that

$$\begin{aligned} h\left[h_*(x)\right] &= h\left[\bigvee_{h(y)\leq x} y\right] \\ &= \bigvee_{h(y)\leq x} h(y) \\ &\leq x \end{aligned}$$

so that $h_*(x) = \max\{y \in L \mid h(y) \leq x\}$ for each $x \in M$.

(b) In fact, h_* commutes with arbitrary meets in the sense that if $x_\alpha \in M$ for each $\alpha \in \mathcal{A}$ then

$$\begin{aligned} h_*(\textstyle\bigwedge_{\alpha\in\mathcal{A}} x_\alpha) &= \max\{y \in L \mid h_*(y) \leq \textstyle\bigwedge_{\alpha\in\mathcal{A}} x_\alpha\} \\ &= \max\{y \in L \mid h(y) \leq x_\alpha, \text{ for all } \alpha \in \mathcal{A}\} \\ &= \max\{y \in L \mid y \leq h_*(x_\alpha), \text{ for all } \alpha \in \mathcal{A}\} \\ &= \textstyle\bigwedge_{\alpha\in\mathcal{A}} h_*(x_\alpha). \end{aligned}$$

(c) In particular, $h_*(e) = e$ - following easily from the definition. However, unlike the frame homomorphism $h : L \to M$ the right adjoint h$_*$ does not commute with arbitrary joins.

A function $f : X \to Y$ between topological spaces is said to be *closed* if $f(K)$ is closed for each closed $K \subseteq X$. We note that closedness so defined *is not* expressible in terms of topologies and the inverse function $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$

derived from a continuous function $f : X \to Y$ between topological spaces X and Y. For instance, consider an infinite topological space (Y, τ_Y) with a fixed point, say $y_0 \in Y$ and take the closed subsets of Y to be Y itself and the finite subsets U such that $y_0 \notin U$. Set $X = Y - \{y_0\}$ and let $f = i : X \hookrightarrow Y$, the inclusion map. Then i is not closed but $i^{-1} : Y \to X$ is an isomorphism of topologies. To show that $i : X \to Y$ is not closed, we need only produce a closed subset K of X for which $i(K)$ is not closed. Given a closed subset A of X, say $A = X \cap A$, set $K = Y$. (Note that Y is closed implies that $X \cap Y$ is closed in X.) Then

$$\begin{aligned} i(A) &= i(X \cap Y) \\ &= i(X) \\ &= X \\ &\neq Y \end{aligned}$$

which is finite, and therefore not closed. Therefore i is not closed. See Dowker and Strauss [4].

Example 1.1.7

(a) For a continuous function $f : (X, \tau_X) \to (Y, \tau_Y)$ between topological spaces, the frame homomorphism $\mathcal{O}f : \mathcal{O}X \to \mathcal{O}Y$ defined in the usual sense $(\mathcal{O}f)(U) = f^{-1}(U)$ for $U \subseteq Y$ is closed if and only if whenever $K \subseteq X$ is closed and H is a closed subset of $f\left(\bar{K}\right)$ then $H \bigwedge f(K)$ is dense in H. Consequently, if f is a closed function so is f^{-1}.

(b) In particular, if Y is a T_1-space and f^{-1} is closed so is f. Suppose that

$K \subseteq Y$ is closed. Claim that $f^{-1}(K)$ is closed (i.e., f^{-1} is closed). But we have $Y - K$ is open in Y, so continuity of f implies that $f^{-1}(Y - K)$ is open, i.e. $X - f^{-1}(K)$ is open, making $f^{-1}(K)$ closed as desired. □

1.2 Closed Homomorphisms and Complemented Sublocales

Because every complemented sublocale has a unique (dual) complemented sublocale it follows from the Duality Principle that the dual result of the one proved hereunder is equally true.The following two results are due to Johnstone [9].

Definition 1.2.1 (Johnstone [9])
Given a frame L, a *nucleus* on L is a function $j : L \to L$ satisfying the following conditions:

(N1) $a \leq j(a), \forall\, a \in L$.

(N2) $j(j(a)) \leq j(a), \forall\, a \in L$.

(N3) $j(a \wedge b) = j(a) \wedge j(b), \forall\, a, b \in L$.

Remark 1.2.2
(a) Note that $j(j(a)) = j(a)$ because $a \leq j(a) \Rightarrow j(a) \leq j(j(a))$ and by (N2) we have $j(j(a)) \leq j(a)$.

(b) The collection of all nuclei on a frame L will be denoted by $N(L)$. A well known fact about $N(L)$ is Proposition 1.2.4

Proposition 1.2.3

Note that for a subset $S \subseteq N(L)$, the function $\bigwedge S : L \to L$ defined by

$$\bigwedge S(a) = \bigwedge\{j(a), a \in L \mid j \in S\}$$

is a nucleus on L.

Proof:

(i) Let $j \in S$ and a$\in L$. Then since j is a nucleus on S, we have that

$$\begin{aligned} a &\leq j(a) \\ \Rightarrow a &\leq \bigwedge\{j(a) \mid j \in S\} \\ \Rightarrow a &\leq \bigwedge S(a) \end{aligned}$$

(ii) Take $a, b \in L$ and $j \in S$. Then since j is a nucleus on S, it follows that

$$\begin{aligned} j(a \wedge b) &= j(a) \wedge j(b) \\ \Rightarrow \bigwedge \{j(a \wedge b) \mid j \in S\} &= \bigwedge\{j(a) \wedge j(b) \mid j \in S\} \\ &= \bigwedge\{j(a) \mid j \in S\} \wedge \bigwedge\{j(b) \mid j \in S\} \\ \Rightarrow \bigwedge S(a \wedge b) &= \bigwedge S(a) \wedge \bigwedge S(b) \end{aligned}$$

(iii) Finally, we have

$$\begin{aligned} \bigwedge S[\bigwedge S(a)] &= \bigwedge\{j(\bigwedge S(a)) \mid j \in S\} \\ &\leq \bigwedge\{j(j(a)) \mid j \in S\} \\ &= \bigwedge\{j(a) \mid j \in S\} \\ &= \bigwedge S(a). \end{aligned}$$

□

Proposition 1.2.4 (See e.g. Simmons [15], Yongming [18])
The collection $N(L) = \{j : L \to L \mid j \text{ is a nucleus}\}$ is a frame.

Proof:

Suppose that $j, k : L \to L$ are nuclei on L. Define

$$(j \to k)(a) = \bigwedge\{(j(b) \to k(b)) \mid b \geq a\}.$$

(i) Let $a, b \in L$ such that $b \geq a$. Now since k is a nucleus on L, it follows that $k(b) = b \geq a$ and hence $j(b) \rightarrow k(b) \geq a$, so that $a \leq (j \rightarrow k)(a)$.

(ii) Suppose that $a, b, c \in L$ satisfy $c \geq a \wedge b$. Then setting $c = d \wedge e$ with $d \geq a$ and $e \geq b$ we find that

$$\begin{aligned} j(c) \rightarrow \quad k(c) &= j(d \wedge e) \rightarrow k(d \wedge e) \\ &= j(d \wedge e) \rightarrow (k(d) \wedge j(e)) \\ &= [j(d \wedge e) \rightarrow k(d)] \wedge [j(d \wedge e) \rightarrow k(e)] \\ &= [(j(d) \wedge j(e)) \rightarrow k(d)] \wedge [(j(d) \wedge j(e)) \rightarrow k(e)] \\ &\geq [j(d) \rightarrow k(d)] \wedge [j(e) \rightarrow k(e)] \end{aligned}$$

So that

$$\bigwedge\{(j(c) \rightarrow k(c)) \mid c \geq a \wedge b\} \geq \bigwedge\{(j(d) \rightarrow k(d)) \mid d \geq a \text{ } or \text{ } d \geq b\}.$$

i.e, $(j \rightarrow k)(a \wedge b) \geq (j \rightarrow k)(a) \wedge (j \rightarrow k)(b)$.

Conversely, for $c \geq a \wedge b$, we find from the definition of $(j \rightarrow k)$ above that

$$\begin{aligned} (j \rightarrow \quad k)(c) \leq &\quad (j \rightarrow k)(a \wedge b) \\ = &\quad (j \rightarrow k)(a) \wedge (j \rightarrow k)(b) \end{aligned}$$

so that

$$\bigwedge\{(j \rightarrow k)(a \wedge b) \mid c \geq a \wedge b\} \leq \bigwedge\{(j \rightarrow k)(d) \wedge (j \rightarrow k)(d) \mid d \leq a \text{ } or \text{ } d \leq b\}.$$

Hence

$$(j \rightarrow k)(a \wedge b) = (j \rightarrow k)(a) \wedge (j \rightarrow k)(b).$$

(iii) To verify $(N2)$ we observe that for $a, b \in L$ such that $b \geq a$, then the definition of $j \to k$ ensures that $(j \to k)(a) \leq j(b) \to k(b)$, so that

$$j[j(b) \to k(b)] \to k[j(b) \to k(b)] \geq (j \to k)(j \to k)(a)$$

But $j(b) \to k(b) \in L_k$, because for any subset Y of a locale X is a sublocale if and only if for $y \in Y, x \in X$ implies that $(y \to_X x) \in Y$, it follows that

$$\begin{aligned} j[j(b) \to k(b)] &\to k[j(b) \to k(b)] = j[j(b) \to k(b)] \to j(b) \to k(b) \\ &= [j[j(b) \to k(b)) \wedge j(b)] \to k(b) \\ &= [j(j(b) \to k(b)) \wedge b] \to k(b) \\ &= j(b) \to k(b) \end{aligned}$$

since $b \leq k \leq j(b) \to k(b)$. But this is true for any $b \geq a$, so

$$(j \to k)(j \to k)(a) \leq \bigwedge \{j(b) \to k(b) \mid b \geq a\}.$$

Thus $(j \to k) \in N(L)$. Now for any $a \in L$ we have

$$\begin{aligned} j(a) \wedge (j \to k)(a) &\leq j(a) \wedge (j(a) \to k(a)) \\ &\leq k(a) \end{aligned}$$

i.e, $j \wedge (j \to k) \leq k$ in L. But if l is any nucleus with $j \wedge l \leq k$ then for any $b \geq a$ we have $j(b) \wedge l(b) \leq k(b)$ and so $l(a) \leq l(b) \leq j(b) \to k(b)$. Hence $l(a) \leq (j \to k)(a)$. □

Definition 1.2.5 (Plewe [12])

(a) Given a nucleus j on L, the set $L_j = \{a \in L \mid j(a) = a\}$ is called a *sublocale* of L. (The set of (all) sublocales on a locale X will be denoted by

$Sub(X)$.)

(b) Let $R, S \in Sub(X)$. Then we define

(i) $R/S = \bigvee\{T \in Sub(X) \mid T \leq R, T \wedge S = 0\}$.

(ii) $Sup(S) = \bigwedge\{R \in Sub(X) \mid R \vee S = X\}$.

(iii) A sublocale Y is *complemented* if $Y \wedge Sup(Y) = 0$.

(iv) Let Y be a sublocale of X. Then

$$Y = \bigwedge\{N \in Sub(X) \mid N \wedge Sup(X) = 0, Y \leq N\}.$$

Observation 1.2.6

$S \vee Sup(X) = X$

Proof:

We have

$$\begin{aligned} S \vee Sup(X) &= S \vee \bigwedge\{R \in Sub(X) \mid R \vee S = X\} \\ &= \bigwedge\{S \vee R \in Sub(X) \mid R \vee S = X\} \\ &= \bigwedge\{X \mid R \vee S = X\} \\ &= X. \end{aligned}$$

□

Remark 1.2.7

Given a locale L, each element $a \in L$ induces *open* (respectively, *closed*) *sublocales* as follows:

$$\hat{a} : L \to \downarrow a,\ x \mapsto a \wedge x \quad \check{a} : L \to \uparrow a,\ x \mapsto a \vee x$$

which are known to be *complements* to each other. See Herrlich and Pultr [5].

Motivating Example 1.2.8

Let $f : M \to L$ be any locale map in $\mathbb{L}\mathbf{oc}$, C a complemented sublocale of L and S a sublocale of M. Then $f[S \wedge f^{-1}(C)] = f(S) \wedge C$.

Proof:

(i) $f[S \wedge f^{-1}(C)] \leq f(S) \wedge C$: Take a complemented sublocale $D \in Sub(L)$ such that $f(S) \wedge C \leq D$. We claim that $f[S \wedge f^{-1}(C)] \leq D$. We have that

$$\begin{aligned} f(S) \wedge C &\leq D \\ \Rightarrow f^{-1}\left[f(S) \wedge C\right] &\leq f^{-1}(D) \\ \Rightarrow f^{-1}[f(S)] \wedge f^{-1}(C) &\leq f^{-1}(D). \end{aligned}$$

But $S \leq f^{-1}[f(S)]$ and $f[f^{-1}(D)] \leq D$, so (with $f(S) \wedge C \leq D$)

$$\begin{aligned} S \wedge f^{-1}(C) &\leq f^{-1}[f(S)] \wedge f^{-1}(C) \leq f^{-1}(D) \\ \Rightarrow f[S \wedge f^{-1}(C)] &\leq f[f^{-1}(D)] \leq D. \end{aligned}$$

(ii) $f(S) \wedge C \leq f[S \wedge f^{-1}(C)]$: Suppose that $f[S \wedge f^{-1}(C)] \leq E$, for a complemented $E \in Sub(L)$. We must show that $f(S) \wedge C \leq E$. Then

$$
\begin{aligned}
f[S \wedge f^{-1}(C)] &\leq E \\
\Rightarrow f^{-1}[f(S) \wedge f^{-1}(C)] &\leq f^{-1}(E) \\
\Rightarrow S \wedge f^{-1}(C) \leq f^{-1}[f(S \wedge f^{-1}(C))] &\leq f^{-1}(E) \\
\Rightarrow f[S \wedge f^{-1}(C)] \leq f[f^{-1}(E)] &\leq E \\
\Rightarrow f(S) \wedge f[f^{-1}(C)] &\leq E.
\end{aligned}
$$

□

We now have

Definition 1.2.9

A frame homomorphism $h : L \to M$ is said to be *closed* if

$$h_*[h(x) \vee y] = x \vee h_*(y)$$

for any $x \in L$ and $y \in M$.

Proposition 1.2.10

The following are equivalent for a frame homomorphism $h : M \to L$:

(a) h is closed.

(b) If $h(x) \leq y \vee h(z)$ then $x \leq h_*(y) \vee z$ for any $x, z \in M$ and $y \in L$.

Thus, $h_* : Sub(L) \to Sub(M)$ maps closed sublocales to closed sublocales.

Proof:

$(a) \Rightarrow (b)$: Suppose that h is closed and assume that $h(x) \leq y \vee h(z)$. We must show that $x \leq h_*(y) \vee z$: It follows from the assumption that

$$\begin{aligned} h(x \vee z) \vee y &= [h(x) \vee h(z)] \vee y \\ &= h(x) \vee [y \vee h(z)] \\ &= y \vee h(z) \end{aligned}$$

But h is closed, so (applying this twice) we find that

$$\begin{aligned} h_* [h(x \vee z) \vee y] &= (x \vee z) \vee h_*(y) \\ &= h_*[y \vee h(z)] \\ &= z \vee h_*(y) \end{aligned}$$

from which the first and the third equalities imply that $x \leq z \vee h_*(y)$.

(b)$\Rightarrow$(a) : Suppose that $h(x) \leq y \vee h(z)$ and $x \leq h_*(y) \vee z$ for all $x, z \in M$ and all $y \in L$. We must show that h is closed. That h is closed follows from the following calculations:

(i) $x \vee h_*(y) \leq h_* [h(x) \vee y]$: We have (by definition of h_*)

$$
\begin{aligned}
h \circ h_*(y) &\leq y \\
\Rightarrow h\,(x) \vee h \circ h_*(y) &\leq h(x) \vee y \\
\Rightarrow h\,[x \vee h_*\,(y)] &\leq h(x) \vee y \\
\Rightarrow x \vee h_*(y) &\leq h_*\,[h\,(x) \vee y]
\end{aligned}
$$

(ii) $h_*\,[h(x) \vee y] \leq x \vee h_*\,(y)$: Since $h \circ h_*\,(y) \;\leq\; y$ we have

$$h\,[h_*\,(h(x) \vee y)] \;\leq\; h(x) \vee y,$$

and together with the assumption we find that (with x replaced by $h_*[h(x) \vee y]$ on the left side of the above inequality)

$$h_*\,[h(x) \vee y] \;\leq\; h_*\,(y) \vee x.$$

By (i) and (ii), we conclude that h is closed. □

In view of this result, the following is immediate:

Corollary 1.2.11

The following are equivalent for a frame homomorphism $h : M \to L$:

(i) h is open.

(ii) $h_* : Sub(L) \to Sub(M)$ maps open sublocales to open sublocales. □

Remark 1.2.12

According to Isbell [8], an element $a \in A$ (where A is a complete lattice) is said to be *linear* if it holds that $a \wedge \left(\bigvee_\alpha b_\alpha\right) = \bigvee_\alpha (a \wedge b_\alpha)$. Thus A is a locale if and only if every element of A is linear.

(i) If A is a locale and $a_1,\ a_2 \in A$ are linear, then

$$\begin{aligned}
(a_1 \wedge a_2) \wedge (\bigvee_\alpha b_\alpha) &= a_1 \wedge \left[a_2 \wedge \left(\bigvee_\alpha b_\alpha\right)\right] \\
&= a_1 \wedge \left[\bigvee_\alpha (a_2 \wedge b_\alpha)\right] \\
&= \bigvee_\alpha [a_1 \wedge (a_2 \wedge b_\alpha)] \\
&= \bigvee_\alpha [(a_1 \wedge a_2) \wedge b_\alpha]
\end{aligned}$$

so that finite meets of linear elements in A are linear.

(ii) Linear sublocales of a locale are the complemented ones: For given an open x in X, and any a, b in the underlying lattice on X satisfy

$$\begin{aligned} & a \wedge \left[x \wedge \left(\bigvee_\alpha y_\alpha\right)\right] &=& \; b \wedge \left[x \wedge \left(\bigvee_\alpha y_\alpha\right)\right] \\ \Leftrightarrow \; & (a \wedge x) \wedge \left(\bigvee_\alpha y_\alpha\right) &=& \; (b \wedge x) \wedge \left(\bigvee_\alpha y_\alpha\right) \\ \Leftrightarrow \; & \bigvee_\alpha [(a \wedge x) \wedge y_\alpha] &=& \; \bigvee_\alpha [(b \wedge x) \wedge y_\alpha] \end{aligned}$$

Dually, the same argument holds for any closed x in X and any a, b in the underlying lattice X. Now we also note that a frame homomorphism $h : L \to M$ is open iff it has a left adjoint (i.e., it preserves joins) and preserves the Heyting implication $(\to)$ (see Johnstone [9]).

Theorem 1.2.13

Suppose that X is any locale and A, $A_i \in sub(X)(i \in I)$ are arbitrary sublocales of X. Then

(1) $A \wedge \bigvee A_i = \bigvee A \wedge A_i$ iff A is complemented.

(2) $A \wedge \bigvee A_i = \bigvee A \wedge A_i$ if all the A_i are open.

(3) $X/A = Sup(A)$.

(4) If C and D are complemented sublocales of X, then

$$C \vee A \;=\; D \vee A \; iff \; C \wedge (X/A) \;=\; D \wedge (X/A).$$

Proof:

(1) This follows from the above Remark 1.2.12 (ii).

(2) Since the order-theoretical dual of the collection $N(X)$ of all nuclei on X is the poset $Sub(X)$ of all sublocales of X we will prove the result for $N(X)$. Recall that for every $i \in I$ we have that $j \vee u(i) = i \rightarrow j(-)$ from which it follows that in $N(X)$

$$\begin{aligned}
\bigvee_{i\in I}[j \vee u(i)] &= \bigwedge_{i\in I}[i \rightarrow j(-)] \\
&= \left(\bigvee I\right) \rightarrow j\,(-) \\
&= j \vee u\left(\bigvee I\right) \\
&= J \vee \left(\bigwedge_{i\in I} u(i)\right)
\end{aligned}$$

a clear dual of what was to be proved.

(3) $Sup(A) = \bigwedge\{T \in Sub(X) \mid T \vee A = X\}$. Suppose that $T \vee A = 0$ and $R \vee A = X$. Then we have that

$$
\begin{aligned}
T &= T \wedge X \\
&= T \wedge (R \vee A) \\
&= (T \wedge R) \vee (T \wedge A) \\
&= (T \wedge R) \vee 0 \\
&= T \wedge R \\
&\leq R.
\end{aligned}
$$

Since T and R are any sublocales of X, it follows that

$$
\begin{aligned}
X/A &= \bigvee\{T \in Sub(X) \mid T \wedge A = 0\} \\
&\leq \bigwedge\{R \in Sub(X) \mid R \vee A = X\} \\
&= Sup(A).
\end{aligned}
$$

It remains to show that $Sup(A) \leq X/A$. Take a complemented sublocale $C \in Sub(X)$ such that $X/A \leq C$. We claim that $Sup(A) \leq C$. This proves the result because by contraposition we have $C \leq Sup(A)$ implies that $C \leq X/A$. Note that

$$\begin{aligned} Sup(A) &\leq C \\ \Leftrightarrow A \vee Sup(A) &\leq A \vee C \\ \Leftrightarrow X &\leq A \vee C \\ \Leftrightarrow A \vee C &= X \end{aligned}$$

so we need only show that $A \vee C = X$. To this end suppose that (D, E) is a pair of sublocales of X such that $D \vee E = X$ and $D \wedge E = 0, A \vee C \leq D$. Then

$$\begin{aligned} E \wedge A &\leq E \wedge (C \wedge A) \\ &\leq E \wedge D \\ &= 0. \end{aligned}$$

Hence

$$A \leq A \wedge C \;\Rightarrow\; E \wedge \; A \leq \; E \wedge (C \wedge A) \leq E \wedge D \;=\; 0$$

$$\Rightarrow\; E \wedge A \;=\; 0$$

$$\Rightarrow\; E \leq X/A \leq C \;\leq C \;\wedge A \;\leq\; D.$$

Together with $D \vee E = X$ and $D \wedge E = 0$, this means that $E = 0$, so that $D = X$ and hence $A \vee C = X$.

(4) Suppose that C and D are complemented sublocales of X satisfying $C \vee A = D \vee A$ for $A \in Sub(X)$, and assume that $T \wedge A = 0$ where T is any sublocale of X. Observing that

$$\begin{aligned} C \wedge T &= (C \wedge T) \vee (T \wedge A) \\ &= (C \vee A) \wedge T \\ &= (D \vee A) \wedge T \\ &= (D \wedge T) \vee (A \wedge T) \\ &= (D \wedge T) \vee 0 \\ &= D \wedge T, \end{aligned}$$

we find that (since $A \wedge Sup(A) = 0$)

$$
\begin{aligned}
C \wedge (X/A) &= C \wedge \bigvee \{T \in Sub(X) \mid T \wedge A = 0\} \\
&= \bigvee \{C \wedge T \mid T \in Sub(X) \text{ and } T \wedge A = 0\} \\
&= \bigvee \{D \wedge T \mid T \in Sub(X) \text{ and } T \wedge A = 0\} \\
&= D \wedge \bigvee \{T \in Sub(X) \mid T \wedge A = 0\} \\
&= D \wedge (X/A)\,.
\end{aligned}
$$

Conversely, suppose that $C \wedge (X/A) = D \wedge (X/A)$. Then

$$
\begin{aligned}
C \vee A &= (C \vee A) \wedge X \\
&= (C \vee A) \wedge [(X/A) \vee A] \\
&= [C \wedge (X/A)] \vee A \\
&= [D \wedge (X/A)] \vee A \\
&= (D \vee A) \wedge [(X/A) \vee A] \\
&= (D \vee A) \wedge X \\
&= D \vee A. \qquad \square
\end{aligned}
$$

Chapter 2

Simple Coverings in Locales

We study separated frames and show that these are those that are related to coequalizers (Theorem 2.1.7). Simple coverings (due to Plewe and equivalent to Banachewski's singly generated frames) are studied and conditions under which they are closed surjections (or open surjections) are given. We also show that simple coverings, under suitable conditions, are regular epimorphisms. Unless stated otherwise, the concepts used in this chapter follow from Banachewski [1] and Plewe [12].

2.1 Coproduct and Separated frames

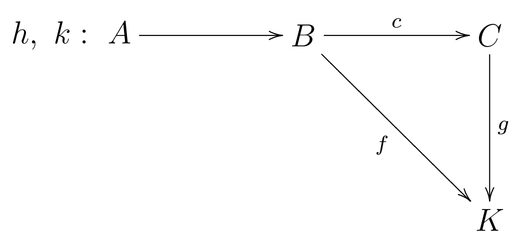
.

Definition 2.1.1 (See Herrlich and Strecker [6])

Given an abstract category $\mathcal{C}$, a $\mathcal{C}$-morphism $c : B \to C$ is called a *coequalizer* of two $\mathcal{C}$-morphisms $h, k : A \to B$ if the following conditions hold:

(i) $c \circ h = c \circ k$;

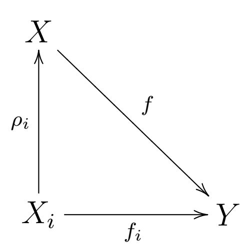
(ii) for any $\mathcal{C}-$morphism $f : B \to K$ with $f \circ h = f \circ k$, there exists a unique $\mathcal{C}-$morphism $g : C \to K$ such that the following diagram commutes: i.e., $g \circ c = f$.



In this case, we write $(c, C) \approx Coeq(h, k)$.

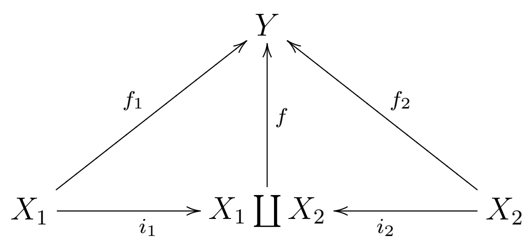
Definition 2.1.2

Let $\mathcal{C}$ be an abstract category and let $\{X_i \mid i \in I\}$ be an indexed family of $\mathcal{C}$-objects. The *coproduct* of the set $\{X_i \mid i \in I\}$ is a $\mathcal{C}$-object X together with a collection $\{\rho_i : X_i \to X \mid i \in I\}$ of $\mathcal{C}$-morphisms which satisfies the property that for any $\mathcal{C}$-object Y and any collection $\{f_i : X_i \to Y \mid i \in I\}$ of $\mathcal{C}$-morphisms there exists a unique $\mathcal{C}$-morphism $f : X \to Y$ such that $f_i = f \circ \rho_i$, that is, the following diagram commutes (for each $i \in I$):



Remark 2.1.3

The coproduct of the family $\{X_i \mid i \in I\}$ is often denoted by $X = \coprod_{i \in I} X_i$ or $X = \bigoplus_{i \in I} X_i$. Sometimes the morphism $f : X \to Y$ may be denoted by $f = \coprod_{i \in I} f_i : \coprod_{i \in I} X_i \to Y$ to indicate its dependence on the individual f_i. If the family of objects consist of only two members the product is usually written $X_1 \amalg X_2$ or $X_1 \amalg X_2$ or sometimes $X_1 \oplus X_2$ and the diagram takes the form



The unique arrow $f : X_1 \amalg X_2 \to Y$ making this diagram commutes is then correspondingly denoted $f_1 \amalg f_2$ or $f_2 \amalg f_1$ or $f_1 \oplus f_2$. It is easily proved that coproducts are unique whenever they exist.

Definition 2.1.4

For a frame homomorphism $h : L \to M$, the *dense decomposition* of h is

$$L \xrightarrow{(.)\vee t} \uparrow t \xrightarrow{\bar{h}} M, \qquad t = \bigvee h^{-1}\{0\}$$

where $[(.) \vee t](x) = x \vee t$ and $\bar{h}(y) = h(y)$ and $\bar{h}$ is called the *dense component* of h. Moreover, if h is onto it is called *closed* whenever $\bar{h}$ is an isomorphism. If $\bar{h}(y) = 0$ then $y = t$, the zero of $\uparrow t$ since the elements of $\uparrow t$ are of the form $y \vee t$. Precisely, $\uparrow t = \{y \in L \mid y \geq t\}$.

Remark 2.1.5

For any frame L, let $L \oplus L$ be the coproduct of L with itself, with the coproduct maps $i, j : L \to L \oplus L$ and the *codiagonal* map $\delta : L \oplus L \to L$ defined by $\delta \circ i = id_L = \delta \circ j$ where

$$i(x) = x \oplus e, \quad j(x) = e \oplus x, \quad i(x) \wedge j(y) = x \oplus y$$

for all x, $y \in L$. This says that

$$\begin{aligned}
\delta(x \oplus y) &= \delta\left[i\left(x\right) \wedge j\left(y\right)\right] \\
&= \delta\left[\left(x \oplus e\right) \wedge \left(e \oplus y\right)\right] \\
&= \delta\left[i\left(x\right)\right] \wedge \delta\left[j\left(y\right)\right] \\
&= \left[\delta \circ i\left(x\right)\right] \wedge \left[\delta \circ j\left(y\right)\right] \\
&= x \wedge y.
\end{aligned}$$

We observe that δ is onto and $(\delta, L) \approx Coeq\,(i.j)$: To show that δ is onto we take $x \in L$ and find $x \oplus e \in L \oplus L$ such that $\delta\left(x \oplus e\right) = x$. Since

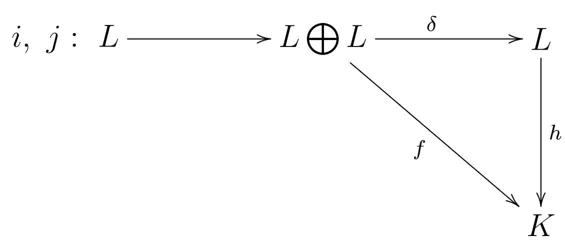
$$\begin{aligned}
\delta\left(x \oplus x\right) &= \delta\left(i\left(x\right) \wedge j\left(x\right)\right) \\
&= \delta \circ i\left(x\right) \wedge \delta \circ j\left(x\right) \\
&= x \wedge x \\
&= x
\end{aligned}$$

it is immediate that δ is onto. To show that $(\delta, L) \approx Coeq\,(i, j)$ we assume that $f : L \to K$, satisfies $f \circ i = f \circ j$. Now define $g : L \to K$ by $f \circ i = g =$

$f \circ j$. Then for any $x \oplus y \in L \oplus L$, we have that

$$\begin{aligned}(g \circ \delta)(x \oplus y) &= g[\delta(x \oplus y)] \\ &= g(x \wedge y) \\ &= g(x) \wedge g(y) \\ &= (f \circ i)(x) \wedge (f \circ j)(y) \\ &= f[i(x) \wedge j(y)] \\ &= f(x \oplus y)\end{aligned}$$

which proves that $g \circ \delta = f$. To see that g is unique such that $g \circ \delta = f$, suppose $h : L \to K$ is another morphism such that $h \circ \delta = f$:



Then $g \circ \delta = h \circ \delta$ so that $g = h$ (since δ is epic being onto); thus $(\delta, L) \approx Coeq\,(i, j)$. □

Definition 2.1.6

A frame L is called *separated* if the codiagonal map $\delta : L \oplus L \to L$ is closed. Explicitly, this says the dense component $\bar{\delta}$ of δ is an isomorphism, and since

$$\begin{aligned} s &= \bigvee \{x \oplus y \mid x \wedge y = 0\} \\ &= \bigvee \delta^{-1}\{0\} \end{aligned}$$

this means that the homomorphism $\bar{\delta} : \uparrow s \to L$ induced by δ is an isomorphism. We call s the *separator* of L.

Theorem 2.1.7

The following statements are equivalent for a frame L:

(i) L is separated.

(ii) In $L \oplus L$ we have that $(x \oplus e) \vee s = (e \oplus x) \vee s$ for all $x \in L$.

(iii) For any $f, g : L \to M$, we have that $f(x) \vee t = g(x) \vee t$, where

$$t = \bigvee \{f(x) \wedge g(y) \mid x \wedge y = 0\}.$$

(iv) For any frame homomorphisms $f, g : L \to M$, we have

$$((.) \vee \uparrow t,\ \uparrow t) \approx Coeq(f, g).$$

Proof:

(i) $\Rightarrow$ (ii): Suppose that L is separated. Then there exist coproduct maps $i, j : L \to L \oplus L$ and the codiagonal map $\delta : L \oplus L \to L$ such that $\delta \circ i = id_L = \delta \circ j$. Since L is separated, there exists a factorization $(.) \vee s : L \oplus L \to \uparrow s$ of δ such that the map $\bar{\delta} : \uparrow s \to L$ induced by δ is an isomorphism. Now we have that

$$\begin{aligned} \delta\,[i(x)] &= \delta(x \oplus e) \\ &= \left[\bar{\delta} \circ (.) \vee s\right](x \oplus e) \\ &= \bar{\delta}[(x \oplus e) \vee s]. \end{aligned}$$

and

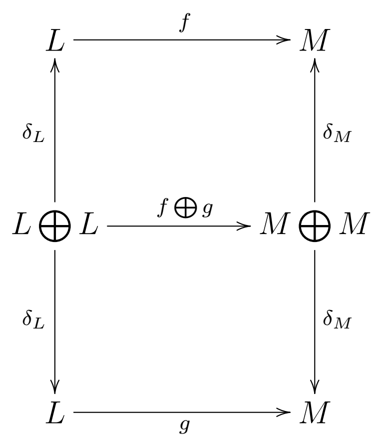
$$\begin{aligned} \delta\,[j(x)] &= \delta(e \oplus x) \\ &= \left[\bar{\delta} \circ (.) \vee s\right](e \oplus x) \\ &= \bar{\delta}[(e \oplus x) \vee s]. \end{aligned}$$

So

$$
\begin{aligned}
(\delta \circ i)\,(x) &= (\delta \circ i)\,(x) \\
\Rightarrow \quad \bar{\delta}[(x \oplus e) \vee s] &= \bar{\delta}[(e \oplus x) \vee s] \\
\Rightarrow \quad (x \oplus e) \vee s &= (e \oplus x) \vee s
\end{aligned}
$$

because $\bar{\delta}$ is an isomorphism.

(ii) $\Rightarrow$ (iii): Consider $f \oplus g : L \oplus L \to M \oplus M$ as in the figure with codiagonal maps $\delta_L : L \oplus L \to L$ and $\delta_M : M \oplus M \to M$:



Set $\delta_M\,(f \oplus g)\,(s) = t$. Then with $(x \oplus e) \vee s = (e \oplus x) \vee s$ we have that

$$
\begin{aligned}
\delta_M\,(f \oplus g)\,[(x \oplus e) \vee s] &= \delta_M\,(f \oplus g)\,(x \oplus e) \vee \delta_M\,(f \oplus g)\,(s) \\
&= (f \circ \delta_L)(x \oplus e) \vee t \\
&= f\,[\delta_L \circ i(x)] \vee t \\
&= f[id_L\,(x)] \vee t \\
&= f(x) \vee t
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_M (f \oplus g) [(e \oplus x) \vee s] &= \delta_M (f \oplus g) (e \oplus x) \vee \delta_M (f \oplus g) (s) \\
&= g [\delta_L (e \oplus x)] \vee t \\
&= g [\delta_L (j(x)] \vee t \\
&= g [id_L (x)] \vee t \\
&= g(x) \vee t.
\end{aligned}
$$

Thus $f(x) \vee t = g(x) \vee t$.

(iii) $\Longrightarrow$ (iv): Suppose that $f, g : L \to M$ satisfy $f(x) \vee t = g(x) \vee t$, where

$$t = \bigvee \{f(x) \vee g(y) \mid x \wedge y = 0\}.$$

Then for $x \in L$, it follows from the assumption that

$$
\begin{aligned}
[((.) \vee t) \circ f] (x) &= [(.) \vee t)] (f(x)) \\
&= f(x) \vee t \\
&= g(x) \vee t \\
&= [(.) \vee t)] (g(x)) \\
&= [((.) \vee t) \circ g] (x) .
\end{aligned}
$$

So take $h : M \to N$ such that $h \circ f = h \circ g$. Now for

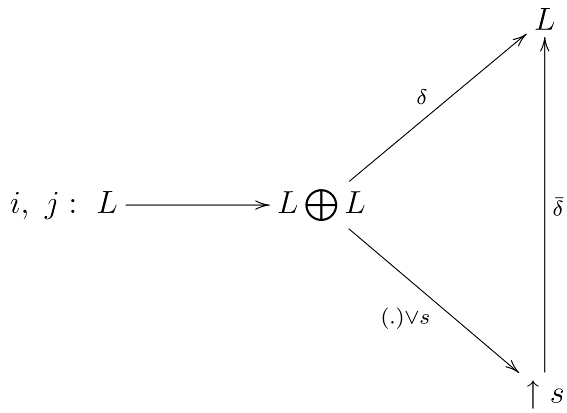
$$t = \bigvee \{f(x) \vee g(x) \mid x \wedge y = 0\}$$

we have

$$\begin{aligned}
h(t) &= h\left[\bigvee\{f(x) \vee g(y) \mid x \wedge y = 0\}\right] \\
&= \bigvee\{(h \circ f)(x) \wedge (h \circ g)(y) \mid x \wedge y = 0\} \\
&= \bigvee\{(h \circ f)(x) \wedge (h \circ f)(y) \mid x \wedge y = 0\} \\
&= \bigvee\{(h \circ f)(x \wedge y) \mid x \wedge y = 0\} \\
&= \bigvee\{h(0)\} \\
&= \bigvee 0 \\
&= 0
\end{aligned}$$

and hence h factors through $(.)\vee \uparrow t$. Thus $(.)\vee \uparrow t$ will be the coequalizer of f and g if and only if it does coequalize f and g.

(iv) $\Longrightarrow$ (i): For the coproduct embeddings $L \overset{i}{\underset{j}{\rightrightarrows}} L \oplus L$, we have that the top and the bottom part of the diagram



are both coequalizer diagrams. Hence $\bar{\delta}$ is an isomorphism. □

2.2 Simple Coverings and Quotient Morphisms

.

Motivating example 2.2.1

If $X \xrightarrow{f} Y$ is a regular epimorphism with a factorization

$$X \xrightarrow{f} Y = X \xrightarrow{g} Z \xrightarrow{h} Y$$

where g is surjective, then h is also a regular epimorphism.

Proof:

Remember (Herrlich and Strecker [6]) that a $\mathcal{C}$-morphism $c : B \to C$ in a category $\mathcal{C}$ is called a *regular epimorphism* if there exists two $\mathcal{C}$-morphisms

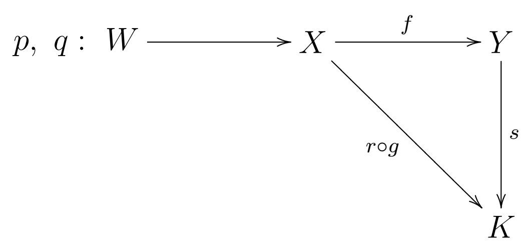
$f, g : A \to B$ such that $(c, C) \approx Coeq\,(f, g)$. Now pick two $\mathcal{C}-$morphisms $p,\ q : W \to X$ such that $(f, Y) \approx Coeq\,(p, q)$. We claim that $(h, Y) \approx Coeq\,(g \circ p, g \circ q)$. For, we have that

(i) $h \circ (g \circ p) = (h \circ g) \circ p = f \circ p = f \circ q = (h \circ g) \circ q = h \circ (g \circ q)$

(ii) Consider a morphism $r : Z \to K$ such that $r \circ (g \circ p) = r \circ (g \circ q)$. Then

$$(r \circ g) \circ p = r \circ (g \circ p) = r \circ (g \circ q) = (r \circ g) \circ q$$

and therefore, since $(f, Y) \approx Coeq(p, q)$, there exists a unique morphism $s : Y \to K$ such that $s \circ f = r \circ g$ as in the figure below:



It remains to show that $s : Y \to K$ is unique such that $s \circ h = r$: Since g is surjective it is well known that it is epimorphic and therefore the equalities

$$(s \circ h) \circ g = s \circ (h \circ g) = s \circ f = r \circ g$$

imply that $s \circ h = r$. On the other hand, if $t : Y \to K$ were another morphism such that $t \circ h = r$ then we would have $t \circ h = s \circ h$ so that $t = s$ (as the second factor of an epimorphism, the morphism h is an epimorphism). Thence $(h, Y) \approx Coeq(g \circ p, g \circ q)$. $\square$

Definition 2.2.2 (See Herrlich and Strecker [6])

A commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle r} \\ B & \xrightarrow[s]{} & C \end{array}$$

is called a *pullback square* if whenever the square

$$\begin{array}{ccc} K & \xrightarrow{h} & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle r} \\ B & \xrightarrow[s]{} & C \end{array}$$

commutes, then there exists a unique $\mathcal{C}$-morphism $u : K \to P$ such that $g \circ u = k$ and $f \circ u = h$.

Definition 2.2.3 (Plewe [12] and Banachewski [1])

A surjection $f : M \to L$ of locales is called a *simple covering* if and only if there exists an embedding $i : M \hookrightarrow L \times \mathbb{S}$ where $\mathbb{S}$ is the Sierpinski space $X = \{0, 1\}$ with the topology $\tau_{\mathbb{S}} = \{\phi,\ X,\ \{0\}\}$, such that $f = \pi_L \circ i$; equivalently, $f : M \to L$ is called a *simple covering* if and only if the corresponding inclusions of frames is a singly generated frame extension. A frame M is called a *singly generated extension* of a frame L if L is a subframe of M, and M is generated by L and some $c \in M$. Such a c is called a *generator* of M over L and the extension M is denoted by $M = L[c]$.

Remark and Notation 2.2.4.

In order to express the properties of simple coverings as far possible as we can, we shall first introduce some notation. Firstly, unless the choice of the factorization through $L \times \mathbb{S}$ matters, we shall assume the fixed factorization $M \xrightarrow{i} L \times \mathbb{S} \xrightarrow{\pi_L} L$ has been chosen. Now pulling back the open sublocale $L \times \{0\}$ along the chosen embedding decomposes M into an open sublocale U and its closed complement $F = M - U$, where U and F depend on the chosen embedding $i : M \hookrightarrow L \times \mathbb{S}$. The images $f(U)$ and $f(F)$ of U and F, respectively, under the map f will be denoted by A_0, A_c, respectively and their meet $A_0 \wedge A_c$ by A_m and the closure of A_m by $A_{\overline{m}}$. Therefore, we have $f(U) = A_0,\ f(F) = A_c,\ A_m = A_0 \wedge A_c$ and $cl(A_m) = A_{\overline{m}}$.

Proposition 2.2.5

If $f : M \to L$ is a simple covering, then f is monomorphic if and only if $A_m = 0$.

Proof:

Suppose that $f : B \to A$ is a simple covering and consider the kernel pair $\pi_1,\ \pi_2 :\ B \times_A B \to B$. Then since projections π_1 and π_2 are isomorphisms (and so f is a monomorphism) iff $U \times_A F = 0$ and $F \times_A U = 0$ iff $A_m = 0$, the result follows. □

The following two results are duals of each other, and so are their proofs.

Proposition 2.2.6

If $f : M \to L$ is a simple covering then f is a closed surjection if and only if A_c is closed in L.

Proof:

Suppose that $A_c = f(F)$ is closed for a closed sublocale $F \in Sub(L)$. Then since L and M are singly generated frame extensions the elements of M are of the form $f^{-1}(V) \vee (f^{-1}(W) \wedge F)$ for closed sublocales $V, W, F \in Sub(L)$. So by Motivating Example 1.2.8 we observe that

$$\begin{aligned} f\left[f^{-1}(V) \vee \left(f^{-1}(W) \wedge F\right)\right] &= V \vee [W \wedge f(F)] \\ &= V \vee (W \wedge A_c) \end{aligned}$$

which is closed because A_c is closed, i.e., f is a closed surjection.

Conversely, suppose that f is a closed surjection. Observe that for closed sublocales $V, W \in Sub(L)$, we have that

$$\begin{aligned} f^{-1}[V \vee (W \wedge A_c)] &= f^{-1}(V) \vee f^{-1}(W \wedge A_c) \\ &= f^{-1}(V) \vee [f^{-1}(W) \wedge f^{-1} \circ f(F)] \\ &= f^{-1}(V) \vee (f^{-1}(W) \wedge F). \end{aligned}$$

□

Proposition 2.2.7

If $f : M \to L$ is a simple covering then f is an open surjection if and only if A_0 is open in L.

Proof:

Let $f : M \to L$ be a simple covering and assume that A_0 is open in L. Then it follows from the definition of a simple covering that f is a surjection. It remains to show that f is an open map. Since M is a singly generated frame extension its elements are of the form $f^{-1}(V) \vee [f^{-1}(W) \wedge U]$, for open sublocales $V, W \in Sub(A)$. So it follows from Motivating Example 1.2.8 that

$$\begin{aligned} f\left[f^{-1}(V) \vee (f^{-1}(W) \wedge U)\right] &= V \vee (W \wedge f(U)) \\ &= V \vee (W \wedge A_0) \end{aligned}$$

which is open because A_0 is open, i.e., f is an open surjection.

Conversely, suppose that f is an open surjection. Observe that for open sublocales $V, W \in Sub(L)$, we have that

$$\begin{aligned} f^{-1}[W \vee (V \wedge A_c)] &= f^{-1}(W) \vee f^{-1}(V \wedge A_c) \\ &= f^{-1}(W) \vee [f^{-1}(V) \wedge f^{-1} \circ f(F)] \\ &= f^{-1}(W) \vee (f^{-1}(V) \wedge F). \end{aligned}$$

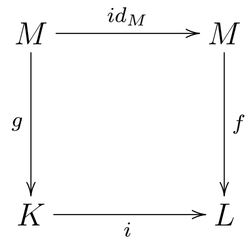
□

Proposition 2.2.8

Surjections are stable under pullback along complemented inclusions. In particular, if C and D are complemented sublocales of A, then $f^{-1}(C) = f^{-1}(D)$ implies that $C = D$ for any surjection $f : M \to L$. Hence $f^{-1} : L \to M$ preserves and reflects inclusions between complemented sublocales.

Proof:

Let $K \in Sub(L)$ be complemented and let $g : M \to K$ be the pullback of $f : M \to L$ along the inclusion $i : K \hookrightarrow L$. Now from the figure below



we observe that

$$\begin{aligned} g\left[(id_M^{-1} \circ f^{-1})(K)\right] &= g\left[id_M^{-1}(f^{-1}(K))\right] \\ &= g\left[f^{-1}(K)\right] \\ &= f\left[f^{-1}(K)\right] \\ &= f\left[M \wedge f^{-1}(K)\right] \\ &= f(M) \wedge K \\ &= K. \end{aligned}$$

On the other hand, we find that for complemented sublocales $C, D \in Sub(L)$

$$f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$$

and

$$f^{-1}(D - C) = f^{-1}(D) - f^{-1}(C)$$

are empty. Since surjections are stable under pullback along complemented inclusions, we must have that $C - D = 0$ and $D - C = 0$. Hence $C = D$. □

Proposition 2.2.9

A simple covering $f : M \to L$ is a regular epimorphism if and only if it satisfies the following condition:

(RE) $\forall A \in Sub(L)$: A is closed $\Longrightarrow A_0 - A_{\bar{m}}$ is open in A.

Proof:

Suppose that $f : M \to L$ is a regular epimorphism and let A be a closed sublocale of L. We must show that $A_0 - A_{\bar{m}}$ is open in A. Since regular epimorphisms are stable under pullback along closed inclusions (Lemma 1.5 of Plewe [12]) we assume that $A = L$. Now we observe that $f^{-1}(A_0 - A_{\bar{m}})$ is open and has equal pullbacks along the projections $\pi_1, \pi_2 : M \times_L M \to M$ because

$$\begin{aligned} A_m \wedge f\left[U \wedge f^{-1}(A_0 - A_{\bar{m}})\right] &\leq A_m \wedge f\left(U\right) \wedge f\left[f^{-1}\left(A_0 - A_{\bar{m}}\right)\right] \\ &\leq A_m \wedge A_0 \wedge \left(A_0 - A_{\bar{m}}\right) \\ &\leq A_m \wedge \left(A_0 - A_{\bar{m}}\right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} A_m \wedge f\left[F \wedge f^{-1}\left(A_0 - A_{\bar{m}}\right)\right] &\leq A_m \wedge f(F) \wedge f\left[f^{-1}\left(A_0 - A_{\bar{m}}\right)\right] \\ &\leq A_m \wedge A_c \wedge \left(A_0 - A_{\bar{m}}\right) \\ &= A_m \wedge A_c \wedge \left(A_0 - A_{\bar{m}}\right) \\ &= A_m \wedge \left(A_0 - A_{\bar{m}}\right) \\ &= 0. \end{aligned}$$

Hence there exists an open sublocale $V \in Sub(L)$ of L such that

$$f^{-1}(V) = f^{-1}\left(A_0 - A_{\bar{m}}\right).$$

Since $A_0 - A_{\bar{m}}$ is complemented, it follows from Corollary 1.2.11 that $V = A_0 - A_{\bar{m}}$, and thus $A_0 - A_{\bar{m}}$ is open.

Conversely, suppose that $f : M \to L$ is a simple covering that satisfies (RE), and take an open sublocale $f^{-1}(V) \wedge [f^{-1}(W) \vee U]$ whose pullbacks along the projections π_1 and π_2 are equal where V and W are open in L. That is,

$$\begin{aligned} A_m \wedge f[f^{-1}(V)] \vee [f^{-1}(W) \wedge U) \wedge U] &= A_m \wedge f[f^{-1}(W) \wedge U] \\ &= A_m \wedge W \wedge f(U) \\ &= A_m \wedge A_0 \wedge W \\ &= A_m \wedge W \end{aligned}$$

and

$$A_m \wedge f[f^{-1}(V) \vee \left(f^{-1}(W) \wedge U\right)] \wedge F \quad = \quad A_m \wedge f[f^{-1}(V) \wedge F]$$

$$= \quad A_m \wedge V \wedge f(F)$$

$$= \quad A_m \wedge V$$

are equal. For $A = L - V$, condition (RE) implies that $A_0 - A_{\bar{m}}$ is open in A, and hence that both $V \wedge \left(A_0 - A_{\bar{m}}\right)$ and $V \vee \left[W \wedge \left(A_0 - A_{\bar{m}}\right)\right]$ are open in L. It remains to show that

$$f^{-1}\{V \vee \left[W \wedge \left(A_0 - A_{\bar{m}}\right)\right]\} \;=\; f^{-1}(V) \vee \left[f^{-1}(W) \wedge U\right].$$

Firstly, we will show that

$$f^{-1}\{V \vee \left[W \wedge \left(A_0 - A_{\bar{m}}\right)\right]\} \;\leq\; f^{-1}(V) \vee \left[f^{-1}(W) \wedge U\right].$$

But this follows from

$$\begin{aligned} f^{-1}(W) \wedge f^{-1}\left(A_0 - A_{\bar{m}}\right) \wedge f^{-1}(A_c) &= f^{-1}\left[(A_0 \wedge A_c) - A_{\bar{m}}\right] \\ &= f^{-1}\left(A_m - A_{\bar{m}}\right) \\ &= 0 \end{aligned}$$

so that $f^{-1}\left(A_0 - A_{\bar{m}}\right) \leq U$. Now for the reverse inequality we have

$$\begin{aligned} V \vee [W \wedge \left(A_0 - A_{\bar{m}}\right)] &= V \vee (W \wedge A_0) \\ &= V \vee (W \wedge A_0) \end{aligned}$$

the first equality holds because $A_m \wedge W = A_m \wedge V$ implies that $A_m \wedge W = A_m \wedge V = 0$, and hence that $A_{\bar{m}} \wedge W = 0$. Now for the second equality we observe that since $A_0 \wedge V \geq A_0$, then

$$\begin{aligned} f^{-1}\left[V \vee W \wedge \left(A_0 - A_{\bar{m}}\right)\right] &= f^{-1}\left[V \vee (W \wedge A_0)\right] \\ &= f^{-1}\left(V\right) \vee \left[f^{-1}(W) \wedge f^{-1}(A_0)\right] \\ &\geq f^{-1}(V) \vee \left[f^{-1}\left(W\right) \wedge U\right]. \end{aligned}$$ □

Chapter 3

Quotient Maps of Locales

In this chapter, we aim to show that in the category $\mathbb{L}\mathbf{oc}$ of locales and continuous maps, closed surjections are also regular epimorphisms at least for those closed surjections whose domains are subfit. Moreover, it is a common knowledge that isomorphisms in any category are precisely those morphisms that are both retractions and monomorphisms (and its dual is equally true). We also study extremal epimorphisms and strong epimorphisms in the setting of locales. Categorical concepts referred to in this chapter, unless stated otherwise, are according to Herrlich and Strecker [6].

3.1 Extremal and Strong Epimorphisms in Locales

Definition 3.1.1

In an abstract category $\mathcal{C}$ a $\mathcal{C}$-morphism $f : X \to Y$ is called an *extremal epimorphism* provided that it satisfies the following:

(i) f is an epimorphism.

(ii) *Extremal condition*: If $f = m \circ e$:

$$X \xrightarrow{f} Y = X \xrightarrow{e} Q \xrightarrow{m} Y$$

is a factorization of f and m is a monomorphism, then m must be an isomorphism.

Proposition 3.1.2

Regular epimorphisms in the category $\mathcal{C} = \mathbb{L}\mathbf{oc}$ are extremal epimorphisms.

Proof:

Consider a regular epimorphism $e : B \to C$ with a factorization, say

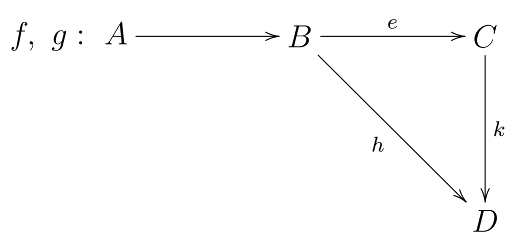
$$B \xrightarrow{e} C = B \xrightarrow{h} D \xrightarrow{m} C$$

where m is a monomorphism. It must be shown that m is an isomorphism. Suppose that $f, g : A \to B$ are two morphisms in the category $\mathbb{L}\mathbf{oc}$ such that

$(e, C) \approx Coeq\,(f, g)$ so that $e \circ f = e \circ g$. We have that

$$\begin{aligned} m \circ (h \circ f) &= (m \circ h) \circ f \\ &= e \circ f \\ &= e \circ g \\ &= (m \circ h) \circ g \\ &= m \circ (h \circ g) \end{aligned}$$

and since m is a monomorphism, it follows that $h \circ f = h \circ g$. But $(e, C) \approx Coeq\,(f, g)$, so there exists a unique morphism $k : C \to D$ such that $k \circ e = h$ as in the figure below:



But e is an epimorphism (being a quotient) and we also have that

$$\begin{aligned} (m \circ k) \circ e &= m \circ (k \circ e) \\ &= m \circ h \\ &= e \\ &= id_C \circ e \end{aligned}$$

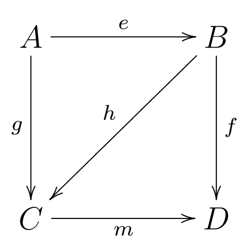
so we must have that $m \circ k = id_C$, making m a retraction. Since a morphism that is both a retraction and a monomorphism is an isomorphism, it follows that the morphism m is an isomorphism. Consequently, the regular epimorphism e is an extremal epimorphism. □

Definition 3.1.3

A $\mathcal{C}-$morphism $e : A \to B$ is a *strong epimorphism* if it satisfies the following conditions:

(i) e is an epimorphism.

(ii) Whenever $f \circ e = m \circ g$ for any $\mathcal{C}$-morphisms $g : A \to C$ and $f : B \to D$, there exists a $\mathcal{C}$-morphism $h : B \to C$ making the following diagram commutative:



Proposition 3.1.4

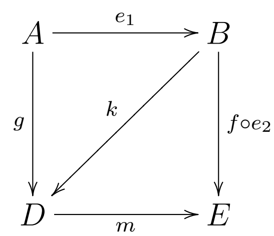
Strong epimorphisms in the category $\mathbb{L}$**oc** are closed under compositions.

Proof:

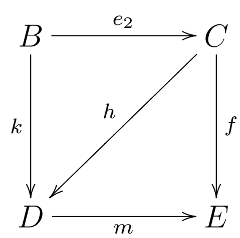
Consider two strong $\mathcal{C}$-epimorphisms $e_1 : A \to B$ and $e_2 : B \to C$ in the category $\mathbb{L}$**oc** of locales. We must show that $e = e_2 \circ e_1 : A \to C$ is a strong epimorphism. Take two $\mathcal{C}$-morphisms $g : A \to D$ and $f : C \to E$ together with a $\mathcal{C}$-monomorphism $m : D \to E$ such that $f \circ e = m \circ g$. We have

$$(f \circ e_2) \circ e_1 = f \circ (e_2 \circ e_1) = f \circ e = m \circ g$$

and so (by definition) there exists a $\mathcal{C}$-morphism $k : B \to D$ making the following diagram commutative: i.e., $k \circ e_1 = g$ and $m \circ k = f \circ e_2$:



But $e_2 : B \to C$ is a strong epimorphism, so $f \circ e_2 = m \circ k$ ensures that the existence of a $\mathcal{C}$-morphism $h : C \to D$ such that $h \circ e_2 = k$ and $m \circ h = f$:



Since $h : C \to D$ and $m \circ h = f$ satisfies

$$\begin{aligned} h \circ e &= h \circ (e_2 \circ e_1) \\ &= (h \circ e_2) \circ e_1 \\ &= k \circ e_1 \\ &= g \end{aligned}$$

it follows that $e : A \to C$ is indeed a strong epimorphism. $\square$

3.2 Characterization of Quotient Maps in Locales

Definition 3.2.1 (See Plewe [12])

A continuous surjection $f : X \to Y$ between topological spaces is called a *quotient map* if it satisfies the following property:

(QO) $\forall S \subseteq Y$, if $f^{-1}(S)$ is open in X then S is open in Y.

Example 3.2.2

A surjection $f : Y \to \mathbb{Q}$ of locales which is not a regular epimorphism, but which satisfies (QO).

Let $f : X \to \mathbb{Q}$, where X is a scattered space and $\mathbb{Q}$ is a space of irrationals such that for any pointless sublocale $S \in Sub(\mathbb{Q})$, $X \times_Q S = 0$. (Recall that a topological space X is said to be *scattered space* if for every closed subset C of X, the set of isolated points of C is dense in X.) Then the map $h = \langle f, i\rangle_{.} : X + \cdot pl(\mathbb{Q}) \to \mathbb{Q}$, where $pl(\mathbb{Q})$ is the largest pointless sublocale of $\mathbb{Q}$ and $X + \cdot pl(\mathbb{Q})$ is a sublocale of X and the pointless sublocale $pl(\mathbb{Q})$ of $\mathbb{Q}$ satisfies (QO) but is not a regular epimorphism. The map h is not a regular epimorphism because $\pi_1^*(X) = \pi_2^*(X)$. But since $h(X) = f(X) = \mathbb{Q}$, so X is not the inverse of any sublocale of $\mathbb{Q}$. To show that h satisfies (QO), we let $S \in Sub(\mathbb{Q})$ be any sublocale of $\mathbb{Q}$ such that $h^{-1}(S)$ is open in $X + \cdot pl(\mathbb{Q})$. Now since $f^{-1}(S)$ is open in X, it suffices to show that S is a spatial sublocale of $\mathbb{Q}$ because then we would use the fact that f is a quotient map of topological spaces to conclude that S is open in $\mathbb{Q}$. Since $h^{-1}(S)$ is open in $X + \cdot pl(\mathbb{Q})$, the set $f^{-1}(S)$ is open in X and

$$i^{-1}\left[\mathbb{Q} - f\left(X - f^{-1}(S)\right)\right] \geq i^{-1}(S) \backsimeq pl(S),$$

hence

$$\mathbb{Q} - f\left(X - f^{-1}(S)\right) \geq pl(S).$$

Since $\mathbb{Q} - f\left(X - f^{-1}(S)\right)$ is an open sublocale of $\mathbb{Q}$, then it is spatial. It remains to show that $S \geq pt(S) \geq \mathbb{Q} - f\left(X - f^{-1}(S)\right)$ to complete the proof because this would imply that $S \geq pt(S) \geq \mathbb{Q} - f\left(X - f^{-1}(S)\right) \geq pl(S)$ and hence that S is spatial since any locale is the join of its largest pointless sublocale and its points provided that the largest pointless sublocale exists. But this follows from the fact that

$$\begin{aligned} S \vee f\left[X - f^{-1}(S)\right] &= f\left[f^{-1}(S)\right] \vee f\left[X - f^{-1}(S)\right] \\ &= f\left[f^{-1}(S) \vee (X - f^{-1}(S))\right] \\ &= f(X) \\ &= \mathbb{Q}. \end{aligned}$$

□

Definition 3.2.3 (See e.g. Siweya [16])

A topological space X is said to be a *sober space* if every irreducible closed subset of X is the closure of exactly one singleton of X. An irreducible closed subset of X is a non-empty closed subset of X which is not the union of two proper closed subsets of itself.

For the proof of the next result, recall that the category $\mathbb{S}ob$ of sober spaces into $\mathbb{L}oc$ is a spatial subcategory of the category $\mathbb{L}\mathbf{oc}$ of locales and locale maps. See Johnstone [9].

Example 3.2.4

A surjection $f : X \to \mathbb{Q}$ of spatial metrizable locales which is a regular epimorphism, but which does not satisfy (QO).

Take any quotient map $f : X = \coprod_{q\in\mathbb{Q}} \mathbb{Q}_q \to \mathbb{Q}$ induced by the natural maps $f_q : \mathbb{Q}_q \to \mathbb{Q}$ in the category $\mathbb{T}op$ of topological spaces from a completely metrizable space $X = \coprod_{q\in\mathbb{Q}} \mathbb{Q}_q$ onto the irrationals $\mathbb{Q}$ where $\mathbb{Q}_q$ (for $q \in \mathbb{Q}$) has the same underlying set as $\mathbb{Q}$, but all the points except q are isolated and q has the same neighborhoods in $\mathbb{Q}$ and $\mathbb{Q}_q$. Since the embedding $\mathbb{S}\mathbf{ob} \hookrightarrow \mathbb{L}\mathbf{oc}$ of the category $\mathbb{S}\mathbf{ob}$ of sober spaces into $\mathbb{L}\mathbf{oc}$ as spatial locales has a right adjoint, it preserves all regular epimorphisms, so $f : X \to \mathbb{Q}$ is a regular epimorphism in $\mathbb{L}\mathbf{oc}$. All pointless sublocales of $\mathbb{Q}$ pullback to the empty sublocales of X because of the following:

(i) $pl(X)$, the largest pointless sublocale is an O_δ in $\mathbb{Q}$ (a countable intersection of open sublocales).

(ii) pulling back along f preserves all meets.

(iii) the O_δ's in complete spaces are spatial (See Isbell [7]), hence pointless O_δ's are empty.

So, if S is any non-zero pointless sublocale of $\mathbb{Q}$, then $f^{-1}(S) = 0$ is open in

X, but S is not open in $\mathbb{Q}$, which means that f does not satisfy (QO). □

Definition 3.2.5 (Plewe [12])

Let $f : X \to Y$ be a surjection of locales. Then f is said to satisfy the condition

(WQ) if, for all complemented sublocales $S \in Sub(Y)$, whenever $f^{-1}(S)$ is open in X then S is open in Y;

(EQC) if, for all sublocales $S = U \vee (V - W)$, where U, V and W are open sublocales of Y, whenever $f^{-1}(S)$ is open in X then S is open in Y.

Proposition 3.2.6

A surjection $f : X \to Y$ in $\mathbb{L}oc$ satisfies (EQC) if and only if it satisfies (WQ).

Proof:

Suppose that $f : X \to Y$ satisfies (WQ). Then by definition it follows that f satisfies (EQC).

Conversely, suppose that $f : X \to Y$ satisfies (EQC) and let C be a complemented sublocale of Y such that $f^{-1}(C)$ is open in X. We want to show that C is open in Y. Let $U = int(C)$, $H = cl(C - U)$ and $P = int_H(C - U)$. Then $U \vee P$ is open in Y and P is locally closed in Y. The set P is dense in H because every dense and complemented sublocale has dense interior (the complement D of $C - U$ is disjoint from $D(H)$ the smallest dense sublocale of H), and hence so is $cl_H(D)$. Now $f^{-1}(U \vee P)$ is open in $f^{-1}(C)$, hence open in X. So applying (EQC) we find that $U \vee P \subseteq C$ is open in

Y. Because U is the interior of C we can conclude that $U \vee P = U$, and then that $P = 0$ since $U \wedge P = 0$. Since P is dense in H, it follows that $H = 0$, so that $C - U = 0$. Hence $C = U$ is open in Y, i.e., f satisfies (WQ).□

Definition 3.2.7

A locale L is called a *subfit locale* if each open sublocale of L is the join of closed sublocales of L or equivalently if each closed sublocale of L is the join of open sublocales of L.

Remark 3.2.8

(a) In [14], Simmons called these locales *conjunctive* whereas Herrlich and Pultr inform [5] us that in spaces the corresponding term used was *weakly regular*

(b) Subfitness is equivalent to the following condition:

$(*)$ *whenever* $a \nleq b$ *in* L *there exists a* $c \in L$ *such that* $a \vee c = 1 \neq b \vee c$

In view of the definition of open and closed sublocales in Remark 1.2.7, we know that $\check{b} \sqsubseteq \hat{a}$ iff $a \vee b = 1$ which implies that the statement "each open sublocale can be written as a join of closed sublocales" can now be rewritten thus:

$$(**) : (x \wedge a \neq y \wedge a) \Rightarrow (\exists c \in L \textit{ such that } a \vee c = 1 \textit{ and } x \vee c \neq y \vee c).$$

(**) $\Rightarrow$ (*) : Therefore, take $a \nleq b$ and set $x = a$ and $y = b$ in (**). Then

$$a \vee c \; (= x \vee c) = 1 \neq y \vee c \; (= b \vee c).$$

(*) $\Rightarrow$ (**) : Given $x \wedge a \neq y \wedge a$, suppose that $x \wedge a \nleq x \wedge y \wedge a$ in (*). Then there exists a $c \in L$ such that $(x \wedge a) \vee c = 1 \neq (x \wedge y \wedge a)$ from which $a \vee c = 1$ and $x \vee c \neq y \vee c$ follow.

Proposition 3.2.9

If $f : Y \to X$ and $g : Z \to X$ are closed surjections where Y and Z are subfit domains, then the induced map $Y \times_X Z \to X$ is again a surjection.

Proof:

Suppose that $f : Y \to X$ and $g : Z \to X$ are closed surjections with subfit domains and consider the map $h : Y \times_X Z \to X$ induced by f and g. We must show that h is a surjection. Note that it is sufficient to show that $Y \times_X Z = 0$ implies that $X = 0$, because of the following:

(a) closed surjections are stable under pullback along complemented sublocales.

(b) subfitness is inherited by complemented sublocales.

(c) the image $h(Y \times_X Z)$ in X is the intersection of complemented sublocales C of X for which the restriction of f and g to $X - C$ have zero pullback.

Now the topology of the pullback $Y \times_X Z$ is presented as follows:

$$T(Y) \otimes_{T(X)} T(Z) = (T(Y) \times T(Z) \mid cov),$$

where $cov(y, z)$ is the union of the following covers:

(i) $\{\{(y_i, z) \mid i \in I\} \mid \bigvee y_i = y\}$

(ii) $\{\{(y, z_i) \mid i \in I\} \mid \bigvee z_i = z\}$

(iii) $\{\{((y \wedge f^*(x)), z)\} \mid g^*(x) \geq z\}$

(iv) $\{\{(y, (g^*(x) \wedge z))\} \mid f^*(x) \geq y\}$

Clearly these covers are stable. Now let the sieve

$$\mathcal{S} = \{(y, z) \mid \forall \text{ closed sublocales } F \leq Y,\ G \leq Z : F \leq U_y \text{ and } G \leq U_z \Longrightarrow f(F) \wedge g(G) = 0\}$$

We claim that the sieve $\mathcal{S}$ is closed:

To this end, under the covering of the first type, let $\{(y_i, z) \mid i \in I\} \subseteq \mathcal{S}$ be arbitrary, and let $F \leq U_{\bigvee y_i}$ and $G \leq U_z$ be closed sublocales of Y and Z, respectively. Then if $U_{y_i} = \bigvee F_{i,j}$ where each $\mathrm{F}_{i,j}$ is closed, then

$$\begin{aligned}
f(F) \wedge g(G) &= f\left[F \wedge \textstyle\bigvee_{i,j} F_{i,j}\right] \wedge g(G) \\
&= f\left[\textstyle\bigvee_{i,j} (F \wedge F_{i,j})\right] \wedge g(G) \\
&= \left[\textstyle\bigvee_{i,j} f(F \wedge F_{i,j})\right] \wedge g(G) \\
&\leq \left[\textstyle\bigvee_{i,j} f(F_{i,j})\right] \wedge g(G) \\
&= \textstyle\bigvee_{i,j} \left[f(F_{i,j}) \wedge g(G)\right] \\
&= 0
\end{aligned}$$

because each $F_{i,j} \leq U_{y_i}$ and $G \leq U_z$.

For covers of the second type, let $\{(y, z_i) \mid i \in I\} \subseteq \mathcal{S}$ be arbitrary and let $F \leq U_y$ and $G \leq U_{z_i}$ be closed sublocales of Y and Z, respectively. If $U_{z_i} = \bigvee G_{i,j}$ where each $G_{i,j}$ is closed, then

$$\begin{aligned}
f(F) \wedge g(G) &= f(F) \wedge g[G \wedge \textstyle\bigvee_{i,j} G_{i,j}] \\
&= f(F) \wedge g[\textstyle\bigvee_{i,j}(G \wedge G_{i,j})] \\
&= f(F) \wedge \textstyle\bigvee_{i,j} [g\,(G \wedge G_{i,j})] \\
&\leq f(F) \wedge \textstyle\bigvee_{i,j} g(G_{i,j}) \\
&= \textstyle\bigvee_{i,j} [f\,(F) \wedge g(G_{i,j})] \\
&= 0.
\end{aligned}$$

because each $F \leq U_y$ and $G \leq U_{z_i}$.

Now for the covers of the third type, let $[(y \wedge f^*(x), z)] \in \mathcal{S}$ and assume that $g^*(x) \geq z$ where $g^* = g : Z \to X$. Let $F \leq U_y$ and $G \leq U_z$ be any closed sublocales of Y and Z, respectively. We want to show that $f(F) \wedge g(G) = 0$. Since X is subfit (subfitness is preserved under closed surjections), there exists a set $\{H_i \mid i \in I\}$ of closed sublocales of X such that $U_x = \bigvee_{i \in I} H_i$. Now

$$[f(F) \wedge H_i] \wedge g(G) \;=\; f\left[F \wedge f^{-1}(H_i)\right] \wedge g(G) = 0$$

because $F \wedge f^{-1}(H_i) \leq U_{y \wedge f^*(x)}$ for all i$\in I$. Since $g^*(x) \geq z$ implies that $g^{-1}(U_x) \geq U_z$ and hence $U_x \geq g(U_z) \geq g(G)$, from which it follows that

$$\begin{aligned} f(F) \wedge g(G) &= f(F) \wedge g(G) \wedge U_x \\ &= f(F) \wedge g(G) \wedge \bigvee_{i \in I} H_i \\ &= \bigvee_{i \in I} [f(F) \wedge H_i \wedge g(G)] \\ &= 0. \end{aligned}$$

Finally, for the covers of the fourth type let $[y, (g^*(x) \wedge z)] \in \mathcal{S}$ and assume that $f^*(x) \geq y$ where $f^* = f : Y \to X$. Let $F \leq U_y$ and $G \leq U_z$ be any closed sublocales of Y and Z, respectively. We want to show that $f(F) \wedge g(G) = 0$. Since X is subfit (subfitness is preserved under closed surjections), there exists a set $\{K_i \mid i \in I\}$ of closed sublocales of X such that $U_x = \bigvee_{i \in I} K_i$. Now

$$[f(F) \wedge K_i] \wedge g(G) \;=\; f\left[F \wedge f^{-1}(K_i)\right] \wedge g(G) = 0$$

because $F \wedge f^{-1}(K_i) \leq U_{g^*(x) \wedge z}$ for all $i \in I$. But $f^*(x) \geq y$ gives rise to $f^{-1}(U_x) \geq U_y$ and hence $U_x \geq f(U_y) \geq f(F)$, so that

$$
\begin{aligned}
f(F) \wedge g(G) &= f(F) \wedge g(G) \wedge U_x \\
&= f(F) \wedge g(G) \wedge \bigvee_{i \in I} K_i \\
&= \bigvee_{i \in I} [f(F) \wedge K_i \wedge g(G)] \\
&= 0.
\end{aligned}
$$

Now $\mathcal{S}$ is maximal if and only if $(1,1) \in \mathcal{S}$, i.e., $f(Y) \wedge g(Z) = 0$. Since f and g are surjections, we have $f(Y) \wedge g(Z) = 0$ if and only if $X = 0$. So we conclude that $Y \times_X Z = 0$ if and only if $X = 0$. □

Theorem 3.2.10

Closed surjections $f : X \to Y$ with subfit domain X are regular epimorphisms.

Proof:

Let $f : X \to Y$ be a closed surjection with subfit domain X. Then $f : X \to Y$ is a quotient map with the kernel pair $X \times_Y X \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X$. To show that f is a regular epimorphism, we need only show that $f \approx Coeq(\pi_1, \pi_2)$. So let $K \subseteq X$ be any closed sublocale such that $\pi_1^{-1}(K) = \pi_2^{-1}(K)$. Now restricting the map $f : X \to Y$ to a map $f^{-1} \circ f(K) \to f(K)$ if necessary (subfitness is inherited by closed subspaces), we may assume that $f(K) = Y$. We want to show that $K = X$. But this will follow if we can show that $K \wedge H = 0$ implies that $H = 0$ for any closed sublocale $H \subseteq X$, because then the subfitness of

X implies that

$$X - K = \bigvee\{H \text{ closed in } X \mid K \wedge H = 0\} = 0.$$

So, let H be any closed sublocale of X such that $K \wedge H = 0$. Then we restrict the domain of f to $K \vee H$. The resulting map $g = f \restriction_{K \vee H} \colon K \vee H \to Y$ is still a closed surjection onto Y and the domain of g is the disjoint union of K and H. Furthermore, both K and H have isomorphic pullbacks along π_1 and π_2 because

$$\begin{aligned} K \times_Y H &\leq K \times_Y (X - K) \\ &= K \times (\bigvee\{H \subseteq X \mid K \wedge H = 0\}) \\ &= F \times_Y 0 \\ &= 0. \end{aligned}$$

Finally, we restrict the codomain Y to $W = g(H)$ to arrive at the closed surjection

$$g^{-1} \circ g(H) \;=\; Z \xrightarrow{h = g \; g^{-1} \circ g(G)} W.$$

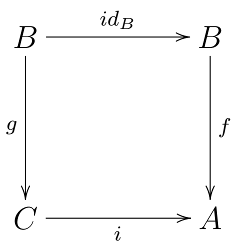
Then both $T = K \wedge Z$ and H are mapped by h onto W and because $T \times_W H = 0$ and both T and H are equalized by the kernel pair (π_1^{-1}, π_2^{-1}) of h. But since the restrictions of h to T are also closed surjections, Proposition (3.2.9) implies that $T \times_W H$ maps onto W, and so $W = 0$ so that $H = 0$. $\square$

Proposition 3.2.11

Closed maps are stable under pullback along complemented inclusions.

Proof:

Suppose that C is complemented in A and let $g : B \to C$ be the pullback of $f : B \to A$ along the inclusion $i : C \hookrightarrow A$. Now from the following figure



we find that if $f : B \to A$ is a closed map and K is a closed sublocale of $f^{-1}(C)$, then (applying Motivating Example 1.2.8)

$$\begin{aligned} g[id_B^{-1}(K)] &= g(K) \\ &= f(F) \\ &= f\,[cl_B(K) \wedge f^{-1}(C)] \\ &= f\,[cl_B(K)] \wedge C \end{aligned}$$

which is closed in C because $f\,[cl_B(K)]$ is closed in A. So g is also closed. □

Proposition 3.2.12

A simple covering $B \xrightarrow{f} A$ is a regular epimorphism if and only if it satisfies (WQ).

Proof: We refer to condition (RE) in Proposition 2.2.9.

Recall that in the category $\mathbb{L}$**oc** of locales regular epimorphisms are always extremal epimorphisms (Proposition 3.1.2), so we need only show that (WQ) implies (RE). To this end, we proceed as follows: if K is any closed sublocale of A then $(A-K)\vee\left(H_0-H_{\bar{m}}\right)$ is a complemented sublocale of A with complement $H_c\vee H_{\bar{m}}$. Now since

$$\begin{aligned} f^{-1}[(A-K)\vee(H_0-H_{\bar{m}})] &= f^{-1}(A-K)\vee f^{-1}\left(H_0-H_{\bar{m}}\right) \\ &= [f^{-1}(A)-f^{-1}(K)]\vee[f^{-1}(H_0)-f^{-1}\left(H_{\bar{m}}\right)] \\ &= [B-f^{-1}(K)]\vee[[f^{-1}(K)-f^{-1}\left(H_{\bar{m}}\right)]\wedge U] \\ &= [(B-f^{-1}(H)]\vee[(B-f^{-1}\left(H_{\bar{m}}\right))\wedge U] \end{aligned}$$

is open in B, condition (WQ) implies that $(A-K)\vee\left(H_0-H_{\bar{m}}\right)$ is open in A and therefore $H_0-H_{\bar{m}}$ is open in K. □

Bibliography

[1] Banachewski, B.: Singly generated frame extensions, *J. Pure Appl. Algebra* **83** (1992), 1-21.

[2] Banachewski, B.: Axiom of countable choice in pointfree topology, *Appl. Cat. Str.* **9** (2001), 245-258.

[3] Chen, X.: Stably closed frame homomorphisms, *Géom. Diff. Cat*, **2** (1996), 123-144.

[4] Dowker, C. H and Strauss, D. P.: Paracompact frames and closed maps, *Symp. Math.* **16** (1975), 93-116.

[5] Herrlich, H and Pultr, A.: Nearness, Subfitness and Sequential Regularity, *Appl. Cat. Str.* **8** (2000), 67-80.

[6] Herrlich, H and Strecker, G. E.: *Category Theory*, Allyn and Bacon, Boston, 1973.

[7] Isbell, J.: Descriptive theory of locales, *Trans. Amer. Math. Soc.* **327** (1991), 353-371.

[8] Isbell, J .: Atomless parts of spaces, *Math. Scand.* **31** (1972), 5-32.

[9] Johnstone, P. T.: *Stone spaces*, Cambridge University Press, Cambridge, 1982.

[10] Johnstone, P. T.: The point of pointless topology, *Bull. of Amer. Math. Soc.* (New series) **8** (1983), 41-53.

[11] Michael, E and Stone, A. H.: Quotients of the space of irrationals, *Pac. J. Math.* Vol. 28, No. 3 (1969), 629-633.

[12] Plewe, T.: Quotient Maps of Locales, *Appl. Cat. Str.* **8** (2000), 17-44.

[13] Plewe, T.: Countable products of absolute C_δ Spaces: *Top. Appl.* **74** (1996), 39-50.

[14] Simmons, H.: The lattice theoretical part of topological separation properties, *Proc. Edinburgh Math. Soc.* (2) **21** (1978), 41-48.

[15] Simmons, H.: *The basic of Frame theory*, University of Manchester, Lecture Notes, 2006.

[16] Siweya, H.J.: On Topology without points: An introduction to spatial locales, *Msc Dissertation*, University of Sussex, 1991.

[17] Vermeulen, J.J.C.: Proper maps of locales, *J. Pure Appl. Algebra* **92** (1994), 79-107.

[18] Li Yongming.: Weak locale quotient morphisms and locally connected frames, *J. Pure Appl. Algebra* **110** (1996), 101-107.