AN EXPLORATION OF FOLDING BACK IN IMPROVING GRADE 10 STUDENTS' REASONING IN GEOMETRY

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AN EXPLORATION OF FOLDING BACK IN IMPROVING GRADE 10 STUDENTS' REASONING IN GEOMETRY

By

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DEDICATION

I dedicate this dissertation to the following people:

- Mrs M.M. Mabotja and Mr K.F. Mabotja, thank you for sowing the seeds of determination and perseverance in my life. I am deeply humbled and blessed to have you as the pillars of my life.
- Dr M.K. Maphoto, words will never be enough to describe your role in my life. Thanks for being there when it mattered most.
- Gloubus Mabotja, may this serve you as a reminder that whatsoever you intend to do in your journey, it can be done.

DECLARATION

I, Koena Samuel Mabotja, declare **AN EXPLORATION OF FOLDING BACK IN IMPROVING GRADE 10 STUDENTS' REASONING IN GEOMETRY** dissertation hereby submitted to the University of Limpopo, for the degree of Master of Education in Mathematics Education has not previously been submitted by me for a degree at this or any other university; that it is my work in design and in execution, and that all material contained herein has been duly acknowledged.

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Mr Koena Samuel Mabotja   Date

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ABSTRACT

The purpose of this study was to explore the role of folding back in enriching grade 10 students' reasoning in geometry. Although various attempts are made in teaching and learning geometry, evidence from several research studies shows that most learners struggle with geometric reasoning. Hence, this study came as a result of such learners' struggles as shown in the literature as well as personal experiences. The study was a constructivist teaching experiment methodology that sought to answer the following research questions: How does folding back support learners' interaction with geometric reasoning tasks during the lessons? How does a Grade 10 mathematics teacher use folding back to enrich student reasoning in geometry? The teaching experiment as a research design in this study was found useful in studying learners' geometric reasoning as a result of mathematical interactions in their learning of geometry. Therefore, it should be noted that the teaching experiments were not conducted as an attempt to implement a particular way of teaching, but rather to understand the role of folding back in enriching learners' reasoning in geometry.

As a referent to the teaching experiment methodology, the participants in this study were 7 grade 10 mathematics learners' sampled from a classroom of 54 learners. These seven learners did not necessarily represent the whole class in accordance with the purpose of the study. This requirement was not necessary in determining rigour in teaching experiments. Instead interest was on "organising and guiding [teacher-researchers] experience of learners doing mathematics" (Steffe & Thompson, 2000, p. 300). Furthermore, the participants were divided into two groups while working on the learning activities. Participants were further encouraged to share ideas with each other as they solved the learning activities.

Data was collected through video recording while learners were working on mathematical learning activities. The focus was on the *researcher-teacher* – learners and learners-learners interactions while working on geometric reasoning learning activities. Learning activities and observations served as subsets of the video data. Learners were encouraged to share ideas with each other as they

solved the learning activities as recommended by Steffe and Thompson (2000). Likewise, in order to learn the learners' mathematics, the researcher could teach and interact with learners in a way that encourage them to improve their current thinking (Steffe & Thompson, 2000).

In analysing data, the study adopted narrative analysis. The researcher performed verbatim transcription of the video recordings. Subsequently, information-rich interaction for each mathematical learning activity, where folding back was observed was selected. The selections of such information-rich interactions were guided by Martin's (2008) framework for describing folding back.

The main findings of the study revealed that in a learning environment where folding back takes place, learners' reasoning in geometry is enriched. The researcher-teacher's instructional decisions such as discouraging, questioning, modelling and guiding were found to be effective sources through which learners fold back. The findings also revealed that learners operating at different layers of mathematical understanding are able to share their geometry knowledge with their peers. Moreover, the findings indicated that in a learning environment where folding back takes place, learners questioning ability is enriched. Based on the findings of the study, the recommendations were that Mathematics teachers should create a learning environment where learners are afforded the opportunity to interact with each other during geometry problem solving; such is a powerful quest for folding back to take place.

KEY CONCEPTS

Folding back, Growth of mathematical understanding, Geometry, Geometric reasoning.

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CHAPTER 1: INTRODUCTION

This chapter presents the overview of the study. The background of the study; the problem statement; purpose of the study; research questions; research methodology and the significance of the study are captured.

1.1. BACKGROUND TO THE STUDY

Over the past years research has shown that geometry as a content knowledge has been a major challenge in mathematics school curriculum. (Malati [Sa]:1) underscored in their report entitled "A vision for the learning and teaching of school geometry" that most school teachers and learners alike can both testify to the difficulties they face when teaching and learning geometry. Likewise, this shows that teaching and learning of geometry is a major concern (Adolphus, 2011; Naidoo, 2013; Siyepu, 2012). In this regard, my brief personal experiences as outlined in the proceeding paragraphs weaved in the literature and geometry curriculum in South Africa serves as the background to the study.

1.1.1. GEOMETRY LEARNER IN HIGH SCHOOL

Having been a mathematics learner in high school, Euclidean geometry was one of the difficult content to understand. As a result, I prioritised paper 1 as my hope in mathematics. This was mainly because I did not find Paper 1, which consisted of algebra amongst other contents, to be as challenging as Paper 2 which geometry was part of. I always aimed at achieving high marks in Paper 1 so that what I get from Paper 2 would just be a supplement. This has resulted in me paying much attention to Paper 1 both in the classroom and in my study sessions and less to geometry. This according to Atebe (2008) demonstrates learners'

dislike even or outright hatred of geometry. Consequently, I developed to what is normally regarded as "negative-attitude" towards geometry; this attitude was even displayed in geometric reasoning tasks to an extent that I would write "given" and "common" as my reasons to such tasks. Such negative attitude shows that the perception of geometry by learners is embedded in negative terms (Atebe, 2008).

It seemed to me that my teacher realised that I had a challenge in geometry, as he suggested to me that I should do mathematics in standard grade. Despite the fact that he told me that geometry is difficult and allocated more marks in paper 2, I remained content in doing mathematics on the higher grade, as I have earlier mentioned, to me paper 2 was just to supplement paper 1. Although I knew that geometry was difficult for me, I was fascinated to hear my teacher telling me that geometry is difficult; particularly because I thought as a learner that teachers knew everything.

1.1.2. BEING TAUGHT GEOMETRY AT HIGH SCHOOL

Geometry, in most cases during my high school years, was given little preference as compared to other content areas in mathematics and was mostly taught when we were approaching final examinations. This according to Kutama (2002) was the case in most South African mathematics classrooms, whereby learners were and are still taught mainly algebra, and during the last few weeks of the year they are hurriedly taught geometry. This was in no way contrary to De Villiers (1997) who points out that it is well known that learners' performance in geometry is poor as compared to in algebra.

Moreover, the instruction in geometry was often teacher-centred. In most cases my teacher would rely on textbooks and chalk when teaching. This subjected me to being a spectator in a mathematics classroom, yet I enjoyed this particularly because of the attitude I developed towards geometry. For the most

part, the geometry lessons were shaped by the sequence of our learners' textbooks. Thus, my teachers would begin by explaining concepts, doing some examples for us and then finally giving us a classwork or homework as they appeared in our learners' textbooks. Gunhan (2014) views such traditional teaching methods as being characterised by teachers giving definitions; making no use of concrete materials and practical ways to explain mathematical concepts. Such methods hinder learners' geometric conceptual development (Acquah, 2011). Furthermore, De Villiers (1997) blames the traditional approach as the main cause of poor understanding of geometry. This approach in theorems in circle geometry was even more difficult; the teacher would just draw a complete diagram for each theorem as they appeared in the learners' textbook and began explaining and then giving us work to do. Although the textbook was mainly our lessons, the difference was that, unlike the textbook, my teacher would "say" something.

Furthermore, my teachers gave a different attention to Euclidean geometry than any other content in mathematics. Thus, how they taught us geometry was totally different; in algebra lessons they would give us chalk to do problems on the chalkboard, but with geometry it was a different approach. In most cases, much of the work in the classroom was done by our teacher, yet our performance in this content contradicted the efforts invested by our teacher. This was no different from Seroto (2006) who underscores that the performance in geometry section is much lower than other sections in Grade 10, 11 and 12.

Despite my difficulties with learning and being taught Euclidean geometry, I secured exemption in matric. As result, I went on to further my education at an institution of higher learning. However, my geometry knowledge was not adequately developed as compared to other content areas such as algebra. Van Putten, Howie and Stols (2010) noted in their study that learners leaving secondary schools in South Africa do not have an in-depth understanding of geometric concepts.

1.1.3. GEOMETRY LEARNER AS A STUDENT-TEACHER

Being a student-teacher in the Department of Mathematics Science and Technology Education at the University of Limpopo was one of the most important mind openers into seeing the beauty of mathematics. However, just as I thought I was done with Euclidean geometry, in my first year as a student-teacher, as I was going through mathematics for educators' course 1 outline, Euclidean geometry was there. I was disappointed, particularly because I did not have solid understanding of geometry, yet as a teacher in training I realised that at some point I would be required to teach it. It was then that I realised that I had to battle with my "negative-attitude" towards geometry. My expectation was that a lecturer would come and stand in front of us and start explaining, just how my teachers used to teach. However, things were totally different. The habitual tradition of a teacher doing things on our behalf was no more.

It was then that I realised as a student-teacher I had to put on two hats at the same time; as a student and also as a teacher. Firstly, as a student I had to learn to interact with the geometry content material and work on geometry tasks that our lecturer would give us from time to time in the form of worksheets. The tasks were designed from simple to complex and it was my responsibility to get as much information as possible so that I could deal with the demands of each tasks. Not only to deal with the demands of the tasks but to interact with the peers in our groups, whole classroom and with our lecturer. The way the course was structured created and embraced the environment of sharing ideas in the learning geometry and other content areas. Such an environment became central towards my development of Euclidean geometry knowledge. Therefore, I became aware that what we bring to our geometry learning as learners can be used in a meaningful way to enhance our geometry conceptual development. Such was an important experience that the high school environment denied me.

Lastly, as a teacher I realised that at some point I would be required to teach geometry. This meant that I had to be knowledgeable in Euclidean geometry as well as in the methodology of teaching it. As a result, my role as a pre-service teacher was to go beyond the algorithm and search for meaning attached to it. Thus, it was important for me to develop pedagogical content knowledge of Euclidean geometry. This helped me in moving from one level to the next in terms of the geometry understanding. This for me was significantly important since it is an incontestable fact that no one can teach beyond their own understanding (Van Putten et al., 2010).

1.1.4. GEOMETRY TEACHER

As a student-teacher I was exposed to various teaching approaches, such as problem solving, discovery approach, telling method, whole class discussion and others which could be used in the teaching of mathematics. Such various approaches and diverse geometry experiences as a learner and student-teacher established my belief on what creation of meaning in geometry is about. In teaching Euclidean geometry in Grade 10 I preferred mostly problem solving and discovery approach. I would prepare lessons that are aligned to mathematics curriculum work schedule. I avoided as much as possible to rely on learners' textbooks, particularly because of one of the undergraduate Method for Mathematics Education 311 (MMAT 311) modules which focused on "Material Evaluation" exposed us to evaluation of learning materials (textbooks) that learners use. Moreover, based on seven categories suggested by Kulm, Morris and Grier (2000), the textbook that learners were using was not structured in such a way that it can promote effective learning of geometry. As a result, I had to consult various sources in preparing geometry lessons.

However, despite my efforts as well as using instructional approaches that literature considers as effective approaches, I realised that learners in my

classroom were struggling with geometrical reasoning. These difficulties were identified from the learners' responses in classroom interactions, class activities, home activities, and class tests. This was a serious issue and I did not want my learners to develop "negative-attitude" towards geometry. To add to this, the literature reviewed confirmed that learners encounter difficulty in geometric reasoning (Panaoura & Gagatsis, 2009; Naidoo, 2013; Fujita, Yutaka, Kunimune & Jones, 2014). Moreover; this has concerned me since it is happening at an introductory stage of the topic in Further Education and Training (FET) mathematics curriculum. Hence, Makgakga (2011) cautioned that errors and misconceptions ignored at this stage could become recursive later in the proceeding grades for such learners.

When I discussed the issue of learners' geometric reasoning difficulty with my colleague who was teaching geometry in Grade 10 for many years, I found it was a similar issue. To reiterate, the discussion we had reminded me of my geometry experiences. As a result, it appeared to me that learners are not used to explain their thought processes in geometry. This challenged me to think of questions such as: what factors contribute to poor student reasoning in geometry? How can the growth of learners' understanding of geometry be enhanced? How can instructional decisions be restructured in a manner which responds to effective learning of geometry?

Fortunately, I was a registered DMSTE honours student at UL, where the importance of putting theory into practice was highly emphasised. One of the modules (MAED 811) titled "Teaching and Learning Mathematics", exposed me to teaching mathematics for understanding. Hence, my honours research report for the research project module focused on teaching completing a square for conceptual understanding. Furthermore, MAED 811 was the module which introduced me to Pirie and Kierien's (1994) theory of growth of mathematical

understanding. It was my first encounter with the concept of folding back as a vehicle for the growth of mathematical understanding.

The experience I had with MAED 811 provided me with genuine insight as to how some of questions raised while teaching geometry could be addressed. Hence, folding back became a quest for me to undertake the present study in an attempt to improve learners' reasoning in geometry. The difficulties that learners had in geometric reasoning inspired me, showing how I could explore the role of folding back in alleviating them. Ndlovu and Mji (2012) note that such difficulties suggest that efforts should not be spared until we can effectively and creatively teach for the clearer understanding of geometry by a greater number of learners.

1.1.5. GEOMETRY IN SOUTH AFRICAN MATHEMATICS CURRICULUM

The South African education curriculum has undergone various reforms under the democratic dispensation. Such reforms amongst others include the introduction of Outcome-Based Education (OBE), Curriculum 2005 and Curriculum Assessment and Policy Statement (CAPS). These reforms have also affected the content areas of various subjects; for an example, Euclidean geometry is one of the victims of such reforms. In 2008 Euclidean geometry was excluded from the mathematics school curriculum; however, it was assessed in Paper 3 which was an optional paper. According to Bowie (2009), the exclusion of Euclidean geometry from the main curriculum has created a lack of consistency in the study of space and shape. For an example, Siyepu and Mtonjeni (2014) argue its exclusion from the secondary school curriculum presented a problem to learners registering for engineering courses at university.

Moreover, Van Putten, et al. (2010) point out that Euclidean geometry as optional in Paper 3 did not only depend on the teachers' knowledge, but also on their attitudes towards it, whether they choose to teach it or not. Thus, it was up to

the teachers as to whether they want to teach geometry or not. This further appears to disadvantage learners who have interest in this content. Furthermore, this denied learners' geometry benefits such as reasoning skills which help learners in developing and evaluating deductive arguments about figures and their properties that help them make sense of geometric situations (NCTM, 2002). In addition, NCTM further indicates that emphasis on geometric reasoning can help learners organise their knowledge in ways that enhance their geometry conceptual development. As a result, international calls have been made for reasoning and proving to permeate school mathematics (Otten, Gilbertson, Males & Clark, 2014). Fortunately, the latest South African National Curriculum Statements for mathematics (DBE, 2011) has reintroduced Euclidean geometry in the main school curriculum. Hence, one of the aims of mathematics as highlighted in CAPS mathematic policy document is to provide the opportunity to develop in learners the ability to be methodical, to generalize, to make conjectures and try to justify or prove them (DBE,2011). This reintroduction rendered geometry to be compulsory and be given equal attention as with other content areas in mathematics. Thus, this compelled teachers not to choose but teach geometry in mathematics curriculum. However, despite introducing geometry back to the school curriculum it remains a threat to many learners and teachers (Siyepu, 2012).

1.2. STATEMENT OF THE RESEARCH PROBLEM

Reasoning in geometry helps learners in developing and evaluating deductive arguments about figures and their properties that help them make sense of geometric situations (NCTM, 2002; Brown, Jones, Taylor, & Hirst, 2004). However, evidence from numerous research studies makes it clear that most learners struggle with geometric reasoning (Kutama, 2002; Panaoura & Gagatsis, 2009; Nassar, 2010; Naidoo, 2013; Fujita et al., 2014). Approaches to teaching geometry have been shown to be highly affecting learners' reasoning ability (Nassar, 2010). Traditional teaching approaches that Gunhan (2014) views as

characterised by teachers giving definitions, making no use of concrete materials and practical ways to explain mathematical concepts; hinder learners' geometric conceptual development (Acquah ,2011). However, in the literature little has been done on the growth of mathematical understanding with particular emphasis on geometric reasoning. Therefore, the reported study sought to explore how teaching for understanding could enrich learners' reasoning in geometry. Teaching that is framed around folding back as represented by Pirie and Kieren (1994) could enrich learners' reasoning in geometry.

1.3. PURPOSE OF THE STUDY

The purpose of the current study is to explore the role of folding back in enriching Grade 10 learners' reasoning in geometry.

1.4. RESEARCH QUESTIONS

In order to pursue the purpose of this study, the reported study sought to answer the following research questions:

- How does folding back support learners' interaction with geometric reasoning tasks during the lessons?
- How does a Grade 10 mathematics teacher use folding back to enrich student reasoning in geometry?

1.5. RESEARCH METHODOLOGY

This study adopted a constructivist teaching experiment methodology as a relevant research design in which interactions were used to organise and guide in experiencing the effect of folding back on learners' learning geometry (Steffe & Thompson, 2000). Thus, the teaching experiment methodology was found useful in studying learners' geometric reasoning as a result of mathematical interactions

in their learning of geometry. The participants in the study were Grade 10 learners in Limpopo Province. As a reference to the research design adopted, the data was collected through video recording. In analysing data, information-rich interaction for each mathematical learning activity, where folding back was observed was selected. The selections of such information-rich interactions were guided by the Pirie and Kirien (1994) model of growth of mathematical understanding, which served as a theoretical framework. Replicability and generalizability as recommended by Steffe and Thompson (2000) ensured rigour for the reported study. Detailed account on research methodology will be discussed later in Chapter 3.

1.6. SIGNIFICANCE OF THE STUDY

This part of the study discusses the role of folding back in enriching learner's geometric reasoning and the anticipated contributions of the study are as follows: The results of the study are likely to provide teachers with deeper understanding of how they can create a learning environment where folding back takes place. Such a learning environment enriches learners' reasoning in geometry. Moreover, the results of this study could inspire teachers to provide appropriate stimulations to their learners. The pedagogy of mathematics will be further informed. The study is likely to add to the body of knowledge in geometry, since there is limited literature dealing directly with the role of folding back in geometric reasoning.

1.7. RESEARCH SETTING

For the purposes of adhering to the ethics in this study, pseudonyms have been used. This was done for the purpose of anonymity and confidentiality of the school as well as the participants as suggested by Creswell (2007). A letter of permission was submitted to the school after having been granted permission by

the Limpopo provincial Department of Education to conduct the study. The permission was also approved by the management of the school.

The school is located in one of the villages in Moletjie, approximately 25 kilometres north of Polokwane in Limpopo Province. The school is populated by over 600 Sepedi speaking learners from nearby villages as well as those from where the school is located. The school offers commercial, science and agricultural subjects. Amongst the grades, there were two grades 10 classes, the first being the grade 10 A with 30 learners doing Mathematics and Accounting, and the second being the grade 10 B with 54 learners enrolled in Mathematics and Science streams. Of particular relevance to the study was grade 10B class. Thus, the teaching experiments were conducted in Grade 10B.

1.8. STRUCTURE OF THE STUDY

This dissertation is divided into five chapters outlined as follows: The first chapter presents a background of the research study. The purpose of the study along with research questions guiding the study is also captured. The second chapter provides a theoretical framework and reviews pertinent literature for the study. The third chapter outlines the methodology of the study. The rationale for choosing a constructivist research paradigm and a teaching experiment methodology as a research design are discussed. How data was collected and issues of rigour in teaching experiments are captured. Issues of ethical considerations were also discussed. The fourth chapter contains analyses of the learners and researcher-teacher interactions from the collected data. The last chapter provides a discussion of the findings, with particular attention to each of the two research questions that guided the reported study. Recommendations are also provided.

1.9. CHAPTER SUMMARY

In this chapter, I provided a background for the study by reflecting on my experiences, literature, geometry school curriculum and how these have resulted in my undertaking of this study. The purpose of the study, research questions, methodology and significance has also been captured. Moreover, the setting of the study was also illustrated. The chapter concludes by providing the structure of the dissertation.

CHAPTER 2: LITERATURE REVIEW

This chapter presents a literature review of a study that explored the role of folding back in improving learners' reasoning in geometry. Presented in the literature are the following aspects: Mathematical reasoning, reasoning in geometry, factors contributing to a low level of geometric reasoning and approaches to teaching geometric concepts. The theoretical framework through which the study was conducted is also captured. An appraisal in the illustration for the purposes of composing the study was informed by the argument that teaching that is framed around folding back as presented by Piere and Kieran (1994) improves Grade 10 learners' reasoning in geometry. A number of studies that used folding back are also captured.

2.1. REASONING IN MATHEMATICS

The concept of mathematical reasoning has been widely researched in mathematics education (English, 2013; Carpenter, Franke & Levi, 2003; Kilpatrick, Swafford & Findell, 2001; NCTM, 2009; Sumpter, 2013; Sundstrom, 2013). Mathematical reasoning is, according to English (2013), reasoning with structures that emerge from our bodily (concrete or base level) experiences as we interact with our environment and transform such experiences into models of abstract thought. This is supported by Kilpatrick, Swafford and Findell (2001) who perceive mathematical reasoning as the capacity for logical thought. Although Carpenter, Franke and Levi's (2003) views on mathematical reasoning is limited, in that it does not talk about concrete experiences and logical thought, its main value lies in explaining why a particular statement is true. From these views on mathematical reasoning, it can be argued that mathematical reasoning is the logical

communication of the developed mathematical constructs or thoughts constructed through learning experiences.

Mathematical thought is central to mathematical reasoning. The vehicles for such a thought involve: Analogy; metaphor and imagery. These vehicles, according to English (2013), are central to the conceptual development of mathematical understanding. They are described in detail below.

2.1.1. REASONING WITH ANALOGY

Reasoning by analogy is a fundamental human trait and the human ability to find analogical correspondences is a powerful reasoning mechanism (English, 2013; Gentner, 2003). English (2013) defines analogical reasoning as the transfer of structural information from one system (the source/base), to another system (the target) through mapping relational correspondences between the two systems. This is similar to Gentner and Smith's (2012) views that analogical reasoning is the ability to perceive and use relational similarity between two systems. These views illustrate that in an analogical reasoning, prior constructs are foundations upon which new constructs build. Therefore, in consideration of these ideas, it could be said that the strength of the new constructs depends on the prior understood constructs.

Furthermore, the transfer of knowledge between the systems is achieved through matching process, which entails finding the relational correspondences between the two systems (English, 2013). It has been widely documented that analogical mapping requires the correspondences between the source domain and the target domain and projecting inferences from the base to the target (Gentner & Colhoun, 2010; Gentner, 2003). Moreover, Leuzzi and Ferilli (2013) held the same view as (English, 2013), that, the analogy is important in the transfer of knowledge and inferences across different concepts, situations, or

domains. In addition, Amir-Mofidi, Amiripour and Bijan-Zadeh (2012) echoed that analogical reasoning allows learners to apply commonalities between mathematical relations to help grasp new concepts through contributing to integral components of mathematical proficiency. Ideally, the result of analogical reasoning is to permit rapid learning of a new domain by transferring knowledge from a known domain, and it promotes noticing and abstracting principles across domains (Gentner & Loewenstein, [Sa]:1423) .

However, Dindyal and Pang (2009) insightfully recognised that, in problem solving, the exercise of analogical reasoning is not merely the memorizing of the solution of previously solved problems. Dindyal and Pang further add that it involves purposeful mathematical thinking in determining the structural similarities and relational properties between the source and target problems. Although analogical reasoning is important in the development of mathematical constructs, it does not always lead to valid conclusions (Dindyal & Pang, 2010). In addition, Dindyal and Pang (2010) attest that without adequate relational understanding on both the source and the target systems, learners simply focus on the surface similarities and neglect the structural differences.

2.1.2. REASONING WITH METAPHOR

Reasoning with metaphors is generally considered a fundamental way of human thinking (Chiu, 2002; English 2013). It has also been documented that mathematical ideas are essentially metaphorical in nature (Lakoff & Núñez, 2000). This seems to imply that mathematics learning and problem solving can be enriched through using metaphors (Lai, 2013). According to Font, Bolite and Acevedo (2010), metaphor allows us to generate a new system of practices (systemic perspective) as a result of our understanding of the target domain in terms of the source domain. Metaphors in this study are defined as understanding knew concepts (the target) on the basis of another prior understood knowledge

(the source). Metaphorical reasoning is generally defined as comparing two domains of experience through giving meaning to elements of one of these domains (the target domain) by reference to structural similarities in the other (the source domain) (English , 2013).

The importance of metaphors in mathematics education has been well documented (Chiu, 2000; English, 2013; Lai, 2013). According to English (2013), learners can generate new concepts metaphorically through building on their prior knowledge. This happens when learners are able to reason metaphorically, that is, to see connection between mathematical ideas (Chiu, 2000). In so doing, they enrich their memory to recall connected mathematical ideas and make sense of their mathematical representations (Chiu, 2000). Furthermore, metaphorical reasoning provides learners with a tool for thinking (Lai, 2013). When learners' thinking is enriched, they can develop mathematical understanding of concepts and procedures that are difficult with concrete analogies (English, 2013). Thus, it could be said that metaphors improve learners' conceptual understanding in mathematics. Although there is a substantial agreement about the advantages of metaphors on mathematics learning, some researchers argue that it also has several potential disadvantages (Lai, 2013).

2.1.3. REASONING WITH IMAGE

The importance of image and visualisation in the teaching and learning of mathematics, particularly on the development of geometric concepts and reasoning has been well emphasised (Battista, 2007; Handscomb, 2007; Bansilal & Naidoo, 2012). An Image in this study is used interchangeably with an object and implies any form of representation-concrete or imagined, and either way they are usually represented as lines on paper, the geometric diagram (Handscomb, 2007). An image, according to Handscomb (2007) is an essential component of the image-based reasoning process. This corresponds with the ideas of Chambers

and Timlin (2013). They perceive the role of imagery in terms of helping strengthen the hierarchy of ideas that learners encounter in their geometric tasks, thus developing a chain of reasoning that build up their constructs using images. I cannot but agree with Fischbein (1987) who attest that imagery not only organises the data in meaningful structures, but also provides the important lenses for guiding the development of a solution. These views illustrate that images play an important role in assisting learners' geometry reasoning when solving geometric tasks.

2.2.   GEOMETRIC REASONING

Battista (2007) views geometric reasoning as the use formal conceptual systems to investigate shape and space. He further indicates that these systems use concepts such angle measure, length measure, congruence, and parallelism to conceptualise spatial relationships within and among shapes. Geometric reasoning is further viewed as a dialectical process between a figure and its conceptual characteristics of geometry (Mariotti, 1995). Looking at geometric reasoning in these two ways involves "abilities to give logical explanations, argumentations, verifications, or proofs to arrive at convincing solutions to geometrical problems" (Budi, 2010, p. 7).

Greenstein (2011) holds the same view as Budi (2010), that geometric reasoning involves the application of geometric properties and relationships in problem solving. Learners must be able to use properties and relationships of geometric shapes in solving geometric tasks that require them to reason. Conceptual understanding of figures and their properties is significant in geometric reasoning. Such properties may be given or determined by the inspection of the diagram, or established by logical deduction (Handscomb, 2007).

Reasoning in geometry helps learners in developing and evaluating deductive arguments about figures and their properties that help them make sense of geometric situations (NCTM, 2002). NCTM further indicates that emphasis on geometric reasoning can help learners organise their knowledge in ways that enhance their geometry conceptual development. Similarly, Brown et al., (2004) illustrate that geometric reasoning is important in developing learners' knowledge and understanding and the use of geometrical properties. In consideration of these ideas, it is without a doubt that geometric reasoning is a tool which can be used to develop learners' geometric knowledge. It can further be argued that learners can use geometric reasoning to communicate their developed geometric knowledge. Learners should not only develop conceptual understanding of geometric properties of figures, but also be able to apply such properties in related problem solving environment. For example, learners can use geometric properties to communicate why a particular proof is valid. Hence, Evans (2007) indicated that development of geometric reasoning enables learners in the process of working with mathematical proof.

Although reasoning in geometry is beneficial in the development of geometrical understanding, numerous studies, internationally and locally have documented the difficulties that learners encounter in this key area in geometry (Panaoura & Gagatsis, 2009; Naidoo, 2013; Jones, Kunimune & Jones, 2014; Gunhan, 2014). Van Putten, Howie and Stols (2010) conducted a case study entitled "Making Euclidean geometry compulsory: Are we prepared?" Their study involved pre-service teachers (student-teachers) who came from diverse secondary schools reflecting a mix of rural and urban, private and government, well-resourced and under-resourced institutions. They concluded that based on their case study, learners leaving secondary schools in South Africa to pursue a career in teaching mathematics do not have an in-depth understanding of geometric concepts. Similarly, Naidoo (2013) found in his study that while the learners from both working-class and middle-class backgrounds employed similar

techniques when solving geometry problems, their methods, logic and geometric reasoning differed considerably. His findings also revealed that learners in working-class contexts struggle with demonstrating well-defined logic in their geometric reasoning as compared to their middle-class counterparts.

Moreover, the results of the study conducted by Fujita et al. (2014) indicate that although learners discovered various methods to construct a square, they had difficulties in using properties of shapes to reason why their construction would be correct. Their findings further indicated that learners rely on the visual perception of the given geometrical shape rather than on a mathematical deduction based on the properties of the geometrical object involved. Similarly, Gal and Linchevski (2010) asserted that children prefer to rely on a visual prototype rather than a verbal definition when classifying and identifying shapes. In addition, Gunhan (2014) carried out a qualitative case study investigating how learners use reasoning skills in geometry-related tasks. The results showed that learners sometimes have insufficient geometrical knowledge and visual perception, and at times, they are not able to provide a mathematical argument for its validity. Gunhan further concluded that school curriculum should place more emphasis on reasoning skills, and when it comes to geometrical concepts, learners ought to be presented problems that allow them to use different reasoning skills.

2.2.1. GEOMETRIC REASONING IN PROOFS

Mathematical reasoning and proof offer powerful ways of developing and expressing insights about a wide range of phenomena (NCTM, 2000). Similarly, Evans (2007) indicates that geometric reasoning needs to be developed in learners in order to enable them to begin the processes of mathematical (geometric) proof. Geometric proof is central to geometric reasoning in the sense

that learners cannot do or solve geometric proofs without using geometric reasoning as part of it.

Fetisov and Dubnov (2012) define proof as a chain of deductions through which the truth of the propositions to be proved is derived from axioms and previously established propositions. Similarly, Hanna and De Villiers (2008) assert that proof consists of explicit chains of inference following agreed rules of deduction. It is said that such explicit chains are often characterized by the use of formal notation, rules of manipulation. Hanna and De Villiers further recognize that mathematicians' proof is also a sequence of ideas and insights with the goal of mathematical understanding with emphasis on understanding why a claim is true. From these views, it can be argued that geometric proof is a form of geometric reasoning.

Battista and Clements (1995) suggest that there is a need to shape the curriculum in order to develop learners' explanation and argumentation skills and so that learners use proof to justify their ideas. Such skills are forms of mathematical reasoning (Brown, Jones & Taylor, 2003). Fortunately, this is supported by CAPS policy document for mathematics curriculum, which specifies that one of the aims for mathematics is to provide the opportunity to develop in learners the ability to be make conjectures and justify them (DBE ,2011). This is supported by Steen (1999), who perceives proof to be central to geometric reasoning. To this, Hanna and De Villiers (2008) add that one of the principles of proof is to specify clearly the assumptions made and to provide an appropriate argument supported by valid reasoning so as to draw the necessary conclusions.

In another view, Herbst and Brach (2006) attested that the essence of mathematics is to make abstract arguments about general objects and to verify these arguments by proofs. Similarly, Stevens (2012) demonstrates that a core part of creating proof lies in the ability to articulate the reasoning that leads to

conclusions. Stevens further indicates that explaining one's approach and justification can be essential to developing the skills required to developing mathematical arguments. I cannot but agree with Weber (2001) who alludes that by examining a proof, a learner can understand why a certain statement is true. Additionally, De Villiers (2012) emphasises that logically explaining (proving) why a result is true gives one deeper insight into its premises.

Although proofs occupy a central role in the teaching and learning of geometry, they present a lot of difficulties to learners (Battista, 2007). Furthermore, learners encounter difficulties in understanding proof in geometry (Kunimune, 2000). In addition, Stevens (2012) cautions that the advanced thought processes required to construct proofs present difficulties for learners. Such advanced thought processes of learning procedures and proofs without a good understanding will leave learners ill-equipped to use their knowledge in later life (DBE, 2011). Moreover, learners do not know what a mathematical proof is and find it challenging to validate arguments (Boyle, 2012). Ndlovu and Mji (2012) attest that such difficulties suggest that efforts should not be spared until we can effectively and creatively teach for the clearer understanding of proof by a greater number of learners.

2.3. SOME FACTORS CONTRIBUTING TO LOW LEVEL OF GEOMETRIC REASONING

2.3.1. POOR TEACHING OF GEOMETRY

The Curriculum Assessment and Policy Standard (2013) in mathematics curriculum require teachers to use instructional approaches that respond to effective learning. Such teaching of mathematics must focus on encouraging learners to express their reasoning (Brandt & Tatsis, 2009). However, approaches to teaching geometry have been shown to highly affect learners' reasoning ability

(Nassar, 2010). Traditional teaching methods that Gunhan (2014) views as characterised by teachers giving definitions, making no use of concrete materials and practical ways to explain mathematical concepts hinder learners' geometric conceptual development (Acquah ,2011). Furthermore, De Villiers (2012) criticises the traditional approach of focusing primarily on the teaching, learning and practising of standard algorithms for repressing learners' natural inquisitiveness and quest for deeper understanding.

In another view, Battista (1999) emphasises that there are instructional approaches that have been shown to be inadequate for developing a conceptual understanding of geometry for all learners. This is supported by Luneta (2008) who indicates that poor performance is attributed to teachers' instructional strategies. These views demonstrate that learning is strongly and necessarily linked to teaching (Stoke, 2010). Therefore, it can be argued that how teachers teach influences how learners learn, and there is therefore a great need to move from a teacher-centred traditional perspective approach to a learner-centred perspective approach.

2.3.2. LACK OF SPATIAL KNOWLEDGE

The importance of spatial knowledge on the development of geometric concepts and reasoning has been well recognized (Séra, Kárpáti & Gulyás, 2002; Nakin, 2003; Battista, 2007). Spatial knowledge is, according to Séra, Kárpáti and Gulyás (2002), characterised by the ability of solving spatial problems through using the perception of two and three-dimensional shapes and the understanding of the perceived information and relations. Spatial knowledge is used interchangeably with spatial visualization. Spatial visualization is, according to Nakin (2003), important in the teaching and learning of mathematics in the sense that the learning of mathematics can be influenced by what is being visualized and the spatial abilities embedded in such visualization.

Battista and Clements (1995) illustrate that by focusing on helping learners build the visual and empirical foundations for higher levels of geometric thought, learners end up using such thoughts meaningfully as a mechanism for justifying ideas. To add on this, NCTM (2000) considers spatial visualization as an important aspect of geometric thinking. NCTM further illustrates that spatial visualization creates mental images of learners' surroundings and objects in them. Likewise, visual imagery is important in reasoning skills in mathematics (Presmeg, 1992). This seems to indicate that it is important for teachers to realize the need to assist learners in building their spatial visualization. Once educators are able to provide such assistance, learners could develop their geometric reasoning. It is, however; equivocally important for teachers to understand how their learners build their visual imagery. As a result, learners' ability to visualize concepts will be enhanced and eventually, becomes a tool through which they could use when solving geometric reason mathematical activities. It could further be said that how learners develop their visual imagery in geometry, will influence their geometric reasoning. In so doing, visualization becomes the key to successful learning and understanding as perceived by Makina (2010).

Despite the significance of spatial visualization, learners encounter difficulties in this domain. Widder and Gorsky (2014) indicate that spatial visualization is quite difficult for some learners. In addition, Fischbein (1993 cited in Widder & Gorsky, 2014) hold the view that poor visualization skills interfere with the proper construction of the appropriate mental model of the geometric image, thereby leading the learner to rely primarily on a figural image which may be misleading. This is supported by (Munro [Sa]:3) who argues that learners cannot act mentally on a shape or visualize it being changed. He further adds that those learners' associate spatial concepts with inappropriate criteria, for example, the child believes that a change in position or size means a change in shape or concept.

2.4.   TEACHING APPROACHES TO GEOMETRY CONCEPTS

2.4.1.   INDUCTIVE AND DEDUCTIVE APPROACH

Both induction and deduction approaches are important in the teaching and learning of geometric concepts. An inductive reasoning is, according to Annenberg (2004) the process of arriving at a conclusion based on a set of observations. This process is characterized by conjecturing about geometric objects, analysing configurations and reasoning inductively about relationships to formulate conjectures NCTM (2009). Moreover, this approach is often used for introducing new mathematical ideas and concepts (Grigoriadou, 2012). This, according to NCTM (2009), does not only provide learners with the opportunity to develop their understanding of the mathematical relationships, but improve their proficiency to explain them.

On the other hand, deductive reasoning is, according to Grigoriadou (2012) about drawing specific conclusions from true given premises by following certain rules of logic. According to Jones (2002) this is central to mathematics and intimately involved in the development of geometry. Moreover, such development of geometry in this approach is characterized by construction and evaluation of geometric arguments, developing and evaluating deductive arguments about figures and their properties that help make sense of geometric situations (NCTM, 2009). NCTM further designated that deduction approach to help learners to follow up their conjectures with efforts either to justify or disprove them. However, Grigoriadou (2012) notes that it is important when the deductive system of geometry is taught to learners; it is not appropriate to introduce learners to the deductive part of the system without first showing them the importance of the role of induction in the creation of this system.

Although inductive and deductive approaches are beneficial to learners, they have certain disadvantages. To this, Simon (1996) cautions that the characterization of learners' mathematical explorations and justifications as inductive and deductive is incomplete. This seems to foreshadow that these approaches are learner-centred and thus remain a responsibility of teachers to create effective geometry learning environments. Moreover, if a teacher fails to create such an environment, learners' explorative opportunities could be hindered.

2.4.2. DISCOVERY APPROACH

Price (1967) perceives the discovery approach as an inductive and intuitive approach which involves learners in the uncovering of geometrical concepts. This approach seems to be learner-centred in that learners' role is central to their construction of geometry knowledge. In addition, learners take responsibility of their learning. Likewise, it can be said further that this approach seems to reject the notion of learners as recipients of knowledge but rather co-constructors of their geometric knowledge. To add on this, Borthick and Jones (2000) emphasize that in the discovery approach, learners learn to recognize a problem, characterize what a solution would look like, search for relevant information, develop a solution strategy, and execute the chosen strategy. Thus, in the discovery approach, opportunities should be created for learners explore on their own and become attached to their learning. Hence, Acquah (2011) cautions that learners cannot gain a better understanding of any concept if opportunities are not made for them to discover it for themselves.

Moreover, Acquah (2011) demonstrates that this approach is characterized by learners' direct involvement in practically manipulating geometric figures to identify their properties. Additionally, in this approach learners could identify properties of a parallelogram without being told such properties. Once learners are able to directly manipulate geometric figures, they could apply such

properties in problem solving. Furthermore, practically manipulating figures reveals a significant advantage of the discovery learning method, which according to Castronova (2002), has the capacity to motivate and allow learners to seek information that satisfies their natural curiosity. Acquah further argues that learners cannot gain a better understanding of any concept if opportunities are not made for them to discover it for themselves. This approach gives learners the opportunity to discover relationships and draw conclusions, thus improving their conceptual development of geometry.

Although learners explore on their own, Price (1967) recognizes that the role of the teacher in providing guidance to ensure that the learners discover the rule, or principle, or concept, which is the goal of instruction, is important. Thus, learners should not be left on their own as they are exposed to the discovery approach. It is important for teachers to follow the learners' exploratory skills in the discovery approach. This will eventually help them to provide help in cases where learners' fail or are stuck. Teachers could use prompts or questions as tools to help learners in the discovery approach. Tran, Van de Berg, Ellermeijer and Beishuizen (2014) suggest that teachers' help should be in the form of questions to help learners to think about the process of discovery, but not showing learners what they need to do. However, in providing such assistance, Bencze (2009) is of the opinion that teachers could face many of difficulties in channelling and maintaining the interest of learners as they engage themselves in discovery inquiry activities and trying to derive appropriate conclusions.

2.4.3. REPRESENTATIONAL APPROACH

The ways in which mathematical concepts are represented is fundamental to how learners can develop conceptual understanding of such concepts. Representations are mostly used in instructional decisions to emphasize key mathematical concepts and support a learner to learn (Mitchell, Charalambous, &

Hill, 2014). Hence, the use of variety of representations could enable learners to understand mathematical concepts (Panaoura, Gagatsis, Deliyianni & Elia, 2010). This seems to illustrate that through representations, learners could grasp mathematical concepts. Such representations, according to Lesh, Post and Behr (1987), can be classified into pictures, written symbols, spoken language, and relevant situations and manipulations. Thus, one could use drawings as tools to enhance the learners' development of their geometry understanding. Not only will they develop their understanding, but they will be able to communicate effectively using visual, symbolic and/or language skills in various modes (DBE, 2011).

Moreover, the properties of mathematical processes are often discovered by studying the geometric properties of their visual representations (Lehrer & Chazan, 2012). Through the use of representations, learners can work with properties of various geometric figures and improve on their geometric reasoning. For example, when learners are provided with a drawing of a square, they could measure the sides and use its properties to solve the tasks. In so doing they will be manipulating the figure for its properties and engage actively with geometric concepts. Thus in this case, learners can work with external representations such as drawings or pictures to identify properties of a parallelogram. Representing geometric concepts can help learners to improve their visualization in geometry. Thus, the importance of representations in learning of geometry can help learners develop their geometry conceptual understanding through using images. Once learners develop their geometric understanding, they can use it in their geometric reasoning activities. Hence, Panaoura (2011) points out that representation are regarded as a useful tool for understanding geometrical concepts and for solving geometry tasks. Therefore, representation can support learners in making sense and reasoning about mathematical tasks (Mitchell et al., 2014).

The importance of representations in geometry seems to indicate that it would be a grave error for a curriculum to ignore the use of representations in its

instructional decisions. To this, Lehrer and Chazan (2012) caution that ignoring representations, curricular not only fail to engage a powerful part of learners' minds in service of their mathematical thinking, but also fail to develop learners' skills at exploration and arguments. Therefore, once a curriculum recognizes representations as central to the teaching and learning of mathematics, learners would be in a better position to communicate appropriately various representations (DBE, 2011).

Although representational approach is important in the teaching and learning of geometry, the possible setbacks involved in using this approach in the classroom must also be emphasised. Teachers may take for granted that learners will become easily initiated to the representations' structure or that representations will by default illuminate underlying mathematical ideas (Mitchell et al., 2014). This seems to indicate that it is important for teachers to carefully follow learners' representation path. This is important, since representation is not always associated with mathematical accomplishment (Sylianou & Pitta-Pantazzi, 2002).

2.4.4. TECHNOLOGICAL APPROACH

One of the potential uses of technology in mathematics education is using the dynamic geometry software (DGS) in the teaching and learning of geometry. According to Guven (2012), the use of DGS as a didactical tool has permitted teachers and researchers to study different teaching/learning methodologies, including those that improve the communication of mathematical concepts and ideas in traditional classes as well as those that develop learner-centered activities. To this, Bhagat and Chang (2015) further add that the sensible use of DSG is effective in the teaching and learning of geometry. This was no different from Jones (2005), who affirmed that DGS as an important tool for learners and teachers to make conjectures and control them and also understand the relationship between concepts. Furthermore, Stols, Mji and Wessels (2008)

illustrate that DSG allows teachers and learners to work quickly through numerous examples by dragging and enabling them to discover patterns, to explore and to test conjectures by constructing their own sketches.

The use of DGG can serve as an interactive context for making generalizations about geometric objects, thus leading to proof-generating situations, wherein justification and reasoning are encouraged (Govender, 2011). In addition, Christou, Mousoulides, Pittalis and Pitta-Pantazi (2004) equivocally assert that through the use of DSG, learners are able to generate fast, accurate and easy arguments leading to conjectures and suggested ways of reasoning, to problem solutions. This is further supported by Stols and Kriek (2011) who demonstrate that software allows learners to discover patterns, to explore and to test conjectures. Using DGS can provide an opportunity to a link between empirical and deductive reasoning (Guven, 2008). Ideally, the results of using technology of technological software such as Interactive Geometry software is the potential of helping learners understand geometry concepts better, while enhancing their geometric reasoning.

Even though DGS is important in the learners' geometry conceptual development, Bieda (2009) demonstrates that justifications and proof in DSG environment are not enough to develop conceptions of mathematical proofs. Comparably, Sipos and Kosztolányi (2009) have shown that the use of DGS leads the decrease of learners' desires to prove the theorems, and some learners find the computer difficult to use and therefore they become frustrated. The use of technology does not only affect learners but their teachers as well. According to Jenson and Williams (1992), technology complicates rather than simplify a teacher's life in the classroom. To this, Ndlovu, Wessels and de Villiers (2013) conclude that teachers who have no prior experience with computers struggle to get used to the computer hardware components. Similarly, Chan (2015) concludes that teachers have both positive and negative beliefs about the use of DSG in the

classroom. Despite the challenges faced by both teachers and learners, the use of DSG in rural schools, which are characterized by lack of resources, become difficult if not impossible to implement in the teaching and learning of geometry.

2.5. THEORETICAL FRAMEWORK

2.5.1. THE GROWTH OF MATHEMATICAL UNDERSTANDING

The concept of understanding in mathematics education has been widely researched (Skemp, 1976; Hiebert & Carpenter, 1992; Sierpinska, 1994; Pirie & Kieren, 1994). Skemp (1976) distinguished two types of mathematical understanding; relational and instrumental. He described relational understanding as "knowing both what to do and why" (p. 2). Instrumental understanding, on the other hand, was simply described as "rules without reasons" (p. 2). Relational understanding provides ways for obtaining information from learners' memory and develops the growth of mathematical understanding. Instrumental understanding, on the other hand, necessitates memorisation of rules.

Hiebert and Carpenter (1992), in describing what understanding is, emphasised that the degree of mathematical understanding is determined by the number and strength of its connections. They are of the view that a mathematical concept is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. Mathematical understanding develops as the networks become stronger and organized. On the other hand, it becomes limitless if the connections are weak (Hiebert & Carpenter, 1992). This agrees with Haylock's (1982) view which stresses that a simple but useful model for discussing understanding in mathematics is that to understand something means to make connections. Mathematical understanding develops as the networks become stronger and organized. On the other hand, it becomes limitless if the connections are weak (Hiebert & Carpenter, 1992). Skemp, Hiebert and Carpenter underscore

the importance of explaining mathematical knowledge as one of the key features in the development mathematical understanding: "articulating what one knows."

Sierpinska (1994), on the other hand, in examining mathematical understanding, discusses three different ways of looking at understanding: act of understanding, understanding, and processes of understanding. Firstly, an act of understanding is defined as the mental experience associated with the connection of what is to be understood within the basis of that understanding. Secondly, understanding acquired as a result of the acts of understanding. Lastly processes of understanding as characterized by connections that are created between mental concepts through reasoning.

These frameworks of mathematical understanding, and others, draw together mathematical understanding as characterized by creating a cognitive connection between mathematical concepts. They also characterized understanding in mathematics as a resulting connection of mathematical concepts. Although these cognitive connections of mathematical concepts play a central role in mathematical understanding, these frameworks seems to be quiet on how the growth of learners' mathematical understanding develops and how it can be represented. In response to these limitations, Pirie and Kieren (1994) have produced important work which intends to provide light on the growth of mathematical understanding.

The evolution of Pirie and Kieren's (1994) theory started with their first publication of "recursive theory of mathematical understanding" in 1989. In their first publication, they held the view that "Mathematical understanding can be characterized as levelled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication" (Pirie & Kieren, 1989, p. 2). The Pirie-Kieren theory views understanding as something different from an internalized and mental process in which a static

notion is acquired and then applied. Thus, this framework, unlike other frameworks of mathematical understanding, views mathematical understanding as a dynamic, growing and ever changing process (Pirie & Kieren, 1994). Therefore, it is in this framework that I find the lens through which this study is located. This study shares the same view that mathematical understanding can be perceived as growing through non-linear movement across various layers of sophistication. It is in this continual movement that this study sought to enrich the learners' reasoning in geometry.

2.5.1.1. Pirie-kieren model for growth of mathematical understanding

The Pirie and Kieren's theory (1994) describes the growth of mathematical understanding through its eight nested circles of the layers of understanding. The eight different layers of understanding are: primitive knowing, image making, image having, property noticing, formalising, observing, structuring, and inventising (see figure 1). The layers of the model grow outward from primitive to inventizing; however, this does not imply that mathematical understanding's growth is hierarchical.

Moreover, Pirie and Kieren (1989) demonstrate that these various interconnected levels of understanding do not equate with higher or lower levels of mathematical understanding. For example, primitive knowing does not represent low mathematical understanding but rather the starting place for the growth of mathematical understanding (Pirie & Kieren, 1994). Likewise, inventizing is the outer layer characterized by a fully structured understanding. However, this does not mean that a learner at this layer can no longer access the prior layers of understanding (Pirie & Kieren, 1994).

The nesting shows that the growth in understanding is neither linear nor monodirectional (Pirie & Kieren, 1994; Pirie & Martin, 2000). Furthermore, the

nesting shows that each layer contains the preceding layer and is contained in the layer that follows (Pirie & Kieren, 1994). For Pirie and Kieren, this mutual inclusivity of layers symbolize the back and forth movement of growth in understanding across these layers.

The Pirie and Kieren theory offers potential benefits in the teaching and learning of mathematics. For example, the theory provides a framework to map the growth of mathematical understanding by tracing the changing mathematical actions of the learners (Droujkova, Berenson, Slaten & Tombes, 2005; Towers & Martin, 2015). Furthermore, an important strength of the Pirie-Kieren's theory is that it enables the teacher to generate a very detailed account of the mathematical actions of individual and collective learners (Martin & Towers, 2014). Thus, the theory allows for listening and looking at learners' growth of mathematical understanding as it is happening (Kyriakides, 2010).

Ideally, the result of this theory is that it offers a means, through its layered model, for observing and describing the process through which knowledge is organized and reorganized, and how learners think about their understandings (Martin, 2008). To add to this, the back and forth movement confirm the notion that mathematical understanding is an ever changing process (Towers & Martin, 2015). Thus, one could perceive the Pirie-Kieren theory as a way of explaining understanding and considering how learners' mathematical understanding grows.

Although there are eight layers that describe the Pirie-Kieren theory, information about the first five layers of the model was presented in Figure since only the first five layers of the model were discussed in this study.
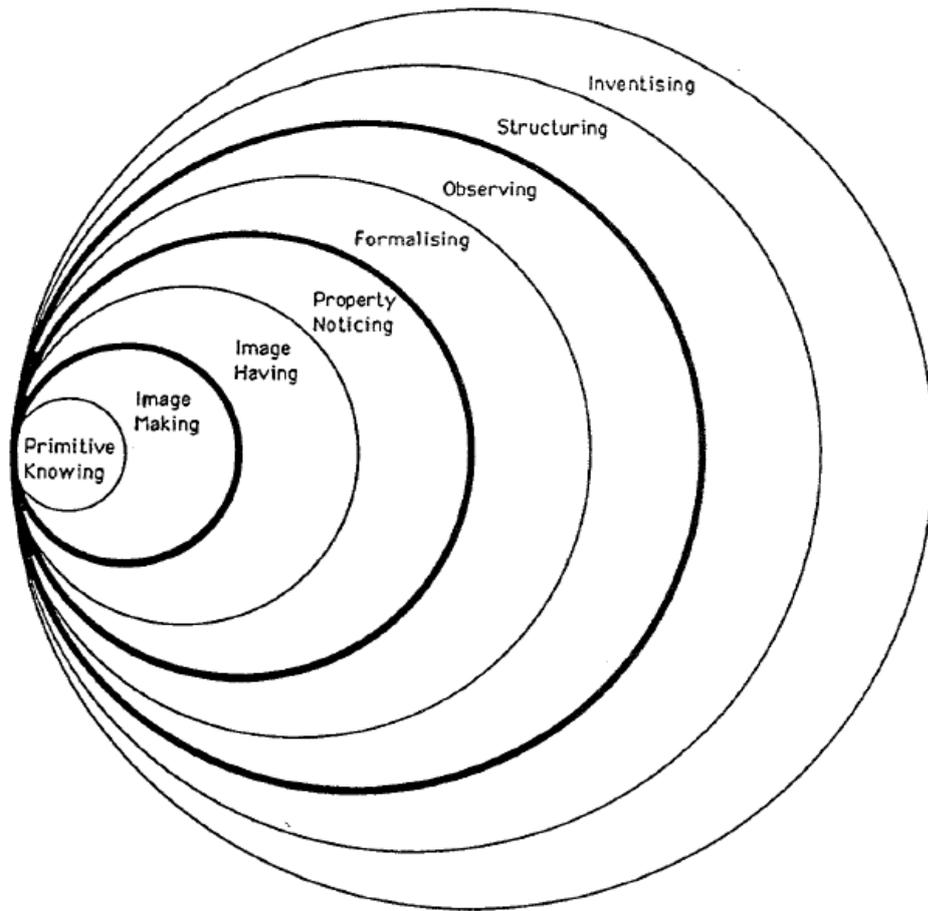
Figure 1: Pirie and Kieren (1994) model of mathematical understanding

2.5.1.2. Primitive knowing

In the Pirie-Kieren theory, Primitive Knowing is the first innermost level, consisting of a learner's previous knowledge brought to the learning environment. Pirie and Kieren (1994) define Primitive Knowing as the knowledge that an individual brings to their learning context. As has been indicated previously, this level does not represent low level mathematics, but the starting place for the growth of any particular mathematical understanding (Pirie & Kieren,1994). Thus, the growth of mathematical understanding begins when learners bring knowledge to the learning situation (Wright, 2014). This seems to show that at this level, prior knowledge that learners possess becomes a quest through which their growth of

mathematical understanding can be developed. Basically, one can view the primitive layer as a foundation through which the growth of mathematical understanding can be built upon. The knowledge that learners bring into their geometry learning is important not only in their geometry conceptual development, but their geometric reasoning as well. Pirie and Kieren (1989) highlight the significance of this layer in saying that the process of coming to know starts at this level. Hence, Warner (2008) adds that all understandings have their roots in primitive knowing. However, Pirie and Martin (2000) assert that it is important for appropriate knowledge to be selected and used as a basis for growth of understanding. This implies that the geometric knowledge that learners bring to their geometry learning may not necessarily be relevant for a certain geometry task at hand. Thus, not everything that a learner brings into the learning environment can be relevant. As a result, what a learner brings into the learning environment becomes a source through which he or she can manipulate and select appropriate knowledge. In so doing, they build subsequent understanding.

2.5.1.3. Image Making

According to Pirie and Kieren (1994), image making is the first layer to be built upon the primitive layer and refers to the level where a learner can make distinctions in his or her previous knowing. The knowledge in this level is used in new ways that include actions and activities with that knowledge (Pirie & Kieren, 1994). This seems to illustrate that image making is strongly linked to the primitive knowing layer in that the knowledge at the image making layer is influenced by the knowledge at the primitive knowing layer. However, at image making learners start distinguishing between the actions in the previous layer (Kaba & Şengül, 2015). At image making, the geometric knowledge that learners bring to their geometric learning is manipulated through engaging at hands-on mathematical tasks. Such tasks are aimed at helping learners develop particular initial conceptions and ideas for the meaning of a mathematical concept (Martin & Towers, 2014). In so

doing, the learner begins to build on his/her knowledge of the topic under discussion, working with this primitive knowing knowledge in an attempt to formulate images that will help them in doing the mathematical activities (Wright, 2014). It can be seen that image making is highly characterized by hands-on activities and such tasks elicit the growth of mathematical understanding as learners work on them. It can further be said that the "selection" of appropriate knowledge characterizes this level. Through manipulating and working from their prior knowledge, learners can select and build upon their understanding. Hence, at the level of image making, the learners begin to form images based on their Primitive Knowing to get an idea of what the concept is about (Cobb, 2012). Thus, once learners are able to manipulate their prior knowledge at image making, they can make sense of it and apply it.

2.5.1.4. Image having

Pirie and Kieren (1994) recognize that at the level of image having, a learner can use mental construct about a topic without having to do the particular activities which brought it about. In addition to this, Pirie and Kieren (1984) demonstrate that image having frees a learner's mathematics from the need to take a particular action. These show that a learner can develop geometry conceptual knowledge without being bound to do the activities that brought it about. Hence, learners at image having can bring in the existing images without the need to create such images (Wright, 2014). Unlike image making, the learner at image having is not necessarily compelled to do particular activities. Although the difference between image making and image having is noticed. Martin and Towers (2014) maintain that both image making and image having are important for developing understanding of a mathematical concept. This seems to indicate that the movement from image making to image having is important in the development of mathematical understanding. Moreover, at the image having layer learners begin to recognize general properties of the mathematical images they

have created (Gibbons, 2012). It could further be said that at this layer a learner is assumed to have developed an understanding and he or she can bring sense of meaning to the activities at hand. However, Martin, LaCroix and Fownes (2005) argue that such understanding may be mathematically limiting and context dependent. From a mathematical point of view, it can be further said that this level is characterized by abstraction. Such abstraction happens when learners are able to bring a mental plan and use it accordingly in solving the activities (Borgen, 2006). Furthermore, it is important to note that it is the learner who makes this abstraction by recursively building on images based on actions (Pirie & Kieren, 1989).

2.5.1.5. Property Noticing

Property noticing is, according to Pirie and Kieren (1989) more concerned with the process of examining an image for specific or relevant properties. This process may involve noticing distinctions, combinations, or connection between images. One can argue that this layer is almost similar to image making, in that they are all characterised by manipulation of an object. Although property noticing and image making are somewhat similar, unlike image making, at property noticing the individual is working at a higher level of sophistication for his or her level of understanding of concept (Borgen, 2006). This higher level of sophistication is, according to Martin (2008), characterized by acts of learners questioning their understanding and examining for what can be said about mathematical activity. Thus, a learner will not just manipulate an object but pursue the meaning attached to that object. For instance, at property noticing, once learners manipulate an iscolesce triangle, they can begin to question themselves about properties of such a triangle. In so doing, a learner could explain how the images are connected (Wright, 2014). Subsequently, a learner would be able to reflect on his or her image and recognize attributes and features of such image (Martin & Towers, 2014). Thus, it could be said that at property noticing, a learner

can identify properties of geometric figures and apply them in problem solving. For example, at this layer, a learner can identify a parallelogram and use its properties of opposite angles to determine the magnitude of an unknown angle. A learner in property noticing may realize that a square is a rectangle and that not every rectangle is a square. Certainly, the property noticing is closely bound to an image constructed, thus the property of an image is dependent upon a formed imagine.

2.5.1.6. Formalising

Formalising is a level which entails thinking consciously about the noticed properties, abstracting common qualities and discarding the origins of one's mental action (Pirie & Kieren, 1994). At this level, full mathematical definitions can occur as learners become aware of the knowledge they have constructed from the formation of images and their properties (Pirie & Kieren, 1989). .This seems to illustrate that in formalising, a learner can use noticed properties of a quadrilateral to prove that it is a parallelogram. Hence, formalising involves reasoning with the properties of the objects (Wright, 2014). For example, a learner could use the properties of the sum of the consecutive angles to show that opposite sides of a quadrilateral are parallel and he or she will eventually be able to define such quadrilateral as a parallelogram. Moreover, a learner in formalising could realize that the sum of angles in a triangle is equal to $180°$. In so doing, the noticed properties and generalizations of concepts made could help learners develop mathematical definitions (Gibbons, 2012).

Apart from the eight of layers of mathematical understanding, the Pirie and Kieren (1994) theory consists of three features: 'Don't need' Boundaries, folding back, and the complementarities of acting and expressing. Whilst it is not necessary in this study to fully elaborate all the features of Pirie and Kieren (1994) theory, the feature of *folding back* is particularly significant. Firstly, the 'don't need' boundaries are reflected in a theory, observable in the model, and illustrated by

the bold rings between some layers of mathematical understanding. The word some is carefully selected to indicate that 'don't need' boundary does not exist between each two consecutive layers of understanding. Beyond these boundaries, according to Pirie and Kieren (1994) theory the learner is able to work with concepts that are no longer obviously dependent upon previous forms of understanding, but these previous forms are rooted in the new level of understanding and readily accessible. They are further of the view that beyond the boundary, a learner does not need the specific inner understanding that gave rise to the outer knowing. For an example, when a learner has an image of a mathematical idea, he or she does not need actions or the specific instances of image making (Pirie & Kieren, 1994). Secondly, folding back as a feature is central to the growth of mathematical understanding in that it reveals the non-unidirectional nature of coming to understand mathematics (Pirie & Kieren, 1994). This feature guided this study and it is detailed as a proceeding subsection.

Lastly, in discussing the complementarities of acting and expressing, they are of the view that each level beyond primitive knowing is composed of a complementarity of acting and expressing. Acting, according to Pirie and Kieren (1994) embraces all previous understanding, providing continuity with inner levels whereas expressing gives distinct substance to that particular level. In addition growth occurs through, at least, first acting then expressing, but more often through to-and-fro movement between these complementary aspects. Acting and expressing are necessary before moving on from any level and as such, play a significant role in the growth of mathematical understanding. The complementarities of acting and expressing within the image making, image having and property noticing ring are labelled by verbs such as: doing, reviewing, seeing, saying, predicting and recording, as labels for the acting/expressing complementarities (Pirie and Kieren, 1994)

2.5.1.2. FOLDING BACK

To reiterate, this study shares the same view as Pirie and Kieren's (1994) that perceived mathematical understanding as growing through the non-linear movement across various layers of sophistication. It is in this continual movement that this study sought to enrich learners' reasoning in geometry. This non-linear movement across various layers is called folding back. Folding back is understood to be an action that is vital to the non-unidirectional nature of coming to understand (Pirie & Kieran, 1994). It happens when learners are faced with a problem or a question that has no immediate solution. In that instance, a learner needs to fold back to an inner layer in order to extend their current understanding that may at the time seem inadequate (Pirie & Kieren, 1991; Pirie & Kieran, 1994). Likewise, Martin (2008) elaborates folding back as an act of going back to an inner layer; an act of revisiting and re-working existing understandings and ideas so that mathematical ideas can be conceptualized.

Although Martin (2008) agrees with Pirie and Kieren's (1994) definition of folding back, he is of the view that "the definition offered by Pirie and Kieren remained essentially undeveloped and unelaborated in their work" (p. 64). Martin's view is that, although the work of Pirie and Kieren (1994) considers folding back as a central feature to the growth of mathematical understanding, how and why folding back occurs is not deeply explored. Pirie and Kieren (1994) explain what folding back is, whereas Martin (2008) expands on elaborating the feature of folding back. Whilst the notion of folding as explained by Pirie and Kieren (1994) theory is significant, in this study the process of folding back is adopted from Martin's (2008) work. Martin's (2008) work elaborates folding back, in the sense that he gives detailed account of this notion as presented in a framework in Figure 2.
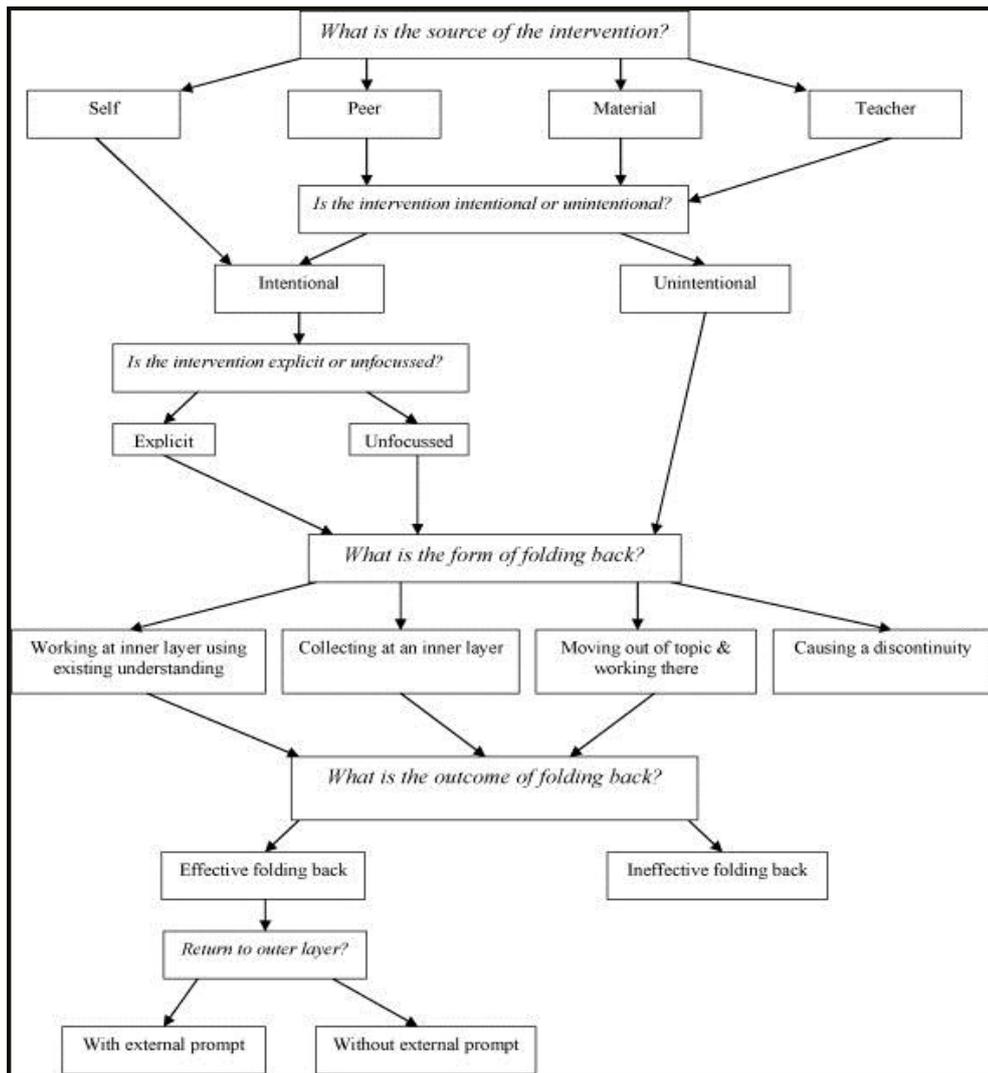
Figure 2: The framework for describing folding back (adapted from Martin,2008; p.72)

Martin (2008) elaborates the notion of folding back through providing three key elements: back: Source, form and outcome. Firstly, according to Martin's work the source is considered as interventions that invoke learners to folding back. Such intervention could be triggered by the teacher, another learner, learning materials or a learner himself or herself. He further outlines two subcategories: intentional intervention and unintentional intervention. He describes Intentional intervention as an intervention "that has an aim of encouraging the learner to fold

back in some way" (p. 72). Unintentional intervention, on the other hand, simply describes "interactions which appear to have caused folding back in the learner without directly meaning to" (p. 72). Furthermore, He subcategorizes the nature of intentional intervention into explicit/focus and unfocused. Focus intentional intervention proposes or even states to the learner which concept the learner should fold back to. In contrast, unfocused intervention stimulates learners to consider earlier understandings without directing them precisely where to fold back and exactly which mathematical idea and existing understanding to work with. Although Martin recognizes four sources of folding back, in this study the focus is on self-intervention, teacher intervention and peer intervention. Teacher interventions can be in various the forms; for example, clue-giving, blocking, modelling (Towers, 1998; Towers & Proulx, 2013).

Secondly, form relates to actions that learners engage in as a result of the source of intervention. In this element, Martin identified four different sub-categories: Working at an inner layer using existing understanding, collecting at an inner layer, moving out of topic and working there, and causing a discontinuity : Working at an inner layer using existing understanding involves the learner shifting to work in a less formal mathematical way. Martin further adds that in form, "the learner either extends his or her current understanding by changing his or her earlier constructs for the concept, or through the generation of new understandings" (p.76). In moving out of the topic and working there, a "learner may need to fold back to his or her Primitive Knowing ...and actually work on extending some aspect of this, effectively separate from the main topic, in order to enable his or her understanding of the current mathematical concept" (p. 77) Collecting at an inner layer occurs when learners "know they know what is needed, and yet their understanding is not sufficient for the automatic recall of usable knowledge" (p. 77).

Lastly, the outcome of the third element involves what happens after a learner has returned to the previous layers. Martin termed this as an outcome. The outcome can become effective or ineffective. Folding back becomes effective when learners are able to use an extended understanding to overcome the original obstacle when returning to the initial problem. Ineffective folding back, on the other hand, is when learners are unable to apply extended understanding to the original problem. Not every act of folding back can result in learners building broader mathematical understanding. Thus, learners can fold back and yet not be able to use knowledge from the inner layers of understanding to the outer layers during the resolution of an activity.

Folding back as described (Pirie & Kieren, 1994) and elaborated (Martin, 2008), differs with Looking back as described by Polya (1985). Folding back focuses on the instances when learners are faced with a problem that has no immediate solution. Looking back, on the other hand, focuses on the completed solution. However, Govender (2013) is of the view that Polya's looking back is similar to Pirie and Kieren's folding back. The similarity of the concepts appears to be on the basis of learners' reflecting on their prior knowledge and consolidating it (Govender, 2013). Although the similarity between folding back and looking back is drawn, the current study is not based latter.

Folding back plays a significant role in learners' mathematical understanding.' It enables learners to build broader mathematical understanding that resides at the inner layers of growth in understanding (Pirie & Kieren, 1994). This happens when learners fold back, carrying with them the demands of the new situation to earlier understanding, and as such informing their new thinking (Martin, LaCroix & Fownes, 2005; Slaten, 2000). This process enables learners to reflect on their prior knowledge as a result of an inadequate current knowledge (Masha, 2004). However, learners need to be self-aware of the nature of their existing understandings if the folding back is to be effective. Thus, learners who

need to fold back must re-evaluate their current knowledge instead of just recalling memorized ideas (Martin, 2008). In so doing, learners thicken their inner level of understanding and gain a broader mathematical understanding of a concept (Pirie & Kieren, 1994). Likewise, the conceptual construct of folding back offers a way to characterize how learners can build a connected understanding through returning to the earlier existing images for a concept (Martin, Allan, Ruttenberg-Rozen, Sabeti, Thomas & Towers, 2014).

2.6.   STUDIES THAT USED FOLDING BACK

Studies that used folding back are based on classroom interactions between teacher-researchers and learners, sometimes during afternoon sessions with individual learners or a group of learners. These studies generally employ teaching experiments and case studies as their preferred research designs. In most cases, mathematics tasks are used as data collecting instruments. Thus, learners are given mathematical tasks to work on. Moreover, video recordings are preferably employed as important tools to capture data during the interactions. Therefore, in this category I present briefly on how folding back has been used in various studies concerning the growth of mathematical understanding (Martin, 2008; Lawan, 2011; Delgado, Code, Monterrubio & Astudillo, 2014; Valcarce et al., 2012; Wright, 2014). The considerations of some of these studies will, hopefully, enhance my understanding of folding back and issues of involved in this phenomenon.

Martin's (2008) study reports on folding back and the dynamic growth of mathematical understanding. In the study, grounded theory approach was used to elaborate an existing theory. Thus, the aim of the study was to "deeply explore the notion and nature of folding back, to elaborate the definition of the phenomena..." (Martin, 2008, p.64). The study used video-recording for data collection. A group of seven learners were videoed over a period of times as they worked on

mathematical activities. The learners involved in the study were observed to be folding back as they worked on the activities. The study found that folding back acts as a necessary mechanism for growth of mathematical understanding in learners. Furthermore, the findings showed that effective folding back is a powerful tool for developing learners' understanding in mathematics.

What I found important and insightful in this study is a developed framework for describing folding back. As I have mentioned earlier on, Martin's framework identifies three important major categories. These categories combined with their subcategories are central to the development of a framework which "allows a researcher or teacher to fully describe any observed act of folding back" (Martin, 2008, p. 71). Although Martin's work identifies three interrelated categories, my interest is in the first category, the source. It is in this category that I learn that folding back can be invoked by various factors during interactions. This informs me to take into consideration such factors in creating a learning environment where folding back takes place.

Lawan (2011) reports on the growth of learners' understanding of part-whole sub-construct of rational number on the layers of Pirie-Kieren theory. Unlike Martin's (2008), Lawan's (2011) study was not concerned with elaborating folding back but rather the role of folding back on the growth of learners' understanding of rational numbers. Moreover, Lawan's study used a teaching experiment methodology as a research design. In the study, two learners were observed during teaching episodes on individual basis in the classroom. Although these studies differ with the research designs, they both used video recordings to record their observations.

Unlike Martin's (2008) study, Lawan's study did not use groups of learners but rather two learners (Sabriyya and Nabil) who were working on an individual basis. Thus, the focus on the aspect of folding back in Lawan's work was limited to

researcher/teacher and learner him/herself as sources of folding back. Peer intervention as source of folding back could not be observed in cases where learners work on an individual basis. Lawan's (2011) findings reveal that folding back occurs with an aim, which is to extend learners' inadequate existing understanding. Lawan's (2011) study further concludes that folding back is central to the growth of learners' understanding in mathematics.

In addition, Valcarce et al. (2012) conducted a study entitled "Growth in the understanding of the concept of infinite numerical series: A glance through Pirie and Kieran Theory". Although, Lawan's (2011) and Valcarce et al.'s (2012) both focused on the growth of mathematical understanding of different mathematical concepts, they differ with respect to the learning environment. In contrast to Lawan's work which focused on individual learners, Valcarce et al.'s (2012) study observed a group of three learners working in a collaborative environment. In the study, they found that during the resolution of the activity, learners moved between different layers of understanding. They further concluded that the change in the understanding of a concept can be produced by the folding back mechanism.

From Lawan's (2011) and Valcarce et al.'s (2012) studies, I have learnt that irrespective of differences in learning environments, learners can fold back to various layers of understanding. Thus, learners can fold back while working on an individual basis as well as with their peers in a collaborative learning environment. This seems to illustrate that learners' growth of mathematical understanding can be enriched either by working alone or with their peers. Although self-intervention is an important source of folding back, it is equally important for learners to be exposed to a collaborative learning environment where peer interactions are encouraged.

Additionally, Delgado, Code, Monterrubio and González Astudillo's (2014) study reports on the role the concept of numerical series and the folding back mechanism. The study focuses on the process followed by a group of learners to build a numerical series and to determine STI convergence. The study is in many ways similar to Valcarce et al.'s (2012) as both involve a group of learners working in a collaborative learning environment while solving mathematical tasks of the same concepts. The difference is that in Delgado et al.'s (2014) study, emphasis was on the folding back mechanism rather than on the Pirie and Kieren theory in general. This enabled them to "prove in several situations that it is necessary to go back using a mechanism called 'folding back'..." (Delgado et al., 2014, p. 2).

Wright (2014) reports on a case study entitled "Frequencies as proportions: Using a teaching model based on Pirie and Kieren's model of mathematical understanding", which focused on folding back. In the reported study, the participants were a group of eight 12-year-old and 13-year-old learners. The teacher/researcher used folding back to help learners to enrich their mathematical understanding of frequencies as proportions. Contrary to Lawan's (2011) and Valcarce et al.'s (2012) studies characterized by learners developing understanding, Wright's (2014) study focused on "a lesson feature that can be manipulated by the teacher to support understanding" (p. 106). The study further adds that folding back has effects on lesson setting. Thus, the focus of the study was on teacher instructional decisions framed on folding back. In conclusion, the study shows that successful implementation of the model is dependent on the teacher noticing and responding to the layers of understanding demonstrated by the learners and the careful selection of materials, problems and situations.

From Wright's (2014) study, I have learnt that the role of a teacher is important during the process of construction of meaning by learners. The nature of teachers' instructional decisions has effect on learners' understanding of mathematical concepts. Moreover, this informs me not only to implement effective

instructional decisions but to follow learners' reasoning path as well. To reiterate, this will eventually enable me to notice layers of understanding shown by learners (Wright, 2014).

In conclusion, several issues that arose in the above studies became useful in the reported study. Firstly, the importance of categories that describes folding back as elaborated by Martin's (2008) study. These categories, particularly the source of intervention, enabled my understanding of how learners could be invoked to fold back. Secondly, the issue of the learners' growth of mathematical understanding in a particular concept, such as numerical series as captured by Valcarce et al. (2012), enlightened my understanding as to how folding back could be used to enhance learners' mathematical understanding. Lastly, the issue of research design as used by Lawan's (2011) study influenced my research methodology as discussed in the proceeding chapter.

2.7. CHAPTER SUMMARY

In this chapter, the literature reviewed was under the following headings: Mathematical reasoning, reasoning in geometry, factors contributing to a low level of geometric reasoning and approaches to teaching geometric concepts. The theoretical framework was also discussed. Some of the studies that used folding back provided an insight of the research methodology as described in the next chapter.

CHAPTER 3: RESEARCH METHODOLOGY

The previous chapter focused on the literature as well as on the theoretical framework pertaining to the issues related to the reported study. In this chapter, I present the methods and steps taken in describing the role of folding back in the teaching and learning of Euclidean geometry, with special focus on improving Grade 10 learners' reasoning in geometry. Presented in this chapter is a rationale for qualitative research paradigm, research design, data collection, data analysis, quality criteria and ethical considerations.

3.1.   CHOOSING A QUALITATIVE RESEARCH PARADIGM

In writing this section I was guided by Guba and Lincoln's (1994) views of a research paradigm as a basic belief or world view that guides the investigation. This is better explained by Johnson and Christensen (2005), who define research paradigm as a perspective that is based on a set of shared assumptions, values, concepts and practice. Such beliefs become a basis for comprehension and interpreting social reality (Cohen, Manion & Morrison, 2000). For the present study, this meant, in order to achieve the purpose of this study, which is to improve Grade 10 learners reasoning in geometry, I had to understand the world view through which the nature of knowledge is developed in the processes of learning. Such knowledge becomes learners' point of view, which serves as a window into their reasoning (Brooke & Brooke, 1999). Thus, once I was able to understand the world view of how knowledge is developed, my inquiry in improving learners' learning could be conducted.

Amongst paradigms employed by researchers are positivism and constructivism (Guba & Lincoln, 1994). The reported study was framed in

constructivist paradigm, where knowledge is created in interaction among the researcher and the participants (Guba & Lincoln, 1994). Although constructivism stresses the development of knowledge as attributed to interactions, Steffe and Thompson (2000) suggest that one should not assume this to include situations where information is transferred directly from teachers to learners. Instead, teachers' roles are to participate in learners' modifications of their mathematical activities. Therefore, teachers who in this case are also researchers need to learn how to use their mathematical knowledge to teach and interact with learners in the classroom (Steffe & Thompson, 2000).

I am of the view that the constructivist paradigm suggests the role of interactions as a quest for development of knowledge. Therefore, it is within this paradigm that the purpose of the reported study is to understand how Grade 10 learners construct their individual and shared geometry meanings while interacting with each other and the researcher-teacher through folding back. Masha (2004) extends constructivism to mean that a constructivist perspective regards knowledge as emanating from our own experiences. It is these experiences that bear understanding of geometry, that are constructed as data for this study.

Contrary to the positivist's search for explanations that are used for predictions and control of phenomena, I was concerned with reconstructions of the geometry that Grade 10 learners initially hold. This was done to allow for new interpretations and constructions of geometry. Such intentions are underscored by Guba and Lincoln (1994) as the aim of a constructivist paradigm in conducting research. In so doing, learners could formulate more informed and sophisticated constructions and become aware of their knowledge (Guba & Lincoln, 1994). This perception on knowledge construction by Guba and Lincoln is similar to that of Von Glasersfeld (1981). He points out that knowledge is not passively received but actively built up by the cognizing subject. Here, one could safely say that constructivism embraces that learners' geometry prior knowledge plays a

significant role in constructing new meaning. Learners bring prior constructions of geometry to their geometry learning environment and through interactions with others, constructions are understood and reconstructed. Hence, individual constructions can be stimulated and developed only through interactions between and among the participants and researcher (Guba & Lincoln, 1994).

Again, in contrast to a positivist paradigm that uses scientific methods and statistical analysis to generalizable findings, the constructivist paradigm rather focuses on methods that will stimulate understanding the phenomena under investigation (Mack, 2010). Furthermore, analysis is done through narrative analysis and analysis of narratives (Polkinghorne, 1995). In the reported study, narrative analysis, where the data was used to illustrate how folding back occurs, was used. This according to Polkinghorne (1995) is the strength of narrative analysis, where data is used to produce a story. It thus becomes invalid to generalize the uniqueness of the findings made on the basis of individual utterances to that of other individuals. Moreover, growth of learners' mathematical understanding and reasoning in geometry cannot be accounted for statistically. Hence, in conducting the reported study, I did not pursue to generalize the findings to a population; I pursued to understand how folding back is used to alleviate learners' challenges in geometric reasoning (Gelo, 2012).

3.2. RESEARCH DESIGN

I mentioned in the previous section that a perspective that one has guides to their investigation (Guba & Lincoln, 1994). It would then have been contradictory to employ positivist research designs, whereas generalizability was not a key concern for this study. Qualitative researchers often choose their study design from the following six types of research designs: phenomenological studies, ethnographic studies, grounded theory studies, historical studies, action research studies and case studies (Flick, Von Kardorff & Steinke, 2004; Cohen,

Manion & Morrison, 2007; Creswell, 2012). Unfortunately, I found none to be more suited for construction of knowledge on learners understanding of mathematical concepts. As a result, I adopted a non-traditional research design, the teaching experiment, which allows researchers to experience first-hand learners' mathematical learning and reasoning (Steffe & Thomson, 2000)

3.2.1. RATIONALE FOR TEACHING EXPERIMENT METHODOLOGY

The choice for a teaching experiment for use in the reported study was that in its conceptualization, the study was to take place as part of a day-to-day occurrence of a classroom. A teaching experiment optimized the changes for development that can to occur in forms that could be observed. Through a teaching experiment the reported study took into consideration what learners were saying and doing in an attempt to understand their geometric reasoning. It is this added advantage that Steffe and Thomson (2000) regarded as a crucial part of teaching experiment that rendered a teaching experiment most suitable for the present study.

The teaching experiment as a research design in this study was found useful in studying learners' geometric reasoning as a result of mathematical interactions in their learning of geometry. This goal of a teaching experiment, as attested by Steffe and Thomson (2000) is for researchers to learn the mathematical knowledge of learners and how they construct it. Besides, the emergence of teaching experiment design in researching classrooms is attributed to progress made by learners as a result of mathematical interaction. As suggested by Czarnocha (1999) on interactions during teaching experiments, I was able to formulate ways and means to foster what a child needs to learn in order to either reach a particular moment of discovery of a geometry idea or to master another. Unlike any other design that subjects learners to experiments without their active participation, the teaching experiment methodology offered

learners opportunity to be actively immersed in co-construction and development of their geometry knowledge. Active involvement of learners in their learning became central to their development of geometry knowledge. Such and similar ideas illustrate that mathematical activity in a classroom happens as a result of learners' active participation in learning (Steffe & Thompson, 2000).

3.2.2. ELEMENTS OF A TEACHING EXPERIMENT

Steffe and Thomson (2000) outlined the following four elements of a teaching experiment:

- a teaching episode
- teaching agent and one or more learners (participants)
- a witness of the teaching episodes
- method of recording (data collection)

Teaching episode: The reported study involved a sequence of teaching episodes that were conducted in a Grade 10 mathematics classroom. In planning for the teaching episode, in collaboration with the practising teacher, we followed the Grade 10 CAPS mathematics work schedule which displays what should be covered in the area of Euclidean geometry. Thus, the teaching episode which included among other things, geometric reasoning activities, was done in consultation with the work schedule as prescribed by DBE (2011). Learning activities were derived from learners' Siyavula textbooks and other sources. The teaching episodes focused on Euclidean geometry. The sub-topics were as follows: Parallel lines and transversal; angles; triangles; congruency and similarity, and quadrilaterals.

The teaching agent: During the teaching episodes, I was the teaching agent (teacher-researcher) who taught lessons on Euclidean geometry to 54 Grade 10 mathematics learners. In this methodology, the distinguishing feature is that as the

researcher took the role of a teacher. Thus, a teaching experiment can be conducted by teacher-researcher to respond to a learning problem in the classroom in a systematic and creative manner for constant improvement (Steffe, 1988; Steffe & Thomson, 2000). In so doing, I was able to interact with Grade 10 learners and experience their geometric reasoning progress in their learning environment.

A witness of the teaching episodes: I have indicated earlier that I took the role of the teacher during the teaching episodes. In accordance with the important element of teaching experiments, the presence of the practising teacher as a witness of the teaching episodes was important (Steffe & Thomson, 2000). In the study, the teaching episodes were witnessed by the Grade 10 mathematics practising teacher. His role of the practising teacher was to offer a more objective view of the interactions that occur during teaching episodes. His presence was necessary because, as a researcher-teacher, I was immersed in classroom interactions, trying to respond to what the learner is saying. This according to Steffe and Thompson (2000) disadvantages the researcher-teacher from being able to reflect on learners' contribution during the teaching. This made it challenging for me to witness elements of learners' actions during teaching. Sometimes I would miss such important leaners' actions during classroom interactions without being aware. In such cases, Steffe and Thompson (2000) urged that the witness could help the teacher-researcher to understand the learner and also suggest action, which was the case in the reported study. This helped in taking into consideration contributions made by learners during classroom interactions. In so doing, together with the observer, we were able to prepare for the next teaching episode.

Communication with the participants is an important aspect of teaching an experiment. I taught Grade 10 mathematics learners for three years. By so doing, a history of interactions with Grade 10 mathematics learners was established.

Communication with learners during teaching experiments can be established readily when the teacher-researcher has a history of interactions with learners comparable to the learners involved in the teaching experiment (Steffe& Thompson, 2000). However, in establishing communication with the participants in this study, three weeks were spent with the participants prior to the teaching experiments. These three weeks offered an exploratory opportunity to discover through immersion and participation, the hows and ways of Grade 10 learners' behaviour in their mathematics classroom.

3.2.3. PARTICIPANTS

The classroom from which the participants in the study came had 54 learners. All these learners were exposed to exploratory teaching, which is used in teaching experiment for four weeks. By so doing none of the learners were disadvantaged by selection bias (Cohen, Manion & Morrison, 2007). From the group of 54 learners 7 were identified as data sources. This choice of 'grainsized' participants from a group is permissible in teaching experiment where either one or a few participants may participate (Steffe & Thompson, 2000) in a study. These seven learners did not necessarily represent the whole class in relation to the purpose of the study. This requirement is not necessary in determining rigour in teaching experiment. Instead interest is in "organising and guiding [teacher-researchers] experience of learners doing mathematics" (Steffe & Thompson, 2000, p. 300). This way of choosing participants is aligned with purposive sampling as described by McMillan (2000). The strength of purposive sampling is that few participants that are studied in depth yield many insights about the topic as noted by McMillan and Schumacher (2001).

3.2.4. DATA COLLECTION

The most predominant sources of qualitative data include interviews, observations, and documents reviews (Creswell, 2007; Silverman & Spirduso, 2010; Mertens, 2014). However, since teaching experiment was a design of choice for this study, the use of video tapes was adopted as a recommended data collection method (Pirie & Kieren, 1996; Steffe & Thompson, 2000; Engelhardt, Corpuz, Ozimek & Rebello, 2004). Video recordings were used to capture teacher-researcher and learners mathematical interactions while working on the learning activities.

3.2.5. VIDEOTAPE

In the study video recordings were employed, particularly focusing on learners' interactions sometimes in the classroom and mainly during the afternoon sessions with varying durations. This was no different from Pirie and Kieren (1994) who used the video recording to collect data from a pair, or small group of learners in different settings, i.e., sometimes in a classroom setting and sometimes in interviews sessions. Two groups of learners were videotaped in separate afternoon sessions. These learners were video-taped to capture how they interact with each other while solving geometric reasoning tasks. Learners were encouraged to share ideas with each other as they solved the learning activities as recommended by Steffe and Thompson (2000). Thus, in order to learn learners' mathematics, the researcher could create situations and ways of interacting with learners that encourage the learners to modify their current thinking (Steffe & Thompson, 2000).

In this study, the use of video recordings offered potential benefits. The video recordings produced large amounts of data concerning mathematical interactions within a short space of time. It also offered an opportunity to closely

observe and record learners as they were solving geometric reasoning activities. Pirie (1996) favourably recommended the use of video recording since folding back could only be observed by close attention to the activities and talk of learners as they worked on mathematical activities. Also, in their conceptualization of the teaching experiment, Steffe and Thomson (2000) urged that focusing on what learners are saying and doing in an attempt to understand their mathematical learning is an essential part of a teaching experiment. In the reported study, through video recordings, not only interviews were captured, mathematics learning tasks were also captured. Furthermore, data on video allowed me to observe classroom interactions repeatedly.

3.2.5.1. Mathematics learning activities.

The role of Euclidean geometry learning activities was of paramount importance to this study. The choice of learning activities on Euclidean geometry was that the reported study was on geometry, particularly geometric reasoning. Learners were given tasks to work on during teaching episodes as well as in clinical interviews. Pirie (1996) indicates that folding back can only be observed by close attention to the activities and talk of learners as they work at mathematical tasks. What learners write is as important as what they say. A part of data collection, learners' written work in the form of responses for the given learning activities were captured through using video recordings.

3.2.5.2. Observations

Observations were conducted during the teaching episodes and also afternoon sessions. The main focus of the classroom observations and afternoon sessions was on the learners' interactions while working on the tasks. Observations offered an opportunity to gain insight into how learners use folding back while learning Euclidean geometry. Learner-to-learner interactions offered an

opportunity to gain insight on the role of peer intervention as one of the sources of folding back (Martin, 2008). The study employed participant observations because folding back as a phenomenon requires interaction with the participant, thus the role of the teacher intervention is also important as one of the sources of folding back (Martin, 2008). Observations were conducted when learners were solving Euclidean geometry activities. Observations offered me with an opportunity to see and hear what was occurring naturally in the learners' learning environment (McMillan & Schumacher, 2010).

3.2.6. DATA ANALYSIS

The analysing of the video data went through various processes. In the first process, firstly I listened to the video recordings for each learning activity interactions through replaying recordings over and over again. This offered an opportunity to make sense of the video data. Secondly, I performed verbatim transcription of the video recordings. This offered an opportunity to produce a written text of the videotapes, as earlier mentioned, the focus on text was important for analysis. In addition to this, I read transcribed data over and over again. Here, the purpose was similar to listening to the video recordings. Lastly, information-rich interaction for each learning activity, were folding back was observed were selected. The selections of such information-rich interactions were guided by the following three key features of folding back: the source of intervention, the form of folding back and the outcome of folding back. These selections are presented as a form of vignettes in the next chapter.

In the second process, the selected information-rich interactions for each learning activity were subjected to translation and interpretation. Here, The Pirie and Kieren theory of the growth of mathematical understanding and the theoretical framework of folding back as presented by Martin (2008) guided the interpretations. Thus, the interpretation of the transcribed information-rich

interactions focused on the three elements of folding back. Firstly, the source of folding back was identified. Here the focus was on what invoked learners to fold back. Attention was paid to sources such as self, peer and teacher interventions. Further attention was given to the nature of intervention on whether the nature of intervention was either explicit or focused intentional intervention or unintentional

Secondly, from the transcribed interactions the form of folding back was also identified. As earlier mentioned, the *form of folding back* relates to actions that learners engage in as a result of the source of intervention. Here the focus was on forms such as working at an inner layer using existing understanding, collecting at an inner layer, moving out of topic and working there, and causing a discontinuity. These forms of folding back are explained in the previous chapter.

Lastly, from the transcribed interactions, the outcome of folding back was identified. Here the focus was on determining whether the outcome of folding back was effective or ineffective folding back.

3.3. QUALITY CRITERIA

In ensuring quality criteria, issues of rigour in teaching experiment methodology as well as some aspects of trustworthiness in qualitative research which were deemed necessary for rigour in the reported study were considered.

3.3.1. RIGOUR IN TEACHING EXPERIMENT METHODOLOGY

As in quantitative research, the importance of ensuring rigour in qualitative research is emphasized. This is no exception with teaching experiment methodology. According to Steffe and Thomson (2000), rigour in teaching experiment is demonstrated through replicability and generalizability. However, it is not suggested for teaching experiments to be replicated (Steffe & Thomson,

2000). In this study, I was able to illustrate in the subsequent chapter, how Grade 10 learners use folding back to improve their geometric reasoning. This was suggested by Steffe and Thompson (2000), pointing out that researchers who make a claim about what learners know are required to make records of the living models of learners' mathematics that demonstrate features of the claim available to the interested public.

As I have earlier mentioned, the participants (seven learners) in this study do not necessarily represent the whole Grade 10 class in relation to the purpose of the study. Instead, they offered me an opportunity to experience how their geometric reasoning is improved through folding back. This in teaching experiment is supported of Steffe and Thompson (2000), who demonstrate that it is pointless to require teaching experiments to "generalize" in the way in which one might hope that claims thought to be true about a random sample, would be true as well about the population from which the sample was drawn. Therefore, I was not in pursuit to generalise to a population but to understand how folding back can be used to improve learners' geometric reasoning. Hence, Steffe and Thomson (2000) are of the opinion that in teaching experiments, it is not a matter of generalizing the findings in a hypothetical way, but of the findings being useful in guiding our experience of learners learning mathematics.

3.3.2. TRUSTWORTHINESS IN QUALITATIVE RESEARCH

3.3.2.1. *Credibility*

Credibility is, according to (Polit & Beck, 2012) the truth of the data or the participants' views and their interpretation of such data or views by the researcher. In ensuring credibility of the study, I provided in the proceeding chapter the participants' direct quotes during the interactions which were observed and video recorded. This in qualitative research is recommended by Cope (2014).

3.3.2.2.   Transferability

Transferability refers to the findings that can be applied to other groups (Houghton, Casey, Shaw, & Murphy, 2013). As I have earlier mentioned the reported study was not meant to generalize its findings. Hence, the criterion of transferability is relevant if the intent of the research is not to make a generalization.

3.3.2.3.   Confirmability

Polit and Beck, (2012) define confirmability as the researcher's ability to demonstrate that the data represent the participant's responses and not the researcher's biases. To ensure confirmability, I described in the subsequent chapter how conclusions and interpretations were established, and exemplify that the findings were derived directly from the data by providing rich quotes from the participants as recommended by Cope (2014).

3.4.   ETHICAL CONSIDERATIONS

Creswell (2013) is of the view that researchers have an obligation to respect the rights, needs, values and desires of the informants. In ensuring that the study is ethically conducted, the following was taken into consideration:

3.4.1. CONFIDENTIALITY

Video recordings were used in the classroom and afternoon sessions and captured the faces of minors on tape. Permission was requested from the school principal, the parents of the participants and as well as from the participants themselves. In ensuring confidentiality, pseudonyms were used to protect the identity of the participants and the research site. The participants were further

informed that the data would be confidential and would not be shared with other individuals outside of the study.

3.4.2. INFORMED CONSENT

Informed consent forms which included the purpose of the study were issued to the participants before this study was conducted. Informed consent guarantees the participants certain rights, and that they are agreeing to be involved in the study and acknowledge the protection of their rights (Creswell, 2012). Parents of the participants were also given consent forms which included the information letter concerning the study. Parents gave the permission for their children to be involved in the study. Informed consent forms were signed and returned back to the researcher.

3.5. CHAPTER SUMMARY

In this chapter, I provided rationale for qualitative research paradigm. An account for choosing teaching experiment as a research design was provided. The selection of participants for this study was also catered. Issues of data collection; How data was collected; analysed data, ethical considerations, and quality criteria were also addressed. In ensuring quality criteria, rigour in teaching experiment methodology as well as trustworthiness in qualitative research was discussed. The next chapter focuses on the analysis of the collected data.

CHAPTER 4: FOLDING BACK AS OBSERVED

The previous chapter illustrated how data was collected. Data collection was done to answer the following research questions which guided the study:

- How does folding back support learners' interaction with geometric reasoning tasks during the lessons?
- How does a Grade 10 mathematics teacher use folding back to enrich student reasoning in geometry?

In this chapter I present the analysis of folding back as observed during the teaching experiments. The data analysis is informed by the theoretical framework of folding back adopted from Martin (2008) as illustrated in the methodology chapter. Descriptions of teaching experiments are also presented.

4.1. TEACHING EXPERIMENTS

In this study, three teaching experiments were conducted. Teaching experiment number 1 focused on the introduction of basic Euclidean geometry knowledge which learners had been exposed to in their previous grades. Teaching experiment 1 consists of two teaching episodes. Teaching episode 1 focused on angles, parallel lines and transversal while teaching episode 2 focused on congruency and similarity. Teaching experiment two, which consists of one teaching episode, focused on the Midpoint theorem. The last teaching experiment focused on quadrilaterals. Learners' sampled responses of tasks are respectively provided for each teaching experiments' episodes. These responses include learners' communication in the form of vignettes. Vignettes provided in the study

illustrate learners' interactions as well as teacher's instructional decisions while learners work on the mathematical tasks.

4.2. TEACHING EXPERIMENT 1: BASIC GEOMETRY INTRODUCTION

4.2.1. TEACHING EPISODE 1: ANGLE MEASUREMENTS

In this teaching episode, learners were given Euclidean geometry tasks that require them to use properties of angles and parallel lines and transversals in finding appropriate values of various angles. The learning activities were taken from learners' Siyavula textbook and other sources. Three learning activities were sampled for the purpose of reporting. They are: learning activity, 1, 2 and 3. What learners were saying and writing as they were working on the given tasks was important and was the key aspect of the data collection. Therefore, sampled vignettes for the teaching episode are provided.

The first activity given to learners was on determining magnitudes of various angles in a geometric figure. It was phrased as indicated learning activity 1.

Learning Activity 1.

An expected response from the learners is presented below.

$$\angle a = 60° \dots vertically\ opp. \angle^s =$$
$$\angle b = 35° \dots alter. \angle^s =$$
$$\angle c = 35° \dots base\ angles\ of\ an\ iscosceles\ triangle\ are\ equal$$
$$\angle d + 60° + 35° = 180° \dots sum\ of\ inter. \angle^s\ of\ a\ \Delta\ is\ supplimentary$$
$$\therefore \qquad \angle d = 85°$$
$$\angle a = e + 35° \dots ext. \angle^s of\ a\ \Delta$$
$$\therefore \qquad \angle e = 25°$$
$$\angle f = 180° - 25° - 85° \dots sum\ of\ \angle^s\ on\ a\ straight\ line$$
$$\angle g = \angle e = 25° \dots corresp. \angle^s =$$

Vignette 1 shows learners' responses of learning activity 1. This is followed by an analysis of such responses for Group A. The group consisted of Lebogang, Lesiba, and Thembi/Sipho

Vignette 1

1.1.	Lebogang:	We are looking for angle B; I think it's corresponding to 35...
1.2.	Lesiba:	Show us why it is corresponding.
1.3.	Lebogang:	Okay [She Started drawing].
1.4.	Thembi:	Isn't 35° not for $angle\ C$.
1.5.	Lesiba:	Wait let's give Lebogang a chance to do it.
1.6.	Lebogang:	[Continues drawing]...it is not correct.
1.7.	Thembi:	Actually $angle\ C$ is it not equal to 35°?
1.8.	Lesiba:	$Angle\ C$ is inside isosceles triangle.
1.9.	Thembi:	No, I was just asking.
1.10.	Thembi:	35° is corresponding to $angle\ C$.
1.11.	Lesiba:	…,we can take out Z [Drawing Z],it is going to be like this, then b is equal to 35°
1.12.	Lebogang:	Yes, that's it, angle B is alternating to 35)
1.13.	Thembi:	Ooh yah…I was not aware of it

The pictures in Figure 3 and Figure 4 illustrate how both Lebogang and Lesiba represented $\angle b$.
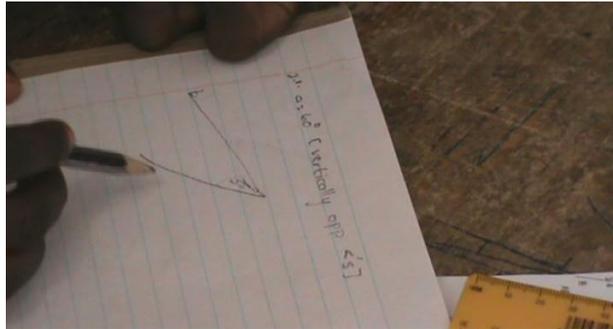
Figure 3: Lebogang's drawing
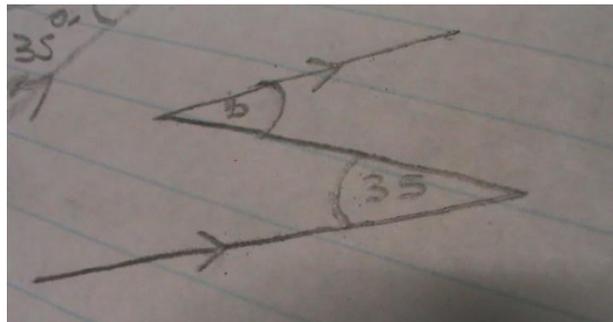
Figure 4: Lesiba's drawing

Learners here were able to determine $\angle a$ through property noticing. They were able to notice that $\angle a$ is vertically opposite to $35°$. This seems to show that once learners are able to identify vertically opposite angles, they consider them to be equal. They then proceeded to determine $\angle b$. In determining $\angle b$, Lebogang considered it to be a corresponding angle (line 1.1.). Here, Lebogang was observed to use her geometry knowledge of the properties of angles informed by her property noticing layer in determining $\angle b$. However, Lesiba's articulation "*Show us why you say it is corresponding angle*" (line 1.2) seems to illustrate that he was interested in knowing why Lebogang considered $\angle b$ being the corresponding angle. Lebogang's response, which was through drawing a diagram (Figure 3), seems to illustrate she wanted to show her peers how corresponding angles are identified. This act indicates that Lebogang has now moved from property noticing to image making. However, she realized that the drawing was incorrect. In responding to Lebogang's actions, Thembi's question (line 1.4) did not consider $\angle b$ to be $35°$; but $\angle c$ to be $35°$. On the other hand,

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Lesiba's response by folding back to primitive knowing through identifying shape Z (line 1.11) was an attempt to find $\angle b$. This eventually led him to draw a correct diagram (Figure 4) as compared to Lebogang's drawing. Consequently learners were able to work with alternating angles. Thus, it appears that once a learner identifies shape Z, he associates it with alternating angles (line 1.11 & line 1.12). In so doing, learners were able to move from image making to property noticing in order to determine $\angle b$ with appropriate reasoning. Here, through folding back to various layers of mathematical understanding, learners were able to build and apply their knowledge of geometry.

The second activity of the teaching episode which was done by Group B, focused on proving the magnitude of an angle. It is illustrated as learning activity 2.

Learning Activity 2.
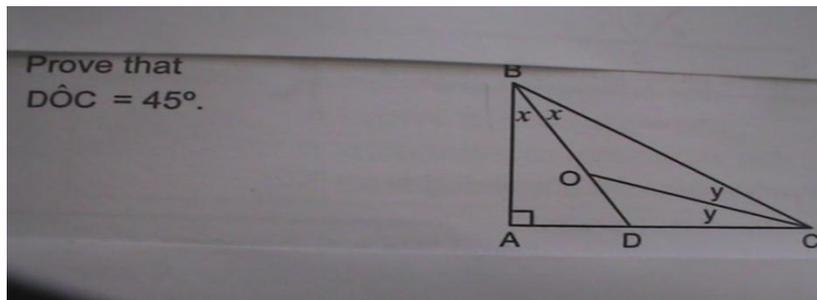
An expected response from the learners is presented as follows:

$$DOC = x + y \dots ext. \angle \text{ of } \Delta$$

but

$$2x + 2y = 90° \dots \angle \text{ sum of } rt. \angle^d \Delta ABC$$
$$\therefore x + y = 45°$$
$$\therefore D\hat{O}C = 45°$$

The following is an analysis of Analysis of task 2 responses for Group B - John, Phuti and Koena.

In their attempt to solve this task, learners appeared to have been making sense of the information provided to them. They realized that they had to use the information provided in order to solve the task, an evidence of image having. This led them to fold back to property noticing layer. In using the information at a property noticing, John was of the view that $\angle C$ is x and $\angle B$ is y, and applied the sum of angles in a triangle, since he considered having three angles ($\angle A, \angle B$ and $\angle C$) of $\triangle ABC$. This in Pirie and Kieren's (1994) theory shows that John was engaging in the acts of property noticing. However, this has not helped them as the given information was incorrectly applied. According to Koena, John's view was going to be correct, provided they were given the value of any angle, either $\angle B$ or $\angle C$ in $\triangle ABC$. This illustrates that Koena was at property noticing layer. Koena further illustrated that since they were given $90°$ and they needed to work on $\triangle ABC$ in order for them to work on the Δ where $D\hat{O}C$ is located. Their difficulty was on trying to find the magnitude of $\angle B$ and $\angle C$, which were already given but they were not able to immediately realize that. As a result, they asked the teacher for assistance. The conversation unfolded as follows:

Vignette 2

2.1.	John:	Sir please help us to find $D\hat{O}C$.
2.2.	Teacher:	What will be the sum of $\angle B$?
2.3.	John:	$2x$…hmmmm $\angle B_1$ is equal to $\angle B_2$ and $\angle B_2$ is equal to x, which means each of $\angle B_1$ and $\angle B_2$ is x,do you understand?
2.4.	Koena:	Eish….[shaking head-indicating that he doesn't understand].
2.5.	John:	It means that here it is $2x$ and $2y$...
2.6.	Koena:	Oh I understand now, meaning $\angle B$ is having two angles...
2.7.	John:	Yes...question is how are we going to calculate x and y
2.8.	Teacher:	What will be the sum of angles in that triangle?
2.9.	John;	$90 + 2x + 2y = 180$.
2.10.	John;	$90 + 2x + 2y = 180 \ldots 2x + 2y = 90$.
2.11.	Koena:	So do we find x first?
2.12.	John;	What if we divide by 2?
2.13.	Koena:	We divide $2x + 2y$ by 2…then we remain with $x + y = 45°$

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2.14.   John: then we find $x + y$ to be $45°$.

Following the learners' conversation closely, here it showed that learners had an idea of what was required to solve the task. However, their current knowledge was insufficient. As a result, the explicit teacher intentional intervention was observed. This form of intervention asked learners a directive question (see line 2.2). Thus, it directed learners to which mathematical concept to focus on. As a result, John was able to fold back to image making to work with existing understanding of angles. Through this intervention John was observed to be concentrating on $\angle B$. Asking learners' directive questions provided them with the opportunity to use their prior knowledge. This intervention invokes the learner to fold back to primitive knowing layer; specifically to the prior knowledge of addition algorithm in this case. For instance, John was then able to determine the sum of $\angle B$ which he found to be $2x$, as well as $\angle C$ which was $2y$ (line 2.3). This shows that the learner (John) was able to move out of the topic. Here John was using his knowledge of algebra and applying it to determine the sum of angles. This shows working out of the topic as a form of folding back.

Despite John's attempt, Koena's response (line 2.4) seems to illustrate that he did not grasp what was implied by his peer. This seems to show that Koena, unlike John has not yet developed the knowledge of the sum of $\angle B$. This further show that he was engaging from a different layer compared to that of John. Thus, according to Pirie and Kieran's (1994) model of growth of mathematical understanding, John was able to transit from primitive knowing to image having, whereas Koena was still at primitive knowing level. As a result, this afforded John the opportunity to further elaborate his thought process (line 2.5). This seems to show that once a learner develops a geometry concept through folding back, he can explain it.

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In consideration of John's explanation, Koena was able to develop concepts of $\angle B \; and \; \angle C$ (line 2.6). Koena was able to fold back from primitive knowing to image making. Here explicit peer intentional intervention was observed. These acts seem to suggest that learning takes place when learners are able to explain their mathematical thinking to each other. It can further be said that learners' geometric reasoning skills develop when they explain their thinking to each other. Thus, through folding back Koena was able to develop an image of $\angle B$ and $\angle C$ respectively with an appropriate reasoning.

Moreover, an explanation of developed geometry concepts through folding back does not only benefit learners who explain their thinking but facilitates growth of mathematical understanding amongst their peers (line 2.5 & line 2.6). Hence, it promotes a collaborative environment where knowledge is shared amongst learners. Thus, folding back appears to create an important learning environment that provides learners with the opportunity to share their geometry understanding. In so doing, their geometric reasoning is enriched.

Furthermore, the explicit teacher intentional intervention in line 2.8, where learners were asked the sum of angles in that triangle, enabled them to determine other two angles (x and y) of ΔABC. They found that $x + y = 45°$ (line 2.13 & line 2.14). Thus, through this intervention, learners were able to move from image making to property noticing by collecting their knowledge of algebra, particularly the division algorithm. Although learners were able to determine x and y, this has not led them to prove that $D\hat{O}C$ is $45°$. As a result, they had to fold back to image making through moving out of ΔABC to ΔDOC. The conversation as it unfolded is captured Vignette 3.

Vignette 3

3.1.	John:	Where is the angle we are looking for?
3.2.	Koena:	It is this one...
3.3.	John:	[he starts referring to the notes pamphlet]

3.4.	John:	Let us say we put it this way…draw it.
3.5.	Phuti:	How?
3.6.	John:	Looking at it the way it is...
3.7.	Phuti:	I'm going to make mistake.
3.8.	John:	[Started drawing]
3.9.	Koena:	Ohoo $\angle O$ is an exterior angle.
3.10.	John:	Yes my friend…so $\angle O$ is equal to $B_2 + C_2$ (exterior…. ohooo….wooowwww! $\angle O$ is equal to $\angle B$, $\angle B$ is the same as, look at it …$x + y$; $\angle O = x + y$, $\angle O = 45°$.
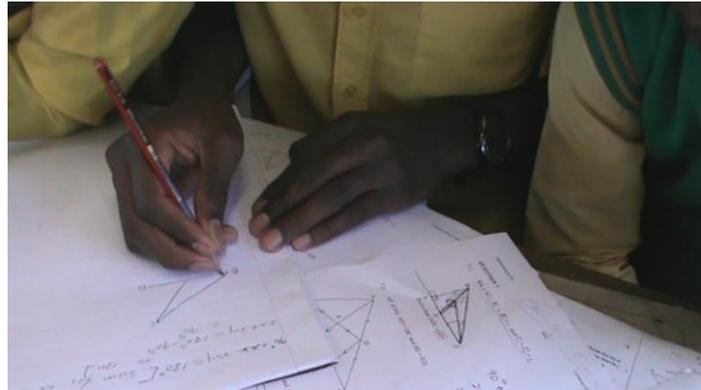
Figure 5: John's drawing of $D\hat{O}C$

In Vignette 5, learners were observed to work in a shape where $D\hat{O}C$ was located. John's question (line 3.1) seems to suggest that having been able to successfully work ΔABC; they have to work ΔAOC. John's question further illustrates that he was engaging in the acts of image making. The idea of John referring to the notes, particularly on types of angles in triangles assisted him to work on his image making. Here explicit material intentional intervention is observed. In image making, John began to draw a shape where $D\hat{O}C$ is located (line 3.8 & Figure 5). Thus, he was engaging in a hands-on task through drawing an image of the figure. Consequently, Koena was able to examine the drawing for its properties. Thus John's act led them to work on property noticing, where they both realized the relationship between exterior angles and the sum of the interior opposite angles (line 3.9 & line 3.10). Here, learners were observed noticing connection between the $\angle^s$ of ΔABC and $D\hat{O}C$. Learners in this case were, through folding back, working with the knowledge of angles in an attempt to formulate

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geometry concepts that would help them determine the meaning of the geometry learning activity. This shows that once learners are able to notice the connection between geometric concepts, they become aware of their properties. This connection through folding back enables them to inform their thought process. As a result of their informed thinking through folding back across various layers, they were able to apply noticed properties of the concepts to prove that $D\hat{O}C = 45°$. Thus effective folding back through learners' interactions with each other was observed.

4.2.2. TEACHING EPISODE 2: SIMILARITY

The focus of teaching episode 2 was on congruency and the similarity of triangles. Conditions for both congruency and similarity were discussed during the episode. Learners were given congruency and similarity tasks that required them to work with their peers. For the purposes of reporting learners' responses to the learning activities during the episode (Learning activity 3) as well as during the afternoon session (Learning activity 2) were sampled. In these learning activities, learners were required amongst other things, to prove that triangles are similar and also to find the unknown sides in triangles. Presented in this task, are learners' responses to proving similarity and finding the unknown sides respectively. Sources of folding back as observed are that of self-invoked intervention (in proving similarity) and explicit intentional teacher intervention (in determining unknown sides).

The first activity (Learning Activity 3) for teaching episode 2 focused on proving that two triangles are similar.

Learning Activity 3
Prove that $\Delta PST \; /// \; \Delta PQR$
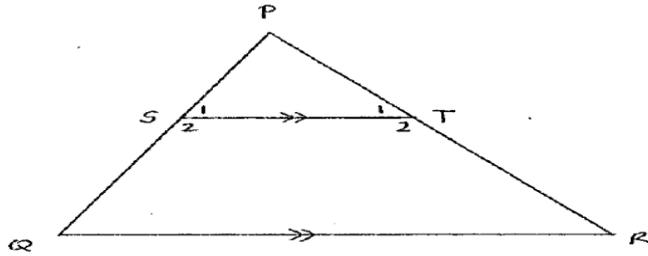
An expected response from the learners is presented below.

In ΔPST and ΔPQR

$\angle P = \angle P \ldots Common$

$\angle S_1 = \angle Q \ldots Corresponding \angle^s =; ST \parallel QR$

$\angle T_1 = \angle R \ldots Corresponding \angle^s =; ST \parallel QR$

$\therefore \Delta PST \;///\; \Delta PQR \ldots AAA$

Vignette 4 shows learners' responses of learning activity 3 and its analysis.

Vignette 4

4.1.	John:	Look...okay...you cannot just say PS. So what are the values of PS? But we cannot use it if we are not given values. If we can determine angles of these triangles, then we can prove that these triangles are similar. So angle P is common, what about other angles? These are angles on a straight line, these are parallel lines. We need transversal; we use those things of corresponding angles.
4.2.	Koena:	$\angle S$ is equal to $\angle Q$, as you see they form F shape.
4.3.	John:	F shape this way?
4.4.	Koena:	Yes.
4.5.	John:	Yes, it is correct.
4.6.	Koena:	If we can put it this way, can't we take it out?
4.7.	John:	Yes it is correct, they are this way, and then $\angle R$ is equal to $\angle T_1$. They are corresponding angles.

Learners in the above vignette were proving that triangles are similar. Koena was of the view that they could use corresponding sides that are in proportion to prove similarity in triangles. John, on the other hand, was of the view

that they could use properties of angles (line 4.1). Although Koena's knowledge of similar triangles is correct, here it appeared to be inadequate and inappropriate for the learning activity at hand. It can be seen that learners can engage in the same layer of understanding and think differently about the same mathematical concepts. Here both Koena and John were both observed to be at property noticing layer. However, unlike Koena, John was able to notice why the property of corresponding sides could not be applicable. This is well captured in his articulation "...but we cannot use it if we are not given values…"

However, they could not resolve the learning activity at hand because they only had one angle which was equal in both the triangles. As a result, John's explicit international self-intervention led him to fold back to image making. In image making layer, John was observed working with his existing understanding of angles. In addition, at image making, John was observed to be questioning and responding to his own knowledge of angles relying of parallel lines as well as transversal. Here John was engaging in actions that would help him to determine angles which would help them in resolving the learning activity at hand. This is well evidenced in his utterances "So $\angle P$ is common, what about other angles? These are angles on a straight line; these are parallel lines" (line 4.1).

On the other hand, John appeared to have caused Koena to fold back to image making layer. In this instance, this intervention became explicitly intentional Thus, John's utterances encouraged Koena to fold back to image making and work with his inner knowledge of corresponding angles. Here Koena considered John's utterances such as "...are parallel lines...transversal; we use...corresponding angles" as interpretative enough for him to realize that $\angle S$ is equal to $\angle Q$. Koena further used mnemonic F to identify equality of such angles (line 4.2). Likewise, John was able to identify that $\angle R$ is equal to $\angle T_1$ with appropriate reasoning.

As a result of working with their existing understanding of corresponding angles in the image making layer, learners were able to fold back to the outer layer of property noticing layer with external prompt. Thus, through interactions with each other, learners were able to return to properties noticing where they used properties of angles to show that Δ^s are similar.

Learning Activity 4: Find the magnitude of a and b

Learning Activity 4, required learners to determine the magnitude of variables a and b respectively.
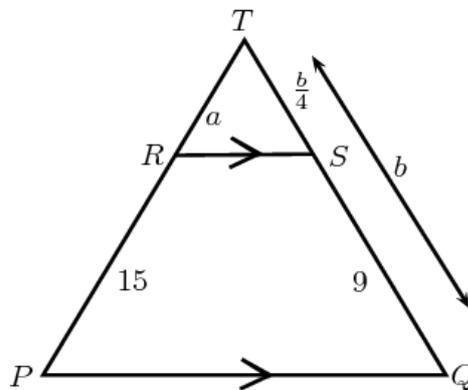
An expected response from the learners is presented below.

$$\frac{TR}{TP} = \frac{TS}{TQ}$$

$$\frac{a}{15+a} = \frac{\frac{b}{4}}{b}$$

$$\frac{a}{15+a} = \frac{b}{4} \div \frac{b}{1}$$

$$\frac{a}{15+a} = \frac{b}{4} \times \frac{1}{b}$$

$$4a = 15 + a$$

$$a = 5$$
$$b = \frac{b}{4} + 9$$
$$4b = b + 36$$
$$b = 12$$
$$\therefore a = 5 \ and \ b = 12$$

The following is an analysis of learning activity 4 learners' responses of Group A. The group consisted of Lebogang, Lesiba and Sipho.

In this task learners were required to determine the unknown sides (a and b) in the figure above using the principles of similarity. Similar triangles are characterized by corresponding sides that are in proportion. Here learners were observed to be at property noticing layer. At property noticing, they began to identify another triangle from the triangle provided, i.e., they were able to discover that there were two triangles on a given figure. They named their triangles, ΔTRS and ΔTPQ respectively. As a result, they were able to identify sides that are in proportion from the resulting triangles. Once learners develop the concept of geometric figure, at property noticing they can identify its properties. However, at property noticing, they incorrectly allocated values of the sides of ΔTPQ. Thus, they considered sides TP and TQ to be 15 and 9 respectively (Figure 6). In this case, learners seemed to have failed to recognize that side TP was formed of sides TR (a) and PR (15); and side TQ (b) formed of sides TS ($\frac{b}{4}$) and SQ (9).
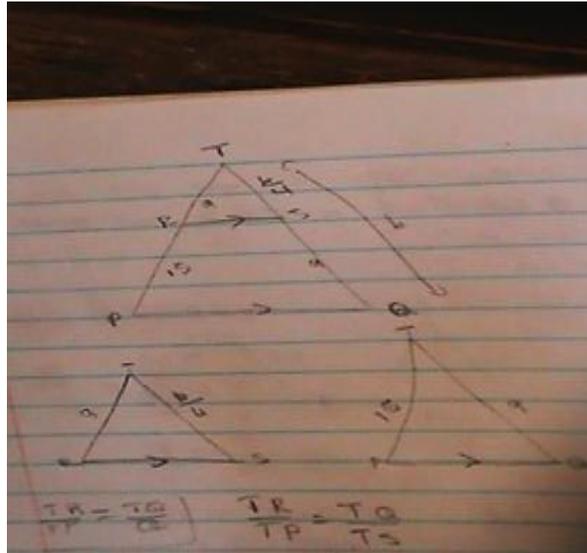
Figure 6: Learners incorrect allocation of values of sides TP and TQ

Although learners had initially allocated incorrect values of the sides, their acts of using the cross multiplication strategy in proportions further show that they were at property noticing layer. Here learners were observed to have difficulty in applying cross multiplication strategy in cases where one of the numerators of the sides that are in proportion is a fraction (Figure 7). As a result of their difficulty in using cross multiplication, they folded back to image making as illustrated in the following vignette 5.
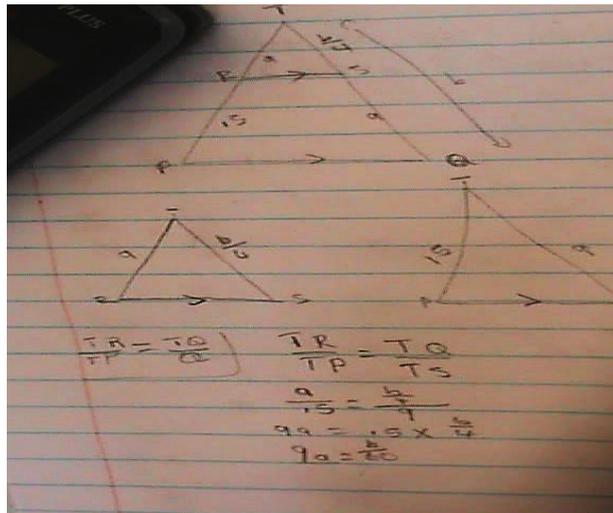
Figure 7: Learners' error of cross multiplication algorithm

Vigntette 5

5.1.	Lesiba:	Actually can't we have…let me ask…on TP can't we have $15 + a$?
5.2.	Lebogang:	TP?
5.3.	Lesiba:	$15 + a$ over…no a over $15 + a$ is equal to
5.4.	Lebogang:	a over 15 times a?[confused of what to write]
5.5.	Lesiba:	[takes the book from Lebogang and begins to write]
5.6.	Lesiba:	Therefore we $say\ a\ over\ 15 + a$ [writes $\frac{a}{15+a}$] is equal to, like here we say $9 + b$ over 4 [writes $\frac{\frac{b}{4}}{\frac{b}{4}+9}$]
5.7.	Lebogang:	The whole of side of TQ is equal to b

Lesiba's utterances "...on TP can't we have 15 + a?" in the above excerpt shows an explicit self-intentional intervention to image making layer. In image making, Lesiba was observed to be working with his existing understanding of algebra. In this instance the form of folding back observed is working at inner layer using existing understanding. Here Lesiba was manipulating sides TP and TQ(line 5.1 & line 5.3). As a result, he was able to realize that side TP is the sum of the sides TR and PR which is $15 + a$ (line 5.1 & line 5.2). Lesiba's utterances, on the

other hand, caused Lebogang to fold back to image making where she realized that line TQ is equal to b (line 5.7). Thus, learners were able through folding back, to re-work sides TP and TQ. Learners' initial conceptions of the sides TP and TQ appeared to have changed. At image making layer, learners were able to enrich their understanding of sides TP and TQ.

Consequently, Lesiba was then able to fold back to property noticing through working with the properties of similar triangles, particularly sides that are in proportion. Such an act of property noticing is illustrated in line 5.6 and in Figure 8, where he was able to set up an equation illustrating sides that are in proportion. However, Lesiba did not consider Lebogang's articulation (line 5.7) of side TB to be equal to b. As a result, Lesiba encountered difficulties in carrying out the cross multiplication algorithm.
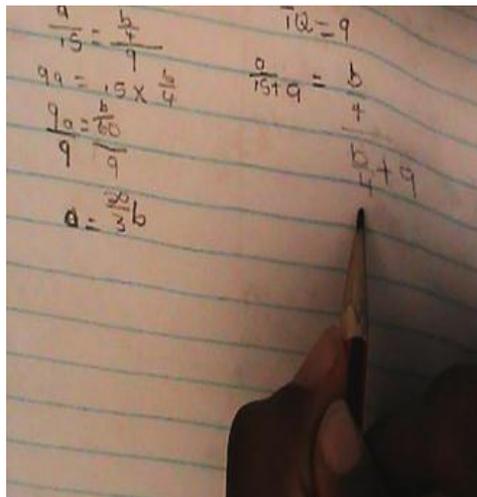
Figure 8: Learner working on proportional sides

Although in property noticing learners were able to identify sides that are in proportion, they were unable to find the magnitude. Their difficulty in this case, as in an earlier attempt, was mainly attributed to their challenges in proportion problems that have either numerator or denominator as a fraction. As a result of such difficulty, Lesiba committed application error of the cross multiplication

strategy. Thus, these are the mistakes that learners make when they know the concept but cannot apply it to a specific situation (Hodes & Nolting, 1998). Instead of firstly simplifying $\dfrac{\frac{b}{4}}{\frac{b}{4}+9}$ through inverting the divisor $(\frac{b}{4}+9)$; Lesiba proceeded with the cross multiplication strategy. Here, Lesiba could have used b as the value of side TQ as raised by Lebogang (in line 5.7) other than $(\frac{b}{4}+9)$. This type of error, shows that Lesiba has a knowledge of cross multiplication strategy but could not apply it in a case where one of the numerators of the sides that is in proportion is a fraction. Furthermore, this seems to reveal that Lesiba's conceptual understanding of fraction was inadequate. This further resulted in another form of intervention, in this instance a teacher's unintentional intervention.

Vignette 6

6.1.	Lesiba:	TR over TP is equal to TS over TQ, TR is a over 15+ a is equal to b over 4 over 9 + b over 4.
6.2.	Teacher:	Yes, that equation is correct.
6.3.	Lebogang:	Cross multiply.
6.4.	Lesiba:	Cross multiply…
6.5.	Teacher:	Okay before we cross multiply.
6.6.	Lesiba:	We can, we can say b over 4 ,this way [starts writing $\frac{b}{4} \div 9 + \frac{b}{4}$]
6.7.	Teacher:	Okay continue.
6.8.	Lebogang:	Write b over 4, do we divide it?
6.9.	Lesiba:	Then we say times, then 4 goes on top…
6.10.	Lebogang:	I don't understand you…
6.11.	Lesiba:	You don't understand me, like, you see now is $\frac{b}{4} \div 9 + \frac{b}{4}$,then we say b over 4,times $9 + \frac{4}{b}$ [writes $\frac{b}{4} \times 9 + \frac{4}{b}$
6.12.	Lebogang:	Times 9? These ones you add them, $9 + \frac{b}{4}$ or you time them?
6.13.	Lesiba:	hmmm, this [division] sign we change it to [multiplication] times.
6.14.	Lebogang:	Yes, write them, but let us replace $9 + \frac{b}{4}$ with b.

In this excerpt, the teacher's unintentional intervention, "Okay before we cross multiply" invoked Lesiba to fold back to his primitive layer. At the primitive

layer, Lesiba was observed to be working with his existing understanding of fractions and cross multiplication. Thus, Lesiba appears to be aware of what was needed, yet his understanding was inadequate. Here the form of folding back observed is collecting at an inner layer. At primitive layer, Lesiba was able to realize that he needed to re-write expression for one of the proportional sides before he could continue working with them. As a result he was able to convert it "$\frac{\frac{b}{4}}{\frac{b}{4}+9}$"into a different form of "$\frac{b}{4} \div 9 + \frac{b}{4}$" (line 6.6). Having successfully converted the expression, he proceeded with "invent" strategy which involves changing division sign to multiplication sign. However, in this case Lesiba considered components of the divisor to be separate entities other than the unified whole, i.e., he inverted only $\frac{b}{4}$ instead of the whole divisor "$9 + \frac{b}{4}$" (line 6.11). This type of error appears to have caused confusion in Lebogang, which is illustrated by in her utterances "Times (multiply) 9? These ones you add them, $9 + \frac{b}{4}$ or you times (multiply) them?" As a result, she suggested that they use b instead of $9 + \frac{b}{4}$ since they both equal to side TQ (line 6.14).

Through folding back, learners were able to reconstruct and rebuild their existing understanding of using cross multiplication strategy in cases where they were dealing with fractions. In this case, the thickening effect of folding back was observed. This resulted in effective folding back where learners were able to return to the outer layer of property noticing with external prompts to resolve the learning activity.

4.3.   TEACHING EXPERIMENT 2: MIDPOINT THEOREM

The teaching experiment two was based on proving the midpoint theorem as well as the application of this theorem. The midpoint theorem states that the line segment joining the two mid-points of two sides of a triangle is parallel and

equal to the third side of a triangle. This experiment consists of two teaching episodes. The first episode focused on the proving of the theorem and the second episode on the application of the theorem.

4.3.1. TEACHING EPISODE 1: APPLICATION OF MIDPOINT THEOREM

In this teaching episode, learners were introduced to the midpoint theorem. As compared to the previous episode, this episode was quite different as learners had not done midpoint theorem in their earlier grades. The lesson started by explaining concepts such as the midpoint of a line segment and bisects. Learners were then given instructional hands on an activity that required them to use paper and pencil approach in proving conjectures. The objective of such an instructional activity was that learners should make conjectures about the line segment joining midpoints of two sides of a triangle. For the purposes of reporting, instructional activity that was given to learners was not included as folding back was not observed. Thus, activities were in such a way that learners were being directed on what to do. Thus, in case of such instructional activities, folding back was not observed. The activity used in this study was given to learners in the afternoon session.

Learning Activity 5 required learners to determine the magnitude of the interior angles of a triangle.

Learning Activity 5.

In the following figure, M, N and T are the midpoints of AB, BC and AC is ΔMNT. $\angle A = 60°$ and $\angle B = 60°$. Calculate the interior $\angle^s$ of ΔMNT.
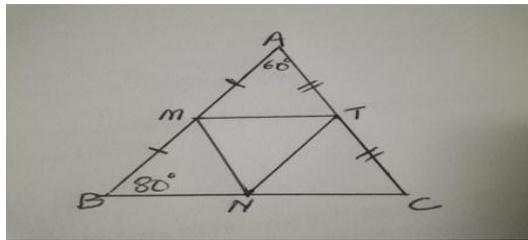
An expected response from the learners is presented below.

$$\angle C = 180° - (80° + 60°) = \cdots sum\ of\ \Delta$$

$$\therefore \quad \angle C = 40°$$

In $\Delta ABC: M\&T\ are\ midpoints\ of\ AB\ and\ AC\ ...\ MT//BC\ ...\ midpoint\ thm$

$$\therefore\ AMT = 80°\ ...\ Corresp.\angle^s =$$

Similarly, $M\&N\ are\ midpoints\ of\ AB\ and\ BC$

$$\therefore\ B\widehat{M}N = 60°\ ...\ Corresp.\angle^s =, MN//AC$$

$$T\widehat{M}N = 180° - (80° + 60°)\ ...\angle^s\ on\ a\ str.\ line$$

$$\therefore T\widehat{M}N = 40°$$

The same method can be followed to determine the other two angles of ΔMNT

Answer: 40°; 80°; 60°

The following is an analysis of activity 5 responses for Group A. The group consisted of Lebogang, Lesiba and Sipho.

The task required learners to determine with reasons, $\angle^s$ of ΔMTN. In their attempts to solve this learning activity, they were observed to be engaging in the acts of property noticing. At property noticing, learners realized that they first had to determine the magnitude of other angles other than the interior angles of ΔMTN. They were observed to be using their existing knowledge of the sum of angles in a triangle. This enabled them to determine $\angle C$ which they successfully found it to be $40°$. Having been able to determine $\angle C$, Sipho's utterance "so how do we find angles M, N and T?" seems to have suggested that they could proceed to find $\angle^s$ of ΔMTN. However, at property noticing layer, their knowledge was inadequate as a result folded back to image making layer.

Vignette 7

7.1.	Lesiba:	Is it not possible to use Midpoint theorem?
7.2.	Sipho:	Isn't this F? This is F…you see…
7.3.	Lebogang:	Then it is corresponding angles...
7.4.	Lesiba:	Then parallel lines
7.5.	Lebogang:	It means that here [pointing $\angle N$], we are going to represent it as $\angle N_1$, $\angle N_2$, here[pointing $\angle T$] $\angle T1$ and $\angle T2$).
7.6.	Lesiba:	But here they didn't give us $\angle N1$ and $\angle N_2$…) [Inaudible] $\angle$ of ΔMTN and then midpoint of AC is T… Midpoint MN and TR are midpoints of...
7.7.	Lebogang:	If we can say $\angle N1$ and $\angle N2$ we will understand that $\angle B$ is equal to $\angle N_2$, then they are corresponding.
7.8.	Sipho:	Ohoo...understand, these are the angles that *Lebogang* is talking about, this if F…
7.9.	Lesiba:	I can see that ..
7.10.	Lebogang:	This means $\angle N_2$ is equal to 80°.
7.11.	Sipho:	Wait a minute! Oh yes I can see that.
7.12.	Lebogang:	Then here it's ΔTNC.
7.13.	Sipho:	Hmmm.
7.14.	Lesiba:	Hmmm.
7.15.	Sipho:	You are using a long way… Ohooo yes continue…
7.16.	Lebogang:	Then in ΔTNC, we are going to do just like the first part, we take $\angle N_2$ and add it with $\angle C$ then subtract 180° to get $\angle T_2$
7.17.	Sipho:	Hmmm I see it.

In this excerpt, Lesiba's initial response shows that he considered the application of midpoint theorem to the learning activity at hand. As a result, he

asked his peers if the midpoint theorem could be applicable or not. However, the responses he got did not align with his question. For example, Sipho's response of identifying mnemonic F was irrelevant to Lesiba's question (line 7.2). In this instance, it appears that Sipho was not actually responding to Lesiba's question but rather was having a different thought process. Sipho was observed to question his own understanding of angles and how they can be identified through mnemonic. The use of mnemonic such as *F*, play a significant role in enhancing learners' conceptual understanding of corresponding angles. Thus, learners use mnemonic to identify various angles (Vignette 1, line 1.11; Vignette 4, line 4.2& line 4.3; Vignette 7, line 7.2 & line 7.8). Sipho was looking for what could be said more generally about the image, an indication that he was at image making. This led Lebogang to fold back to the image making layer. Thus, explicit peer intentional intervention is observed in this instance. At image making Lebogang worked with her existing understanding of corresponding angels. Furthermore, Lebogang continued to manipulate and work on $\angle N$ and $\angle T$. She realized that both $\angle N$ and $\angle T$ consisted of three $\angle^3$, which she named "... N_1, N_2 & N_3; T_1, T_2 & T_3" respectively. In so doing, she was of the view that they "...will understand that $\angle B$ is equal to $\angle N_3$...they are corresponding." Subsequently, at image making learners were able to find the magnitude of $\angle N_2$ to be 80°. They proceeded to determine $\angle T_2$.

How learners, Lebogang in particular, approached $\angle T$ was interesting. This was because $\angle T$ consisted of 3 $\angle^s$; $\angle T_1, \angle T_2$ & $\angle T_3$ with a similar structure as $\angle N$. What is interesting in this case is that learners approached it quite differently from $\angle N_2$. Instead of using corresponding angles as in $\angle N_3$, she used her knowledge of angles in a triangle. This is well demonstrated in her articulation "…here is...ΔTNC". However, Sipho's response "you are taking a long way" (line 7.15) seems to indicate that she could use the same approach of corresponding angles she applied to $\angle N_2$. Here the image that Lebogang had formed is different

from the one formed by Sipho. Lebogang depended on her knowledge of the sum of interior angles of a triangle, whereas Sipho on corresponding angles. Although, Sipho appeared to have developed a different thought process to that of Lebogang, he did not discourage her from trying a different approach. This is evidenced in his response: **"Yes do it, carry on and then? [Pause] Ohooo, hmmm carry on Lebogang …I understand you"**. Here, Sipho's level of excitement seems to indicate that he developed an understanding of the new approach used by Lebogang. However, Lebogang's response "We are going to do just like the first part…" appears to remind her peers they could determine $\angle T_2$ the same way as $\angle C$. Eventually, learners were able to determine the magnitude of $\angle T_2$ which they found to be $80°$.

Vignette 8

8.1.	Lesiba:	We are supposed to find $\angle M, \angle N$ and $\angle T$.
8.2.	Lebogang:	Wait, listen, $\angle N_2$ is alternating to $\angle T_1$.
8.3.	Sipho:	Show [Pause] Yes, carry on *Lebogang.*
8.4.	Lebogang:	[starts writing]
8.5.	Sipho:	So $\angle T_1$ is $80°$.
8.6.	Lesiba:	Alternating angles are equal...
8.7.	Lebogang:	Alternating angles are equal. It means that $\angle T_1$ is $80°$.

Having been able to determine $\angle N_2$ and $\angle T_2$ respectively, they were able to fold back from image making to property noticing layer. Thus, learners were able to return to the outer layer of property noticing through using their inner image making layer of knowledge of angles to determine $\angle T_1$. Here the effective folding back is observed as learners were able to return to their initial layer of property noticing through interacting with each other. This enabled them to determine $\angle T_1$ as one of the interior $\angle^s$ of ΔMTN with appropriate reasoning as shown in the above vignette. They proceeded with other angles as follows:

Vignette 9

| 9.1. | Lebogang: | Then it means $\angle N_1$ is alternating to… |
| 9.2. | Sipho: | It's going to be difficult when we move forward… |

9.3.	Lebogang:	$\angle T_2$ is corresponding to…
9.4.	Sipho:	Lesiba do you have any idea…?
9.5.	Lesiba:	Did we find $\angle N_1$?
9.6.	Lebogang:	No, we found $\angle T_1$ and $\angle T2$…
9.7.	Lesiba:	Can't we use Midpoint theorem?
9.8.	Teacher:	What will be the size of $\angle M_3$?
9.9.	Sipho:	Ah!…it's going to be $80°$.
9.10.	Lesiba:	Do we have $\angle T_2$?
9.11.	Sipho:	This thing is upside down [rotating the worksheet]...
9.12.	Teacher:	Okay let's look at $\angle M_3$.
9.13.	Sipho:	Yes, that's what I was suggesting, that's why I said, I have looked at this and then I saw that they form letter F but I wasn't sure about it.
9.14.	Teacher:	Okay.
9.15.	Sipho:	Then if they form letter F, they are corresponding angles of which M3 will be equal to 80 degrees.
9.16.	Lebogang:	We are going to work with ΔAMT…
9.17.	Sipho:	Yes…
9.18.	Lebogang:	$\angle A$ plus $\angle M$ plus $\angle T$, must give us $180°$.

Although learners were able to return to the outer property noticing to determine $\angle T_1$, they could not determine other remaining interior angles. In this case, the difficulty arose when they were supposed to determine $\angle N_1$. Lebogang was of the view that $\angle N_1$ is an alternating angle. However, she could not realize that $\angle M_2$ was alternating to $\angle N_1$ (line 9.1). It appears that Lebogang wanted to apply the same approach of $\angle T_1$ to $\angle N_1$. This according to Sipho's response: "is going to be difficult for us to go forward" (line 9.2) because they did not have the magnitude of an angle alternating to $\angle N_1$. This led to explicit teacher's intentional intervention (line 9.8). Here the intervention was international and explicit in such a way that it intended to direct learners to $\angle M_3$. $\angle M_3$ was precisely selected in particular because Sipho's actions focused on this angle. As a result, learners were able to fold back to image making layer.

At image making, Sipho was able to manipulate the figure through rotating it in order to identify mnemonic F (line 9.22, line 9.31 & line 9.33). This enabled him to realise that $\angle M_3$ is corresponding to $\angle B$ and was equal to $80°$. Here,

Learners were working at image making using their inner understanding of various angles. Thus, they continued to work with ΔAMT where they determined $\angle T_3$. As a result they were able to fold back to property noticing layer where they found that $\angle M_1$ to be $40°$ with appropriate reasoning that it is alternating to $\angle T_3$. Moreover, at property noticing they were able to use some of angles in a triangle to determine $\angle N_1$ which was $40°$. The also realized that $\angle N_1$ was alternating to $\angle M_2$. Thus learners were able to effectively fold back to the outer layer of property noticing to determine the interior $\angle^s$ of ΔMTN with reasons.

4.4. TEACHING EXPERIMENT 3: QUADRILATERALS

4.4.1. TEACHING EPISODE 1: PARALLELOGRAMS

In this teaching episode, learners were discovering properties of various quadrilaterals. The teaching episode started by the teacher asking learners what a quadrilateral is, and learners were further required to mention types of quadrilaterals they knew. Investigative approach was used to investigate properties of parallelograms. This approach in facilitation is underscored by Mayer and Wittrock (2006) in saying that it gives learners opportunity to explore mathematical properties on their own. Thus, this approach gave learners opportunities to discover relationships and draw conclusions, thus, improving their conceptual development of geometry. This approach was used because the intention was that learners should be able to discover properties of parallelograms themselves. In investigating properties of parallelogram, learners were given instructional activity. From this task, learners were able to discover the properties of the parallelograms without the teacher telling them.

Learners were invited to a session after school where they were working on the geometry tasks that required them to apply properties of quadrilaterals. Folding back reported here was observed during after school sessions. The following activity 6 was given to the learners.

Learning Activity 6

In the accompanying figure, $ABCD$ is a $\|^m$. $AB = BE$. The diagonal AC is produced to E, such that $AD = CE$. If $C\hat{E}B$ is equal to x. Prove, giving reasons that angle $F\hat{D}C = 3x$

An expected response from the learners is presented below.

$E\hat{A}B = x \dots. AB = BE$

$\therefore\ D\hat{C}A = x \dots alt. \angle^s; DC \parallel AB \ in\ \|^m$

$C\hat{B}E = x \dots CB = CE$

$\therefore\ A\hat{C}B = 2x \dots ext. \angle\ of\ \Delta$

$\therefore\ D\hat{A}C = 2x \dots alt. \angle^s; AD \parallel BC \ in\ \|^m$

$\quad F\hat{D}C = D\hat{A}C + D\hat{C}A \dots ext. \angle\ of\ \Delta$

$\qquad = 2x + x$

$\qquad = 3x$

$or: F\hat{D}C = F\hat{A}B = 2x + x = 3x \dots corr. \angle^s; AB \parallel DC$

The following is an analysis of activity 6 learners' responses for Group B. The group consisted of John and Koena.

Vignette 10

10.1.	John:	Line EC is equal to line AD, then C is a midpoint and D is also a midpoint
10.2.	Koena:	Midpoint…
10.3.	Teacher:	Midpoint of?
10.4.	John:	D is the midpoint of AF.
10.5.	Teacher:	Midpoint of?
10.6.	Koena:	D is the midpoint of AF.
10.7.	John:	C is the midpoint of AE.
10.8.	Teacher:	Is DF equal to DA?
10.9.	John:	Hmmm no…

In this task, learners were required to show giving reasons that $F\widehat{D}C = 3x$. In order to carry out this task, the knowledge of various types of angles was important. Learners' initial attempt indicates that they intended to use their knowledge of midpoint theorem which states that the line segment joining the two mid-points of two sides of a triangle is parallel and equal to the third side of a triangle. Here learners were at image making. However, this knowledge was irrelevant to this task. Here the unfocused intentional teacher intervention is observed. This unfocused intervention through asking "Is DF equal to DA?" was to discourage learners from using midpoint theorem. This form of intentional intervention did not indicate the idea or mathematical knowledge learners should consider in working out the learning activity at this stage. The intention was to allow learners to discover an approach that might assist them in manipulating the learning activity. Through this intervention, learners were asked questions that drew their attention to what they were thinking; either its relevance or irrelevance (Line 10.3, 10.5 and 10.8). As a result, John ultimately realized that the midpoint theorem cannot be applied to further manipulate the geometric figure through fold back to property noticing. The conversation that reflects property noticing by John unfolded as follows:

11.1.	John:	Look it's a parallelogram. Look, this one is not a parallelogram, it is a triangle…Look at this triangle can you see it?
11.2.	Koena:	Yes.
11.3.	John:	Being it a triangle, $\angle E$ is equal to $\angle A$ because is an iscosceles triangle.
11.4.	John:	It must be $3x$.
11.5.	Koena:	It must be?
11.6.	John:	$3x$ [pause] Then what is the value of x?
11.7.	Koena:	It is not given, but they have given us $\angle E$ which is x and then $\angle A$ is x.

At property noticing, John was able to identify parallelogram and also triangle (line 11.1) from the figure that was given. Thus John was able to work with his existing understand of parallelograms and triangles. In this instance John was able to discover various properties of an identified isosceles triangle through to manipulation of the geometric figure. Both Koena and John were able to find the magnitude of $\angle A$ by engaging in the acts of property noticing. It was only after they determined angle A that they were able to find other angles. This enabled them to proceed in determining other angles.

Vignette 12

12.1.	John:	But they said this is a parallelogram. We are told that this is a parallelogram…
12.2.	Koena:	Then sides AD and BC are parallel…
12.3.	John:	Let us think clearly…
12.4.	Koena:	It can be parallel lines…
12.5.	John:	Yes…
12.6.	Koena:	$\angle E$ is equal to $\angle A$…
12.7.	Koena:	$\angle D$ is equal to the sum of $\angle A$…they are corresponding angles
12.8.	John:	Ohooo…yes you are right

At property noticing layer learners were observed to be manipulating and identifying the geometric figure they were working with. Here John was able to recall that they are given a parallelogram (line 12.1). It can be seen that John's ability to recall led them to work with the properties of parallelogram. In this

instance, Koena was able to be working with corresponding angles properties of a parallelogram where he realized that "$\angle D$ is equal to the sum of $\angle A$". Additionally, John's articulation "Ohooo…yes you are right" (line 12.8) is an evidence of conceptual formation. As a result, learners were able through property noticing to discover that $\angle A$ is equal to $\angle D$. Although learners were able to determine that $\angle A = \angle D$, they were not able to return to the outer layer to resolve the task. This was as a result of being unable to work on the magnitude of both $\angle A$ and $\angle D$. This led them to fold back to primitive knowing through explicit teacher intentional intervention. The proceedings are as follows:

Vignette 13

13.1.	Teacher:	Let's go for $D\hat{C}A$
13.2.	Koena:	Is x....
13.3.	John:	Here is x, angle is equal to angle
13.4.	Koena:	Is x.
13.5.	Teacher:	Why are you saying they are equal?
13.6.	John:	Because it is a parallelogram and opposite angles of a parallelogram are equal.
13.7.	Teacher:	This angle [Pointing to $D\hat{C}A$] here, not the whole of $\angle D$.
13.8.	Koena:	$\angle E$ equal …alternating angles.
13.9.	John:	$E\hat{A}B$ is equal to $D\hat{C}A$.
13.10.	Teacher:	The reason being?
13.11.	Koena:	Alternating angles...
13.12.	John:	[adds] are equal.

In this excerpt, an explicit teacher intentional intervention was observed. This intervention provided specific angles that learners could work with. For example, "Let's go for $D\hat{C}A$" intentionally directed the learners to work with $D\hat{C}A$. As a result, learners folded back to their primitive knowing layer. At primitive knowing layer, learners were reworking their knowledge of angles. Here it appears that learners had a difficulty in multifaceted nature of the angles that are in the form of $D\hat{C}A$. Such kind of multifaceted nature of the angle, where the angle symbol is followed by three points that define the angle, with the middle letter identifying the vertex, and the other two identifying points on both sides, probably

caused confusion amongst learners. This is well captured in John's response "Because it is a parallelogram and opposite angles of a parallelogram are equal" In this instance, John's response seems to illustrate that he considered $D\hat{C}A$ as $\angle D$. Through explicit teacher intentional intervention, learners became aware that $D\hat{C}A$ represented $\angle C$ of ΔACD. Thus, learners were able to rebuild their knowledge of multifaceted nature of the angle at a primitive knowing layer (line 13.9, line 13.11 & line 13.12).

Having been able to determine angles $E\hat{A}B$ and $D\hat{C}A$ respectively through the intentional teacher intervention, learners had to proceed to determine $C\hat{B}E$. This also took an intentional intervention, which was in the form of clue-giving. The proceedings were as follows:

Vignette 14

14.1.	Teacher:	Can we draw ΔCBE?
14.2.	John:	[begins drawing].
14.3.	Teacher:	Let us look at $ABCD$… which side is equal to DA?
14.4.	John:	Line BC.
14.5.	Koena:	No it is line CE.
14.6.	John:	Look it's CE but even CB is also equal to DA.
14.7.	Koena:	Yes.
14.8.	Teacher:	Why do you think it will be equal?
14.9.	John:	Because is a parallelogram
14.10.	Koena:	Opposite side of a parallelogram
14.11.	John and Koena:	$C\hat{B}E$ and $B\hat{E}C$ are equal ![smile]
14.12.	John:	[Smiles] look at this!
14.13.	Koena:	$C\hat{B}E$ is x.
14.14.	John:	Let us just say $\angle B_2$ is equal to $\angle E$.
14.15.	Koena:	Base angles of an isosceles triangle are equal.

Here the explicit teacher intentional intervention suggested learners to work with ΔCBE and $\parallel^m ABCD$ (line 14.1). Ultimately, this has led John to be able, through property noticing, to give side which was equal to DA (line 14.4). The intervention here appears to have caused John to fold back to his knowledge of parallelograms, particularly properties of equal sides. Thus, John was working with

his existing knowledge at property noticing layer. However, Koena's initial comment "No it's CE" (line 14.5) is evidence that he was having an image, mainly because his answer seemed to be based on the given information and as such did not realize that BC was also equal to DA. As a result, the explicit teacher intentional intervention through asking learners questions "Why do you think it will be equal?" (line 14.8), provided them an opportunity to provide reasons for their thought process. Thus, the role of questioning in this context was to elicit learners' thinking and create an opportunity for them to explain their own geometry thought process.

In addition, this intervention through questioning also provided an opportunity to follow the learners' reasoning path. John's further explanation, "Because it is a parallelogram" probably illustrates that he was aware of his thinking. As a result, Koena was ultimately able to identify opposite sides of a parallelogram, an illustration that he was engaging in property noticing. It appears that once learners are able to identify a parallelogram, they know that it has two opposite sides equal (line 14.10). Moreover, learners further realized the connection between ideas in that they were able to discover BC as the side of ΔCBE. As a result, of noticing the connection between ideas, learners were able to find that $C\hat{B}E$ is equal to $B\hat{E}C$ since they are both base angles of an iscosceles triangle (line 14.11, line 14.12, line 14.13, line 14.14 & line 14.15). In this instance, learners' reasoning ability through using properties of triangles was enriched. Learners then proceeded to find other angles through explicit teacher's intentional intervention.

The results of clue-giving intervention led to another form of explicit intervention known as guiding, which Towers and Proulx (2013) characterized as directing learners' attention towards a specific position. This form of intervention was as a result of being able to follow learners' reasoning path. Once a teacher is able to follow learners' reasoning path, s/he can intervene appropriately to provide

learners with guidance. Following the reasoning path that learners took and asking the questions enabled me to gather more information about the work learners were doing. The intervention unfolded as follows:

Vignette 15

15.1.	Teacher:	Let us go back to that ΔCBE..Can you produce $to\ A$?
15.2.	John:	Here?
15.3.	Teacher:	So that we have $straight\ line\ EAC$.
15.4.	Koena:	Ohooo [starts drawing].
15.5.	Teacher:	What will be that angle?
15.6.	John:	[smiles] x.
15.7.	Koena:	$A\hat{C}B$ is equal to the sum of $C\hat{B}E$ and $C\hat{E}B$.
15.8.	John:	Why are you saying that?
15.9.	Koena:	Exterior angle is equal to the...
15.10.	John:	[Interrupts] sum of the opposite angles.
15.11.	Koena:	Yes.
15.12.	John:	It is not equal to, it is the sum.
15.13.	Koena:	Yes, this one is equal to the sum of these once.
15.14.	John:	Let me show you something.
15.15.	Teacher:	Okay let's go to what *Koena* was saying
15.16.	Koena:	$\angle C$ is equal to the sum.
15.17.	John:	[interrupts] the sum.
15.18.	Koena:	Yes, the sum.
15.19.	Teacher:	So do we know those angles?
15.20.	Koena:	Is x.
15.21.	John:	Is x n x, is going to be $2x$, which means $\angle C$ is equal $2x$?
15.22.	Koena:	Yes.
15.23.	John:	$2x$ only?
15.24.	Koena:	Yes.
15.25.	John:	$2x$ only?
15.26.	Koena:	Yes.
15.27.	John:	$2x$ only?
15.28.	Koena:	Serious.
15.29.	John:	Ohooo…I see it now , the sum

The explicit teacher intentional intervention in this regard enabled learners to fold back from property noticing to their image making layer. At image making learners were working with their existing understanding of angles through explicit teacher's intentional intervention. This intervention was explicit and intentional in the sense that it stated to learners concepts they worked with. For example, "So

that we end up having ECA as a straight line" (line 15.3) directed learners to which mathematical concepts they should work with in their attempts to work with relationship between various angles. As a result, Koena began to draw the line ECA. The intention in this case was to promote the use of external representations. Such usage is important in the development of geometric concepts. This was evident in Koena's situation, as he was able to draw external representation, through the assistance of the teacher, to develop geometry conceptual understanding. At image making, it appears that the role of external representation assists a learner in developing geometric conceptual understanding. Koena was then able to develop a mental construct of $A\hat{C}B$ (line 15.5); hence, he was then able to discover that $A\hat{C}B$ is equal to the sum of $C\hat{B}E$ and $C\hat{E}B$. Once learners develop an image, they can notice the relationship between the exterior and the sum of the interior opposite angles. This further shows that when learners are able to represent geometric figure, they could manipulate it.

John's question "Why are you saying that?" (line 15.8) has led them to persistently engage in a discussion through which they exchanged ideas as they worked on the task. This seems to indicate that learners are confident to ask their peers to explain their thought process. In addition, this appears to suggest that learners are not only interested in what their peers are doing, but also why they are doing it. It can further be seen that through such exchanges, learners are able to articulate their developed geometric understanding of angles. For instance, Koena's articulation, "$A\hat{C}B$ is equal to the sum of $C\hat{B}E$ and $C\hat{E}B$" is an evidence that he was engaging in the acts of property noticing and had developed an understanding of $A\hat{C}B$. When learners exchange their ideas, their reasoning ability of the concept is enriched.

Although, John was able to recall the prior highlighted relationship of angles, his knowledge appeared to be incomplete. This is well articulated in his comment "Is not equal to, is the sum…" (line 15.12), which seems to suggest that in this case, he failed to understand the context of the sum. As a consequence, the explicit intentional intervention as articulated in line 15.16 (okay let's go to what Koena was saying) was aimed to reinforce an idea raised by a learner in line 15.8. This encourages learners to share their ideas and promote active learners' participation during the learning process. Learners often feel that they are in control of their learning when their ideas are emphasised in the teaching and learning. In addition, it helps learners to develop confidence in explaining their ideas. Hence, Koena's further articulation "yes the sum, when we add them together", is an illustration that he was at property noticing layer and able to explain his mathematical thinking. Moreover, it can be seen that such an explanation of an idea was aimed at assisting Koena to develop mathematical insight. Consequently, John was able to engage in image making, particularly to thicken his knowledge of the sum of angles as being used in the context of exterior and interior angles of a triangle. Moreover, learners were through, all the different forms of intervention as observed in this learning activity to return to the initial outer layer of image making to resolve the learning activity.

Learning Activity 7: Prove that ABCD is a parallelogram.

97
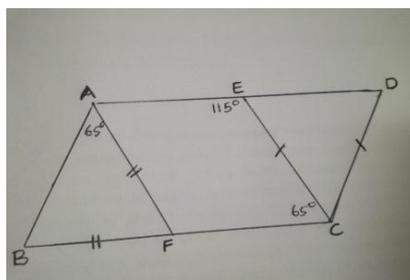
Learning activity 7 required learners to prove that ABCD is a parallelogram. An expected response from the learners is as follows:

$$AEC + FCE = 115° + 65° = 180° \therefore AE(D)$$
$$\parallel (B)FC \dots co - interior \angle^s supplimentary .$$
$$\angle B = 65° \dots FA = FB$$
$$CED = 65° \dots str.line\ AED$$
$$\therefore \quad \angle D = 65° \dots CE = CD$$
$$\therefore ECD = 50° \dots sum\ of\ int.\angle^s of\ \Delta$$
$$\therefore \angle B + \angle C = 65° + 115° = 180°$$
$$\therefore AB \parallel DC \dots co - interior \angle^s supplimentary$$
$$\therefore ABCD\ is\ a\ \parallel^m \dots 2\ pairs\ opp.sides\ \parallel$$

Vignette 16 shows learners' responses of learning activity 7.

Vignette 16

16.1.	Lebogang:	$\angle A$ is equal to $\angle C$, opposite angle of a parallelogram are equal
16.2.	Teacher:	which parallelogram are you looking at?
16.3.	Lebogang:	$ABCD$
16.4.	Teacher:	and then what are you supposed to do?
16.5.	Sipho:	prove that it is a parallelogram [image having]
16.6.	Teacher:	So are saying $\angle A$ is equal to $\angle C$? Or are you saying if you can find $\angle A$ being equal to $\angle C$, then you can conclude that $ABCD$ is a $\parallel^m$?

Learners in this task were required to prove that $ABCD$ is a $\parallel^m$. Lebogang's initial articulation "$\angle A$ is equal to $\angle C$, opposite angle of a parallelogram are equal" appears to be informed by her knowledge of the properties of parallelogram. The evidence seems to suggest that the learner knows the properties of a parallelogram yet finds it difficult to prove that the given figure is a$\parallel^m$. Here Lebogang was observed to be at property noticing layer. However, in this instance Lebogang could not realise that properties of a $\parallel^m$ cannot be used unless it can be

established that they are working a $\|^m$. Thus, Lebogang response could only be correct provided she is given a $\|^m$. Moreover, it can be seen that learners rely on the property of opposite angles when working with parallelogram related tasks. It is important for learners to not only recognise and deduce properties of parallelograms but also know when and how to use them.

Subsequently, this led to explicit teacher's intentional intervention through asking learners question "…what are you supposed to do?"(line16.4), this intended to test whether learners knew and understood what they were supposed to do. This further provided an opportunity to have an insight of what learners knew and did not know. Moreover, "are you saying if you can find $\angle A$ being equal to $\angle C$, then you can conclude that $ABCD$ is a $\|^m$?" was posed to learners in order to get clarity of their thought process. This intervention in facilitation is underlined by Van Zoest and Enyart (1998) in saying that it is important to ask learners to clarify and justify their ideas. Likewise, this intervention served to shed light in clarifying the task that required learners to establish that $ABCD$ is a $\|^m$. This was to say to them if they could establish that $ABCD$ was a $\|^m$, then they could conclude that $\angle A$ is equal to $\angle C$. In addition, the intervention ultimately led learners to fold back to their image making layer. The following illustrate how learners proceeded:

Vignette 17

17.1.	Lebogang:	Is $\triangle ABF$ an Isosceles triangle?
17.2.	Lesiba:	Yes
17.3.	Lebogang:	Then it means... we can say $\angle B$ is equal to $65°$ ... base angles of iscosceles triangle are equal
17.4.	Sipho:	Where is an Isosceles triangle here?
17.5.	Lebogang:	Is not this $\triangle ABF$ an Isosceles?
17.6.	Lesiba:	Two sides are equal
17.7.	Teacher:	yes that will be true

In image making layer, Lebogang was observed manipulating the figure provided. Her question "Is triangle ABF an isosceles?"(line 17.1), is evidence that she was working on the image. At this level of image making, the learner was

observed to be constructing her knowledge on how she could approach the task under discussion. The form of folding back observed here was "working at inner layer using existing understanding". Thus Lebogang was using her existing knowledge of geometry figures and their properties. At image making, such hands-on mathematical tasks are aimed at helping learners to develop particular initial conceptions and ideas for the meaning of a mathematical concept (Martin and Towers, 2014). Hence, the learner was observed to be working with her knowledge of the isosceles triangle. This has then led her to use properties of the isosceles triangle to determine the magnitude of $\angle B$ (line 17.3). The evidence suggests that once learners are able to manipulate images of the task, they can use properties of such images. Hence, at image making Lesiba's explanation (line 17.6) seems to indicate that once they work with a triangle with two sides that are equal, it is an isosceles triangle. In this instance, learners' reasoning with properties of an iscosceles triangle was enriched.

It appears that learners' level of engagement in image making enables them to develop mathematical insight of the task. For instance, a learner's suggestion (line 17.1) to find all angles is evidence that illustrates mathematical insight in terms of how to proceed with the task. Moreover, it appears that the development of mathematical idea at image making enables learners to fold back to other layers of understanding. According to Borgen (2006) and Martin (2008), a learner in image making engage in activities aimed at helping them to develop a particular representation for the task and mathematical idea. Piere and Kieran (1994) explain that a learner must create an image before she/he reaches any other layer. Hence, Lesiba was able to fold back from image making to image having.

Vignette 18

18.1.	Sipho:	Let us find all the angles, in that way we can see if it's a parallelogram...
18.2.	Teacher:	You thinking of finding all the angles?

18.3.   Sipho: Yes...and in that way we will see whether our opposite side are equal or not, and if they are equal and then this means is a parallelogram and if they are parallel

In the above vignette, it appears that the learner has an idea of using $\angle^s$ to determine if opposite sides are equal and parallel (line 18.1). It can further be seen that Sipho was engaging in a higher level of thinking. This according to Pirie and Kieren (1994) model of growth of mathematical understanding is an evidence of image having. This seems to indicate that a learner is aware that, once it can be established that two opposite sides are equal and parallel, then a figure is a parallelogram (line 18.1; 18.3). However, he could not fold back to property noticing, particularly because he seemed uncertain of how he was supposed to use the angles to determine whether lines are parallel or not. It seems his knowledge of two straight lines which are cut by a transversal, such that the interior angles on one side of the transversal are supplementary then lines are parallel was inadequate.

Later in the task, explicit teacher intentional intervention in the form of modelling led learners to engage in their primitive knowledge of parallelograms; in particular, their knowledge of properties of parallelograms. Through such intervention I was able to explain to learners how they could use other properties than opposite angles of a parallelogram to approach the task. According to Towers (1998), this form of intervention provides teachers with an opportunity to explicitly model their thought processes. As a result, this intervention seems to have assisted learners to learn how to use the transversal lines and the sum of consecutive interior angles to determine whether lines are parallel or not. It is important for learners to not only know that opposite sides of a parallelogram are equal and parallel but use angles to prove such equality and parallelism of the opposite sides. Thus at image making, learners were observed to be rebuilding their earlier constructs of parallelograms.

Moreover, this intervention did not only assist learners in solving the task but modified their prior constructs of parallelograms. As a result, learners were able to use the consecutive interior angle sum and the transversal lines and then ultimately folded back to property noticing layer. Thus through various forms of intervention as observed in this learning activity, learners were able to return to the outer property noticing layer with an informed thinking. At property noticing layer, learners were able to prove that figure $ACBD$ is a $\parallel^m$ through applying the consecutive interior $\angle^s$ to find two pairs of opposite sides of figure $ABCD$ being parallel. Thus, learners applied noticed property of consecutive interior sum to define a parallelogram. In this learning activity, learners were able, through folding to prove that a figure is a parallelogram using properties of angles.

Learning Activity 8.

In the accompanying Figure 9, $ABCD$ is a square. The diagonals AC and BD intersect at O. M is the midpoint of BD and $AM = ME$. Prove that $MD \parallel EC$
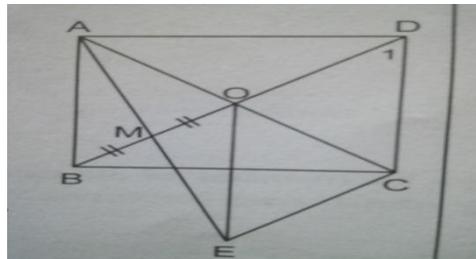
Figure 9 : ABCD is a square

Learning Activity 8 required learners to show that lines are parallel. An expected response from the learners is as follows:

In $\Delta AEC: O \; midpoint \; of \; AC \ldots Diagnal \; bisect \; one \; another$
$\& \; M \; midpoint \; of \; AE \ldots given$

$MO(D)//EC \ldots mipoint \; theorem$

Vignette 19 shows learners' responses of learning activity 8 learners' responses for Group A: The group consisted of Lesiba, Lebogang and Sipho. It is followed by its analysis.

Vignette 19

19.1.	Lesiba:	Can't it be a midpoint theorem? Can't it be? how do we see it?
19.2.	Sipho:	MD parallel to EC?
19.3.	Lesiba:	MD they say its third [pause], what? A third is equal to erh, what do they say?
19.4.	Sipho:	The third side of a triangle, ae, eish I forgot that...
19.5.	Lebogang:	Line segment
19.6.	Lesiba:	Can't ME be equal to EC? ME is equal to EC, midpoint theorem
19.7.	Sipho:	midpoint theorem. I don't know
19.8.	Lesiba:	Can't it be a midpoint, how do we see it?

In this task, learners were required to prove that MD is parallel to EC. Lesiba had an idea of approaching the task; however, his articulation "Can't it be midpoint theorem, how do we see it?"(line 19.1) appears to suggest that he was uncertain whether the midpoint theorem will be applicable or not. This is evidence that Lesiba was engaging in image making, i.e., he was trying to work on his knowledge of midpoint theorem. However, his existing knowledge of midpoint theorem appears to be insufficient which is evidenced in his articulation "MD they are saying its erh....; A third side is equal to…" (line19.2). this further suggested that Lesiba had to re-work his knowledge of midpoint theorem in his primitive knowing. Sipho, on the other hand seems to be aware of the inadequacy of their notion of the midpoint theorem; hence instead of offering an alternative idea, he suggests that midpoint cannot work. At this stage, the explicit teacher's intentional intervention was observed in Vignette 20.

Vignette 20

20.1.	Teacher:	But we can work with midpoint theorem, the one you are talking about. If we look at A, is it A?
20.2.	Sipho:	Yes Sir, oh yes, yah yah, I can see that
20.3.	Teacher:	then continue
20.4.	Sipho:	What is the midpoint of AC?
20.5.	Lebogang;	Midpoint of AC is O
20.6.	Sipho:	and then?
20.7.	Lebogang:	Then it means that here it is ΔAEC
20.8.	Lesiba:	[interrupt] we have to know that EC is parallel to BD and
20.9.	Sipho:	[interrupt Lesiba] Yes...Yes, *Lebogang* is…I think eish
20.10.	Lebogang:	ΔAEC, it means that line MO is equal to line BC, do you understand?
20.11.	Sipho:	What will be the reason?
20.12.	Lesiba:	midpoint theorem? What? What did you say?
20.13.	Sipho:	Are you saying MO is equal to EC?
20.14.	Lebogang:	ke half!
20.15.	Sipho:	MO is half of EC and also parallel

The explicit teacher intentional intervention suggested learners to work on the midpoint theorem (line 20.1). Such intervention in facilitation is supported by Bansilal (2015) who demonstrates that teachers often have to identify blockages in learners' conceptual development and help the learners resolve these gaps in knowledge. Thus, explicit teacher's intentional intervention as illustrated in Vignette 20 intended for learners to fold back and re-work on their primitive knowing layer. In this case, it was realised that Lesiba had an image of resolving the learning activity 8, as such the intervention "But we can work with midpoint theorem, the one you are talking about" intended to explicitly reinforce the notion raised by one of the learners, in this case by Lesiba. This explicit intentional intervention prompted learners to fold back to their primitive knowing, particularly to midpoint theorem. At primitive knowing layer, both Sipho and Lebogang were re-working their existing understanding of the midpoint theorem where they identified midpoint of AC to be O (lines 20.2, 20.4, 20.5). This has led learners to identify midpoint in the triangle as well as in the square. Here learners were able to realise AC is a side of a ΔAEC and also a diagonal of as square $ABCD$. The result of folding back to image making has enabled learners to extend their

knowledge of midpoint theorem. Consequently, learners were able to return to the outer layer of property with external prompt. In this instance, learners were able to use their extended knowledge of midpoint theorem of a triangle as well as diagonals of a square to establish that $MD \parallel EC$. Through folding back across various layers of understanding, leaners' reasoning in midpoint theorem was enriched.

4.5.   CHAPTER SUMMARY

In this chapter, I presented the data in the form of vignettes. Data analysis of the vignettes in the form of discussion illustrating how folding back occurred was also presented. The theoretical framework of folding back was used to analyse the data. The following chapter draws on the discussion of findings in relation to each research question guiding the study.

CHAPTER 5: CONCLUSION AND RECOMMENDATIONS

This study explored the role of folding back in improving Grade 10 learners' reasoning in geometry. In the literature, learners' lack of geometry conceptual understanding and difficulties in geometric reasoning has been well documented. Ndlovu and Mji (2012) echo that such difficulties suggest that efforts should not be spared until we can effectively and creatively teach for the clearer understanding of geometry by a greater number of learners. To this, Acquah (2011) concludes that there is an urgent need to change the traditional mode of geometry instruction to one that is more rewarding for both teachers and learners. Therefore, this study sought to alleviate learners' difficulties in geometric reasoning. Hence, the purpose of this study was to explore the role of folding back in enriching Grade 10 learners' reasoning in geometry. In order to achieve this purpose, the answers to the following research questions were of paramount importance:

- How does folding back support learners' interaction with geometric reasoning tasks during the lessons?
- How does a Grade 10 mathematics teacher use folding back to enrich student reasoning in geometry?

Therefore, this chapter draws on the main research findings as per the research questions which guided the study. Limitations and recommendations are also provided.

5.1. SUMMARY OF THE MAIN FINDINGS OF THE STUDY

5.1.1. HOW DOES FOLDING BACK SUPPORT LEARNERS' INTERACTION WITH GEOMETRIC REASONING TASKS DURING THE LESSONS?

The study reveals that folding back is a powerful tool that supports learners' interactions while working on geometric reasoning tasks. When learners fold back across various layers of their mathematical understanding, their geometry conceptual understanding is developed and in so doing, their geometric reasoning is further enriched. This result concurs with the result of the study conducted by Martin (2008) and the result of the study done by Lawan (2011). The study that Martin (2008) did revealed that folding back is a powerful tool for developing learners' understanding in mathematics. Similarly, the research done by Lawan (2011) showed that folding back is central to learners' growth of mathematical understanding. The growth of mathematical understanding happens when move across different layers of mathematical understanding. Likewise this study shows that learners move across different layers of mathematical understanding when resolving geometry learning activities (Vignette 1, Vignette 15). This agrees with the result of Valcarce et al. (2012) who found that during the resolution of the activity, learners move between different layers of understanding. In so doing, learners were able, through folding back, to improve their reasoning in geometry learning activities.

5.1.1.1. LEARNERS SHARE GEOMETRY KNOWLEDGE WITH EACH OTHER AS A RESULT OF FOLDING BACK

The study reveals that individual learners' engaging in a different layer of mathematical understanding to that of their peers, are able to share their developed geometry knowledge to their peers. This result shows that folding back embraces sharing of geometric ideas amongst learners. Thus, learners are not

only interested in their own geometry conceptual development but those of their peers as well. This was shown in some cases where learners had different approaches to the same learning activity. For example, this was evidenced in learning activity 5, where Sipho realized that Lebogang is using a different approach to the task. Sipho's further articulation "…ohoo yes continue…" (line 7.15) seems to demonstrate that he appreciates Lebogang's thought process. This was also evidenced in learning activity 1, where Lesiba's articulation "Wait let us give Lebogang a chance to do it" (line 1.5) is an illustration of respecting Lebogang's thought process. This results in a pathway through which learners can modify and build their mathematical understanding (Pirie & Kieren, 1994; Cobb, 2002; Weia & Ismail, 2010; Martin & Towers, 2011). Similarly, this according to Towers and Martin (2014), show that individual learners' actions and statements interweave constructively to build understanding for a group of learners. Consequently, this further creates a learning environment where ideas are not only shared but respected. Thus, where folding back takes place, individual learners' respect and appreciate their peers' geometry thought processes.

5.1.1.2. FOLDING BACK ENRICHES LEARNERS' QUESTIONING SKILLS IN GEOMETRY

The results of the study show that in a learning environment where folding back takes place, learners' questioning ability in geometry learning activities is enriched. When learners fold back, they stimulate their peers questioning skills. For example, as a result of folding back to various layers of mathematical understanding, learners were observed to question each other while working on geometry learning activities (Vignette 1, line 1.2; Vignette 15, line 15.8; Vignette 20, line 20.11, line 20.12 & line 20.13). This shows that folding back enhances learners to develop their questioning skills while interacting with each other. Moreover, learners' knowledge of geometry becomes enriched when they question each other. Learners where observed questioning each other on

instances where they did not understand their peers' approach on working on the geometry learning activities, they ask them why they do things in a certain manner. For an instance, this is evidenced in Vignette 1, line 1.2, where Lesiba was asking Lebogang to show them why it is a corresponding $\angle$. This is compatible with Makgakga's (2011) study which indicates that questioning plays an essential role in developing learners' mathematical understanding. Similarly, Wong (2012) holds the view that asking questions is a critical step to advancing one's learning. In addition, Warner and Schorr (2004) demonstrate that as learners reflect on their own thinking in response to questions that are posed by their peers they have the opportunity to revise, refine, and extend their ways of thinking about the mathematics.

5.1.1.3. LEARNERS JUSTIFY THEIR GEOMETRY THOUGHT PROCESS AS A RESULT OF FOLDING BACK

This study reveals that in a learning environment where folding back takes place, learners' reasoning skills in geometry is enriched. This shows that in a learning environment where folding back takes place, learners are able to justify their geometry thought process. For example, this was observed in various vignettes such as Vignette 7, Vignette 14 and Vignette 15. In so doing, their reasoning in geometry is enriched. As learners fold back, opportunities are created for them to justify their approaches and ideas in resolving geometry learning activities. Learners' ability to justify their decision making in resolving geometry plays a significant role in their development of geometry knowledge. Carpenter and Lehrer (1992) make a similar point when they state that the ability to articulate one's idea is a benchmark of understanding. This is what Jones (2004) also indicates, that geometric reasoning is important in developing learners' knowledge and understanding of geometrical properties. Thus, folding back does not only assist learners in developing questioning skills, but also in advancing their reasoning skills.

5.1.1.4. LEARNERS USE MNEMONIC TO IDENTIFY ANGLES

This study indicates that at property noticing, learners use mnemonic to identify the shape of a geometric figure in order to justify their reasoning. Thus, learners use mnemonic to understand different types of angles. For example, it was observed in the previous chapter that when learners' recognize mnemonic alphabets such as F on a geometry figure, they associate it with corresponding angles. They further use it to justify why angles associated with such figure are equal (Vignette 1, line 1.11; Vignette 4, line 4.2 & line 4.4; Vignette 7, line 7.2 & line 7.8). Chambers and Timlin (2013) point out that this helps learners to strengthen the hierarchy of ideas they encounter in their geometric tasks and develop a chain of reasoning building up their constructs using objects. Moreover, Valcarce et al. (2012) indicates that in carrying out these actions learners notice that the objects they had manipulated had certain properties. Similarly, Martin and Towers (2014) point that learners at property noticing are able to reflect on the object and recognize attributes and features of such object. At property noticing,; the properties of the constructed object are identified (Duzenli-Gokalp & Sharma, 2010). Likewise, this study shows that at property noticing learners are able to use shapes of alphabets to identify properties and relationships of geometric shapes in solving geometry learning activities that require them to reason.

5.1.2. HOW DOES A GRADE 10 MATHEMATICS TEACHER USE FOLDING BACK TO IMPROVE STUDENT REASONING IN GEOMETRY?

The findings of this study reveal that the teacher's instructional decisions affect learners' growth of geometry understanding. The nature of the teacher's interventional decisions both explicit intentional and unintentional interventions cause learners to fold back. In so doing, learners' geometric knowledge is developed; hence, their geometric reasoning is enriched. The teacher's various instructional decisions such as questioning, modelling, leading amongst others

(Vignette 13, Vignette 14, Vignette 15, and Vignette 20) appeared to have caused the learners to fold back. This agrees with the findings of Towers (1998), and Towers and Proulx (2013) which show that teachers' interventions enhance learners' learning. Similarly, Wu (2014) found that teacher's interventions provide learners with opportunities for them to independently construct or modify their mathematical understanding.

5.1.2.1. WHEN A TEACHER FOLLOWS LEARNERS' REASONING PATH, HE IS ABLE TO IMPLEMENT EFFECTIVE INSTRUCTIONAL DECISIONS

The findings of the study show that when a teacher follows learners' reasoning path, he is able to implement effective instructional decisions. Following the reasoning path that learners take and asking them questions enables a teacher to gather more information about the work learners are doing and leads to effective folding back (Vignette 16, line 16.4; Vignette 16 and Vignette 20). This result concurs with the study conducted by Wright (2014). Wright's (2014) study concludes that folding back becomes effective teaching strategy if aligned to the noticing of the layers of understanding learners are actually operating at. For example, when a teacher follows learners' reasoning path, he can be able to intervene through leading. Leading as a form of explicit intentional intervention enable learners to fold back and enhance their geometry knowledge. This form of intervention in facilitation is underscored by Towers (1998) in saying that it involves learners in explanation through questioning. Likewise, teacher's intervention by leading through asking questions, learners are able think differently through folding back across various layers of their mathematical understanding. This seems to show that a teacher's instructional decision plays a significant role in invoking learners' thought process.

5.1.2.2. TEACHERS' EXPLICIT INTENTIONAL INTERVENTION INFLUENCES LEARNERS' RECONSTRUCTION OF GEOMETRY KNOWLEDGE

This study found that teacher' explicit intentional intervention as instructional decision does not only offer a teacher to be aware of learners' geometry knowledge, but becomes a powerful tool upon which learners can re-construct geometry knowledge through folding back. For example, questioning as a form of teacher's explicit intentional intervention (Vignette 16 & Vignette 18) enables learners to fold back and extend their prior knowledge of geometry. Likewise, through leading as a form of the teacher's explicit intentional intervention, learners were able to reconstruct and extend their knowledge of proving some geometry facts on some geometric figures (vignette 16). This result agrees with various studies which show that when learners fold back, they extend their existing mathematical understanding, which have proved to be inadequate for solving a newly encountered problem (Pirie & Kieren, 1994; Martin, 2008; Lawan, 2011; Valcarce et al., 2012).

Furthermore, asking learners frequent questions (leading) as an instructional decision plays a significant role in extracting learners' misconceptions in geometric reasoning. For example, it is through questioning as an instructional decision that this study reveals that learners possess misconceptions concerning the multifaceted nature of angle conception (vignette 13 line 13.6). The instructional decision in this case, assisted learners to fill the gap in multifaceted nature of angle conception. Bansilal (2015) affirms that teachers often have to identify blockages in learners' conceptual development and help the learners resolve gaps in knowledge. As a result of questioning, learners are able to extend their existing knowledge of multifaceted angles through folding back. This is what the literature indicates, that a change in understanding a concept can come from folding back mechanism (Pirie & Kieren, 1994, Martin, 2008; Valcarce et al., 2012). Similarly, Martin et al. (2014) asserts that through folding back learners can

build a connected understanding through returning to earlier existing images of a concept. Likewise, this study shows that questioning enables them fold back and the ability to work on their geometry misconceptions.

Moreover, this study shows that clue-giving as a form of teacher explicit intentional stimulates learners to fold back to specific geometry concepts. For example, "Let's go for $D\hat{C}A$" (Vignette 13, line 13.1), "Can we draw ΔCBE?" (Vignette 14, line 14.1) and "...we can work with midpoint theorem, the one you are talking about...?" Vignette 20 (line 20.1) shows learners a preferred route. Showing learners a preferred way when resolving geometry learning activity enriches their reasoning in geometry. As a result, learners were observed to fold back to image making layer to work on their knowledge of various geometry concepts. In so doing, they were able to work with their knowledge of alternating angles and their reasoning in geometry was naturally enriched as they were able to determine various $\angle^s$ (Vignette 13, line 13.9, line 13.11 & line 13.12; Vignette 14, line 14.1 ,line 14.3). This is what Towers and Proulx (2013) indicate that clue-giving technique enables learners to look at the issue from their teacher's view and helps them make more sense of the mathematical concepts. Similarly, Martin (2008) asserts that learners need to be reminded of a particular approach in order to allow them in the building of mathematical concepts. This intervention purposefully directs learners to a line of thinking (Towers, 1998; Towers & Proulx, 2013).

5.1.2.3. Teacher's unintentional interventions stimulate learners to fold back and thicken their existing geometry concepts

The findings of the study indicate that teachers' unintentional interventions stimulate learners to fold back and thicken their understanding of the algorithm for cross multiplication and equivalent fractions (Vignette 6, line 6.12). This intervention helps learners to fold back in order to enrich their geometry

knowledge; in particular, their knowledge of cross multiplication and fractions in similarity learning activities. This form of intervention provides learners with opportunities to further engage in their   process and also re-working their existing understanding of geometry concepts. This in the literature shows the thickening effect of folding back (Pirie & Kieren, 1998; Martin, 2008; Martin et al., 2014). The intervention in this case did not prescribe to learners which knowledge is important in resolving the geometry learning activities. This concurs with Martin (2008) who shows that this intervention encourages learners to be self-aware of the limitations of their current understanding without prescribing a solution.

5.2. LIMITATIONS OF THE STUDY

The study has a number of limitations that should be taken into consideration when interpreting the results. Firstly, the main source of data collection was video recording. Although observations and written work (learning activities) were subset of video data, the use of a test at the end of all teaching experiments would have been comparatively more informative. Information from the test could have strengthened the data gathered.

Secondly, the sample involved few participants. As a result, the findings of this study may not be generalized to larger populations as it is the case with teaching experiment research design.

Lastly, the study was conducted in one school setting which might have influenced the findings of this study. Had this been conducted in two or more different schools, it would have provided a good picture of folding back in enriching Grade 10 learners' reasoning in geometry.

5.3. RECOMMENDATIONS

Notwithstanding its limitations, this study does suggest that creating a learning environment where folding back takes place and enriches learners reasoning in geometry. Therefore, in consideration of the findings of this study, the following recommendations are made:

- Mathematics teachers should follow learners' reasoning path in order to implement effective instructional decisions that will facilitate learners' growth of geometry knowledge.
- Mathematics teachers should create a learning environment where learners are given an opportunity to interact with each other during geometry problem solving; such is a powerful quest for folding back to occur. Hence, in a learning environment where folding back takes place, learners' reasoning in geometry is enriched.
- Further studies on the role of folding back in the teaching and learning of mathematics should be conducted.

5.4. EXPERIENCES IN UNDERTAKING THE STUDY

Qualitative researchers are according to Stiles (1993), often study concepts that are personally significant and thereby involve them in self-examination, significant personal learning and change. As a novice researcher, having conducted this study, I position myself within Stile's notion. This is particularly so because of many experiences gained when conducting this study. Of such many experiences, I ponder upon the following: my professional growth.

The experience of undertaking this study is certainly unique in my professional growth. As a result of this study, I have become aware of various issues that are significant for my professional growth. Firstly, this study exposed

me to what it means to do a research. Prior to undertaking the study, I had been exposed to the theoretical aspects of what research is. My first encounter with research and a research proposal was during my third year in MMAT 3 class. Briefly, as one of the assessments component for MMAT 3 we were tasked to write a research proposal. I would go to the computer lab and browse aspects of a research proposal. As expected, I found such aspects but I did not understand their meanings. Hence, after writing the title of my research proposal I went straight to write an abstract. Well, I did not find anything wrong with this until our lecturer took us through the process of what a research proposal was about. In my final year MMAT 4 module, the task was now to report on what we proposed. Here, my understanding of the research report was just to write about what I did in carrying out the research proposal.

Secondly, in my Honours modules, issues of research where highly given major attention. Being grounded on what a research is theoretically, I did not find it difficult carrying out an exam equivalent research project for a research project module and other modules as well. Thus, at Honours level my understanding was that there is a need to move from a research as a theoretical explanation to research as an experienced process. These modules from both my undergraduate and Honours classes were in a way equipping me with necessary skills to undertake a detailed research. As a result, I undertook this study.

Thirdly, in undertaking the current study, the process was totally different from what I had been exposed to. For example, after registering as a Master's student I had to prepare for a research proposal with the assistance of my supervisor. The prepared research proposal was defended at a departmental level and then to the school level. Most questions that were asked during my title defence revolved around the issues of research methodology. This was particularly because of the proposed research design. As I have mentioned earlier in Chapter 3, the adopted teaching experiment methodology as a research design

is not discussed in most traditional qualitative research books. As a result, various research designs such phenomenological study were even suggested for my study. However, what gave me the courage was the fact that my supervisor understood from the onset what I wanted to do. Hence, teaching experiment methodology as a research design was appropriate for my research questions.

Fourthly, having gone through the defence stage, the next task was to collect the data (this is captured in Chapter 3) that would respond to the research questions. Data collection was not as difficult as I thought it would be. However, it was a challenging yet great experience. This was particularly so because I had to do two things at the same time: That of being a teacher and researcher in the learning environment. Firstly, as a teacher I have become aware of the importance of creating a learning environment where folding back takes place. This helped me to implement effective instructional decisions that responded to the learners' learning of geometry. Subsequently, such instructional decisions appeared to be a quest for creative a learning environment where folding back takes place. Moreover, I have become aware of the importance of learners' interaction in a learning environment. Such interactions are a powerful source through which learners could fold back and re-work on their existing geometric understanding. Lastly, as a researcher I became aware of the importance of putting a theory into practice. This awareness came as a result of applying the Pirie and Kieren theory to explain the process of folding back as observed in the study. Thus, through applying the theory, I was able to understand the role of folding back in enriching Grade 10 learners' reasoning in geometry. This particularly stood out during the process of data analysis as captured in chapter 4.

Lastly, writing a research methodology chapter was a profound experience. As I was writing this section two key issues came out: *what* and *why*. Firstly, *what* focused on steps taken to respond to the research questions. Here, the challenges were minimal as I could explain what I did. To be specific, what I

did was just to respond to the components of research methodology. This seemed to be not what was expected from a Master's student; hence, my supervisor would often tell me that I should not write as if I am an Honours student. As a result, I had to rewrite this section over and over again. I should mention that this process was frustrating and painful yet rewarding. Secondly, things changed in one of the consultations with my supervisor, she asked me a key question: *why* did you choose the teaching experiment methodology against other research designs? This for me meant that there are other research designs that could be relevant to the purpose and research questions of the study. This took me back to my title defence presentation and I have now become aware of the importance of why those questions were asked. As a result, I have become aware of the importance of defending my choice of research design over other designs. Moreover, I have also become aware that it is not just about steps being taken but why they were taken. Hence, this awareness led me to understand the importance of a research design as the overall plan to conduct a study. This overall plan joins all the components of a research together. It is actually through this chapter that I have become aware of what it means to undertake a research. This was in a way preparing me for future research studies.

5.5. CHAPTER SUMMARY

In this chapter, I presented the overview of the study findings as per the research question which guided the study. The findings show that in a learning environment where folding back takes place, learners' reasoning in geometry is enriched. Limitations for the study were also addressed. In addition, recommendations for classroom teaching practice and future research has been provided. Lastly, personal experiences in undertaking this study have been captured.

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Appendixes

APPENDIX A: LEARNING ACTIVITIES

Learning Activity 1

.1 Find the size of angles a to g (in that order), giving reasons.

An expected response from learners is presented below.

$\angle a = 60° \dots vertically\ opp. \angle^s =$

$\angle b = 35° \dots alter. \angle^s =$

$\angle c = 35° \dots base\ angles\ of\ an\ iscosceles\ triangle\ are\ equal$

$\angle d + 60° + 35° = 180° \dots sum\ of\ inter. \angle^s\ of\ a\ \Delta\ is\ supplimentary$

$\therefore \qquad \angle d = 85°$

$\angle a = e + 35° \dots ext. \angle^s of\ a\ \Delta$

$\therefore \qquad \angle e = 25°$

$\angle f = 180° - 25° - 85° \dots sum\ of\ \angle^s\ on\ a\ straight\ line$

$\angle g = \angle e = 25° \dots corresp. \angle^s =$

Learning Activity 2
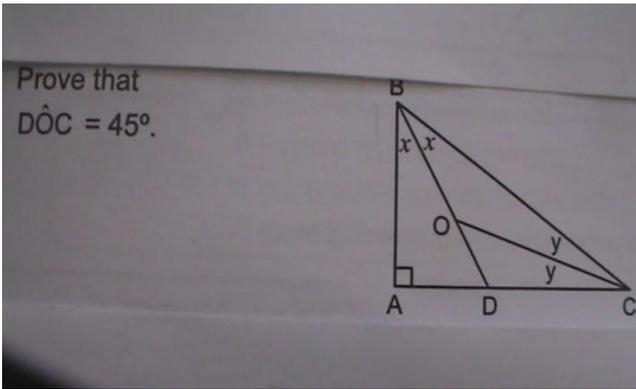
Prove that
DÔC = 45°.

An expected response from learners is presented as follows:

$DOC = x + y \dots ext. \angle \ of \ \Delta$

$2x + 2y = 90° \dots \angle \ sum \ of \ rt. \angle^d \Delta ABC$

$\therefore x + y = 45°$

$\therefore D\hat{O}C = 45°$

Learning Activity 3: Prove that $\Delta PST \ /// \ \Delta PQR$
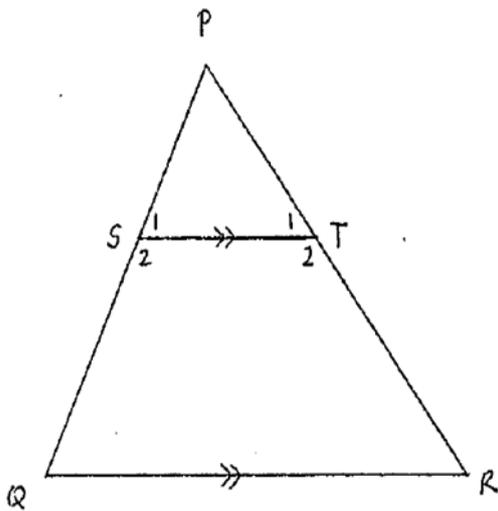
An expected response from learners is presented below.

In $\triangle PST$ and $\triangle PQR$

$\angle P = \angle P \ldots Common$

$\angle S_1 = \angle Q \ldots Corresponding\ \angle^s =; ST \parallel QR$

$\angle T_1 = \angle R \ldots Corresponding\ \angle^s =; ST \parallel QR$

$\therefore \triangle PST \; /// \; \triangle PQR \ldots AAA$

Learning Activity 4: Find the magnitude of a and b
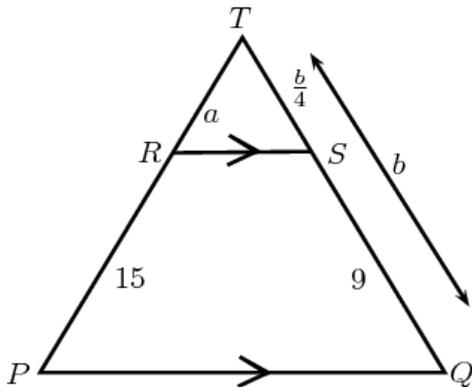
An expected response from learners is presented below.

$$\frac{TR}{TP} = \frac{TS}{TQ}$$

$$\frac{a}{15 + a} = \frac{\frac{b}{4}}{b}$$

$$\frac{a}{15 + a} = \frac{b}{4} \div \frac{b}{1}$$

$$\frac{a}{15 + a} = \frac{b}{4} \times \frac{1}{b}$$

$$4a = 15 + a$$

$$a = 5$$

$$b = \frac{b}{4} + 9$$

$$4b = b + 36$$

$$b = 12$$

$\therefore a = 5 \; and \; b = 12$

Learning Activity 5

In the following figure, M, N and T are the midpoints of AB, BC and AC is ΔMNT. $\angle A = 60°$ and $\angle B = 60°$. Calculate the interior $\angle^s$ of ΔMNT.

An expected response from learners is presented below.

$\angle C = 180° - (80° + 60°) = \cdots sum \; of \; \Delta$

$\therefore \quad \angle C = 40°$

In ΔABC: $M\&T$ are midpoints of AB and AC ... $MT // BC$ ... midpoint thm

$\therefore \; AMT = 80° \ldots Corresp. \angle^s =$

Similarly, $M\&N$ are midpoints of AB and BC

$\therefore \; B\widehat{M}N = 60° \ldots Corresp. \angle^s =, MN // AC$

$\quad T\widehat{M}N = 180° - (80° + 60°) \ldots \angle^s \; on \; a \; str. line$

$\therefore T\widehat{M}N = 40°$

The same method can be followed to determine the other two angles of ΔMNT

Answer: $40°; \; 80°; \; 60°$

Learning Activity 6

In the accompanying figure, $ABCD$ is a $\|^m$. $AB = BE$. The diagonal AC is produced to E, such that $AD = CE$. If $C\hat{E}B$ is equal to x. Prove, giving reasons that angle $F\hat{D}C = 3x$

An expected response from learners is presented below.

$$E\hat{A}B = x \dots. AB = BE$$
$$\therefore\ D\hat{C}A = x \dots alt.\angle^s; DC \parallel AB\ in\ \parallel^m$$
$$C\hat{B}E = x \dots CB = CE$$
$$\therefore\ A\hat{C}B = 2x \dots ext.\angle\ of\ \Delta$$
$$\therefore\ D\hat{A}C = 2x \dots alt.\angle^s; AD \parallel BC\ in\ \parallel^m$$
$$F\hat{D}C = D\hat{A}C + D\hat{C}A \dots ext.\angle\ of\ \Delta$$
$$= 2x + x$$
$$= 3x$$
$$or: F\hat{D}C = F\hat{A}B = 2x + x = 3x \dots corr.\angle^s; AB \parallel DC$$

Learning Activity 7

Prove that ABCD is a parallelogram.

An expected response from learners is presented below.

$$AEC + FCE = 115° + 65° = 180° \therefore AE(D) \parallel (B)FC \ldots co-interior \angle^s supplimentary .$$

$$\angle B = 65° \ldots FA = FB$$

$$CED = 65° \ldots str. line\ AED$$

$$\therefore \quad \angle D = 65° \ldots CE = CD$$

$$\therefore ECD = 50° \ldots sum\ of\ int.\angle^s of\ \Delta$$

$$\therefore \angle B + \angle C = 65° + 115° = 180°$$

$$\therefore AB \parallel DC \ldots co-interior \angle^s supplimentary$$

$$\therefore ABCD\ is\ a\ \parallel^m \ldots 2\ pairs\ opp.sides\ \parallel$$

Learning Activity 8

In the accompanying figure, $ABCD$ is a square. The diagonals AC and BD intersect at O. M is the midpoint of BD and $AM = ME$.
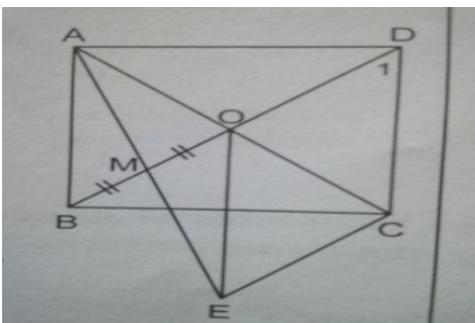
Prove that $MD \parallel EC$

An expected response from learners is presented below.

In ΔAEC: O midpoint of AC … Diagnal bisect one another & M midpoint of AE … given

$MO(D)//EC$ … mipoint theorem

APPENDIX B: APPROVAL FROM DEPARTMENT OF EDUCATION

LIMPOPO
PROVINCIAL GOVERNMENT
REPUBLIC OF SOUTH AFRICA

DEPARTMENT OF
EDUCATION

Enquiries: MC Makola PhD, Tel No: 015 290 9448 . E-mail: MakolaMC@edu.limpopo.gov.za

BOX 231
GA-SETATI
0720

MABOTJA

RE: Request for permission to Conduct Research

1. The above bears reference.
2. The Department wishes to inform you that your request to conduct research has been approved. Topic of the research proposal: **"AN EXPLORATION OF FOLDING BACK IN IMPROVING GRADE 10 STUDENTS REASONIND IN GEOMETRY"** .
3. The following conditions should be considered:

   3.1   The research should not have any financial implications for Limpopo Department of Education.

   3.2   Arrangements should be made with the Circuit Office and the schools concerned.

   3.3   The conduct of research should not anyhow disrupt the academic programs at the schools.

   3.4   The research should not be conducted during the time of Examinations especially the fourth term.

   3.5   During the study, applicable research ethics should be adhered to; in particular the principle of voluntary participation (the people involved should be respected).

   3.6   Upon completion of research study, the researcher shall share the final product of the research with the Department.

Cnr. 113 Biccard & 24 Excelsior Street, POLOKWANE, 0700, Private Bag X9489, POLOKWANE, 0700
Tel: 015 290 7600, Fax: 015 297 6920/4220/4494

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4. Furthermore, you are expected to produce this letter at Schools/ Offices where you intend conducting your research as an evidence that you are permitted to conduct the research.

5. The department appreciates the contribution that you wish to make and wishes you success in your investigation.

Best wishes.

14/04/2015

Mashaba KM   Date

Acting Head of Department.

APPENDIX C: SAMPLED VIGNETTES

Vignette 1

1.14.	Lebogang:	We are looking for angle B; I think it's corresponding to 35...
1.15.	Lesiba:	Show us why it is corresponding.
1.16.	Lebogang:	Okay [starts drawing].
1.17.	Thembi:	Isn't $35°$ not for $angle\ C$.
1.18.	Lesiba:	Wait let's give Lebogang a chance to do it.
1.19.	Lebogang:	[Continues drawing]...it is not correct.
1.20.	Thembi:	actually $angle\ C$ is it not equal to $35°$?
1.21.	Lesiba:	$Angle\ C$ is inside isosceles triangle.
1.22.	Thembi:	No, I was just asking.
1.23.	Thembi:	$35°$ is corresponding to $angle\ C$.
1.24.	Lesiba:	…,we can take out Z [draws Z]. It is going to be like this, then b is equal to $35°$
1.25.	Lebogang:	Yes, that's it, angle B is alternating to 35)
1.26.	Thembi:	Ooh yah…I was not aware of it.

Vignette 2

2.15.	John:	Sir, please, help us to find $D\hat{O}C$.
2.16.	Teacher:	What will be the sum of $\angle B$?
2.17.	John:	$2x$…hmmmm $\angle B_1$ is equal to $\angle B_2$ and $\angle B_2$ is equal to x, which means each of $\angle B_1$ and $\angle B_2$ is x,do you understand?
2.18.	Koena:	eish….[shaking head-indicating that he doesn't understand]
2.19.	John:	it means that here it is $2x$ and $2y$...
2.20.	Koena:	oh I understand now, meaning $\angle B$ is having two angles...
2.21.	John:	Yes...the question is how are we going to calculate x and y
2.22.	Teacher:	What will be the sum of angles in that triangle?
2.23.	John;	$90 + 2x + 2y = 180$
2.24.	John;	$90 + 2x + 2y = 180 … 2x + 2y = 90$
2.25.	Koena:	So, do we find x first?
2.26.	John;	What if we divide by 2 ?
2.27.	Koena:	We divide $2x + 2y$ by 2…then we remain with $x + y = 45°$
2.28.	John:	then we find $x + y$ to be $45°$

Vignette 3

3.11.	John:	Where is the angle we are looking for?
3.12.	Koena:	It is this one...
3.13.	John:	[starts referring to the notes pamphlet]
3.14.	John:	Let us say we put it this way…draw it.
3.15.	Phuti:	How?
3.16.	John:	Looking at it the way it is...
3.17.	Phuti:	I'm going to make mistake.
3.18.	John:	[Starts drawing]
3.19.	Koena:	Ohoo $\angle O$ is an exterior angle.

| 3.20. | John: | Yes my friend…so $\angle O$ is equal to $B_2 + C_2$ (exterior…. ohooo….wooowwww! $\angle O$ is equal to $\angle B$, $\angle B$ is the same as, look at it …$x + y$; $\angle O = x + y$, $\angle O = 45°$. |

Vignette 4

4.8.	John:	Look...okay...you cannot just say PS. So what are the values of PS? But we cannot use it if we are not given values. If we can determine angles of these triangles, then we can prove that these triangles are similar. So, angle P is common, what about other angles? These are angles on a straight line, these are parallel lines. We need transversal; we use those things of corresponding angles.
4.9.	Koena:	Angle is equal to, as you see they form F shape.
4.10.	John:	F shape this way?
4.11.	Koena:	Yes.
4.12.	John:	Yes, it is correct.
4.13.	Koena:	If we can put it this way, can't we take it out?
4.14.	John:	Yes, it is correct, they are this way, and then $\angle R$ is equal to $\angle T_1$

Vigntette 5

5.8.	Lesiba:	Actually can't we have…let me ask…on TP can't we have $15 + a$?
5.9.	Lebogang:	TP?
5.10.	Lesiba:	$15 + a$ over…no a $over$ $15 + a$ is equal to
5.11.	Lebogang:	a $over$ 15 times a?[confused of what to write]
5.12.	Lesiba:	[Takes the book from Lebogang and begins to write]
5.13.	Lesiba:	therefore we say a $over$ $15 + a$ [writes $\frac{a}{15+a}$] is equal to, like here we say $9 + b$ over 4 [writes $\frac{\frac{b}{4}}{\frac{b}{4}+9}$]
5.14.	Lebogang:	The whole of side of TQ is equal to b

Vignette 6

6.1.	Lesiba:	TR over TP is equal to TS over TQ, TR is a over 15+ a is equal to b over 4 over 9 + b over 4
6.2.	Teacher:	Yes, that equation is correct.
6.3.	Lebogang:	Cross multiply.
6.4.	Lesiba:	Cross multiply…
6.5.	Teacher:	Okay before we cross multiply.
6.6.	Lesiba:	We can, we can say b over 4 ,this way [starts writing $\frac{b}{4} \div 9 + \frac{b}{4}$]
6.7.	Teacher:	Okay continue
6.8.	Lebogang:	Write b over 4, do we divide it?
6.9.	Lesiba:	Then we say times, then 4 goes on top…
6.10.	Lebogang:	I don't understand you…

6.11.	Lesiba:	You don't understand me, like, you see now is $\frac{b}{4} \div 9 + \frac{b}{4}$, then we say b over 4, times $9 + \frac{4}{b}$ [writes $\frac{b}{4} \times 9 + \frac{4}{b}$
6.12.	Lebogang:	Times 9? These ones you add them, $9 + \frac{b}{4}$ or you time them?
6.13.	Lesiba:	Hmmm, this [division] sign we change it to [multiplication] times.
6.14.	Lebogang:	Yes, write them, but let us replace $9 + \frac{b}{4}$ with b

Vignette 7

7.18.	Lesiba:	Is it not possible to use Midpoint theorem?
7.19.	Sipho:	Isn't this F? This is F…you see…
7.20.	Lebogang:	Then it is corresponding angles...
7.21.	Lesiba:	Then parallel lines
7.22.	Lebogang:	It means that here [pointing $\angle N$], we are going to represent it as $\angle N_1$, $\angle N_2$, here[pointing $\angle T$] $\angle T1$ and $\angle T2$).
7.23.	Lesiba:	But here they didn't give us $\angle N1$ and $\angle N_2$…) [Inaudible] $\angle$ of ΔMTN and then midpoint of AC is T… Midpoint MN and TR are midpoints of...
7.24.	Lebogang:	If we can say $\angle N1$ and $\angle N2$ we will understand that $\angle B$ is equal to $\angle N_3$, then they are corresponding.
7.25.	Sipho:	Ohoo...understand, these are the angles that *Lebogang* is talking about, this if F…
7.26.	Lesiba:	I hear you…
7.27.	Lebogang:	This means $\angle N_2$ is equal to 80°.
7.28.	Sipho:	Wait a minute!
7.29.	Lebogang:	Then here it's ΔTNC.
7.30.	Sipho:	Hmmm.
7.31.	Lesiba:	Hmmm.
7.32.	Sipho:	You are using a long way… Ohooo yes continue…
7.33.	Lebogang:	Then in ΔTNC, we are going to do just like the first part, we take $\angle N_2$ and add it with $\angle C$ then subtract 180° to get $\angle T_2$
7.34.	Sipho:	Hmmm, I see it

Vignette 8

8.1.	Lesiba:	We are supposed to find $\angle M, \angle N \ and \ \angle T$
8.2.	Lebogang:	Wait, listen, $\angle N_2$ is alternating to $\angle T_1$
8.3.	Sipho:	Show [pause]. Yes, carry on *Lebogang*
8.4.	Lebogang:	[starts writing]
8.5.	Sipho:	So $\angle T_1$ is 80°
8.6.	Lesiba:	Alternating angles are equal...

8.7.	Lebogang:	Alternating angles are equal. It means that $\angle T_1$ is 80°

Vignette 9

9.19.	Lebogang:	Then it means $\angle N_1$ is alternating to…
9.20.	Sipho:	It's going to be difficult when we move forward…
9.21.	Lebogang:	$\angle T_2$ is corresponding to…
9.22.	Sipho:	Lesiba do you have any idea…?
9.23.	Lesiba:	Did we find $\angle N_1$?
9.24.	Lebogang:	No we found $\angle T_1\ and\ \angle T2$…
9.25.	Lesiba:	Can't we use Midpoint theorem?
9.26.	Teacher:	What will be the size of $\angle M_3$?
9.27.	Sipho:	Ah!…it's going to be 80°
9.28.	Lesiba:	Do we have $\angle T_2$?
9.29.	Sipho:	This thing is upside down [rotates the worksheet]. We are looking for cannot be possible.
9.30.	Teacher:	okay let's look at $\angle M_3$
9.31.	Sipho:	yes, that's what I was suggesting, that's why I said, I have looked at this and then I saw that they form letter F but I wasn't sure about it.
9.32.	Teacher:	Okay
9.33.	Sipho:	Then if they form letter F, they are corresponding angles of which M3 will be equal to 80 degrees.
9.34.	Lebogang:	we are going to work with ΔBMT…
9.35.	Sipho:	Yes…
9.36.	Lebogang:	$\angle B$ plus $\angle M$ plus $\angle T$, must give us 180°

Vignette 10

10.10.	John:	Line EC is equal to line AD, then C is a midpoint and D is also a midpoint
10.11.	Koena:	Midpoint…
10.12.	Teacher:	Midpoint of?
10.13.	John:	D is the midpoint of AF
10.14.	Teacher:	Midpoint of?
10.15.	Koena:	D is the midpoint of AF
10.16.	John:	C is the midpoint of AE
10.17.	Teacher:	Is DF equal to DA?
10.18.	John:	Hmmm no…

Vignette 11

11.8.	John:	Look it's a parallelogram. Look, this one is not a parallelogram, it is a triangle…Look at this triangle can you see it?
11.9.	Koena:	Yes.
11.10.	John:	Being it a triangle, $\angle E$ is equal to $\angle A$ because is an iscosceles triangle.
11.11.	John:	It must be $3x$
11.12.	Koena:	It must be?

| 11.13. | John: | $3x$ [pause] Then what is the value of x? |
| 11.14. | Koena: | It is not given, but they have given us $\angle E$ which is x and then $\angle A$ is x |

Vignette 12

12.9.	John:	But they said this is a parallelogram, we are told that this is a parallelogram…
12.10.	Koena:	Then sides AD and BC are parallel…
12.11.	John:	Let us think clearly…
12.12.	Koena:	It can be parallel lines…
12.13.	John:	Yes…
12.14.	Koena:	$\angle E$ is equal to $\angle A$…
12.15.	Koena:	$\angle D$ is equal to the sum of $\angle A$…they are corresponding angles
12.16.	John:	Ohooo…yes you are right.

Vignette 13

13.13.	Teacher:	Let's go for $D\hat{C}A$
13.14.	Koena:	Is x....
13.15.	John:	Here is x, angle is equal to angle
13.16.	Koena:	Is x
13.17.	Teacher:	Why are you saying they are equal?
13.18.	John:	Because is a parallelogram and opposite angles of a parallelogram are equal
13.19.	Teacher:	This angle [Pointing to $D\hat{C}A$] here, not the whole of $\angle D$
13.20.	Koena:	$\angle E$ equal …alternating angles
13.21.	John:	$E\hat{A}B$ is equal to $D\hat{C}A$
13.22.	Teacher:	The reason being?
13.23.	Koena:	Alternating angles...
13.24.	John:	[Adds] are equal

Vignette 14

14.16.	Teacher:	Can we draw ΔCBE.
14.17.	John:	[Begins drawing].
14.18.	Teacher:	Let us look at $ABCD$… which side is equal to DA?
14.19.	John:	Line BC.
14.20.	Koena:	No, it is line CE.
14.21.	John:	Look its CE but even CB is also equal to DA.
14.22.	Koena:	Yes
14.23.	Teacher:	Why do you think it will be equal?
14.24.	John:	Because it is a parallelogram.
14.25.	Koena:	Opposite side of a parallelogram
14.26.	John and Koena:	$C\hat{B}E$ and $B\hat{E}C$ are equal![smile]
14.27.	John:	[Smiling] look at this!
14.28.	Koena:	$C\hat{B}E$ is x.

| 14.29. | John: | Let us just say $\angle B_2$ is equal to $\angle E$. |
| 14.30. | Koena: | Base angles of an isosceles triangle are equal. |

Vignette 15

15.30.	Teacher:	Let us go back to that ΔCBE...Can you produce $to\ A$?
15.31.	John:	Here?
15.32.	Teacher:	So that we have $straight\ line\ EAC$.
15.33.	Koena:	Ohooo [starts drawing]
15.34.	Teacher:	What will be that angle?
15.35.	John:	[smiling] x
15.36.	Koena:	$A\hat{C}B$ is equal to the sum of $C\hat{B}E$ and $C\hat{E}B$.
15.37.	John:	Why are you saying that?
15.38.	Koena:	Exterior angle is equal to the...
15.39.	John:	[Interrupting] sum of the opposite angles.
15.40.	Koena:	Yes.
15.41.	John:	It is not equal to, it is the sum.
15.42.	Koena:	Yes, this one is equal to the sum of these once.
15.43.	John:	Let me show you something.
15.44.	Teacher:	Okay let's go to what *Koena* was saying
15.45.	Koena:	$\angle C$ is equal to the sum.
15.46.	John:	[Interrupting] the sum
15.47.	Koena:	Yes, the sum.
15.48.	Teacher:	So, do we know those angles?
15.49.	Koena:	It is x
15.50.	John:	Is x n x, is going to be $2x$, which means $\angle C$ is equal $2x$?
15.51.	Koena:	Yes.
15.52.	John:	$2x$ only?
15.53.	Koena:	Yes.
15.54.	John:	$2x$ only?
15.55.	Koena:	Yes.
15.56.	John:	$2x$ only?
15.57.	Koena:	Serious.
15.58.	John:	Ohooo…I see it now , the sum.

Vignette 16

16.7.	Lebogang:	$\angle A$ is equal to $\angle C$, opposite angle of a parallelogram are equa.l
16.8.	Teacher:	Which parallelogram are you looking at?
16.9.	Lebogang:	$ABCD$.
16.10.	Teacher:	And then what are you supposed to do?
16.11.	Sipho:	Prove that it is a parallelogram [image having]

| 16.12. | Teacher: | So, are saying $\angle A$ is equal to $\angle C$? Or are you saying if you can find $\angle A$ being equal to $\angle C$, then you can conclude that $ABCD$ is a $\parallel^{m}$? |

Vignette 17

17.8.	Lebogang:	Is ΔABF an Isosceles triangle?
17.9.	Lesiba:	Yes.
17.10.	Lebogang:	Then it means... we can say $\angle B$ is equal to $65°$ ... base angles of iscosceles triangle are equal
17.11.	Sipho:	Where is an Isosceles triangle here?
17.12.	Lebogang:	Is not this ΔABF an Isosceles?
17.13.	Lesiba:	Two sides are equal
17.14.	Teacher:	Yes, that will be true.

Vignette 18

18.4.	Sipho:	Let us find all the angles, in that way we can see if it's a parallelogram...
18.5.	Teacher:	You thinking of finding all the angles?
18.6.	Sipho:	Yes...and in that way we will see whether our opposite sides are equal or not, and if they are equal and then this means it is a parallelogram and if they are parallel.

Vignette 19

19.9.	Lesiba:	Can't it be a midpoint theorem? Can't it be? How do we see it?
19.10.	Sipho:	MD parallel to EC?
19.11.	Lesiba:	MD they say its third [pause], what? A third is equal to erh, what do they say?
19.12.	Sipho:	The third side of a triangle, ae, eish I forgot that...
19.13.	Lebogang:	Line segment
19.14.	Lesiba:	Can't ME be equal to EC? ME is equal to EC, midpoint theorem.
19.15.	Sipho:	Midpoint theorem. I don't know.
19.16.	Lesiba:	Can't it be a midpoint, how do we see it?
Vignette 20		
20.16.	Teacher:	But we can work with midpoint theorem, the one you are talking about. If we look at A, is it A?
20.17.	Sipho:	Yes Sir, oh yes, yeah yeah, I can see that
20.18.	Teacher:	Then continue.
20.19.	Sipho:	What is the midpoint of AC?
20.20.	Lebogang;	Midpoint of AC is O
20.21.	Sipho:	And then?
20.22.	Lebogang:	Then it means that here it is ΔABC.

20.23.	Lesiba:	[Interrupting] we have to know that EC is parallel to BD and
20.24.	Sipho:	[interrupting Lesiba] Yes...Yes, *Lebogang* is…I think eish…
20.25.	Lebogang:	ΔAME, it means that line MO is equal to line BC, do you understand?
20.26.	Sipho:	What will be the reason?
20.27.	Lesiba:	Midpoint theorem? What? What did you say?
20.28.	Sipho:	Are you saying MO is equal to EC?
20.29.	Lebogang:	Ke half!
20.30.	Sipho:	MO is half of EC and also parallel.