ON STRONGLY PARACOMPACT AND UNIFORMLY PARACOMPACT LOCALES

By

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Declaration

I declare that the dissertation hereby submitted to the University of Limpopo, for the degree of Master of Science in Mathematics has not previously been submitted by me for a degree at this or any other university; that it is my work in design and in execution, and that all material contained herein has been duly acknowledged.

Signature of Student: *Gwizo* **Date:** November 20, 2023

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Abstract

This study investigates the properties of strongly paracompact and uniformly paracompact spaces into locales. The research builds on the work of Rice [31], Frolik [15], and Borubaev [3], who expanded the concept of paracompact spaces, making it applicable to a wider range of topological situations and allowing for the use of more ideas. The main interest lies in exploring the strongly uniformly paracompact property. The dissertation looks into how recently introduced paracompactness concepts such as R-paracompactness Rice [31], can be adapted into locales. In the context of a uniform space (X, U), the study defines a space as uniformly R-paracompact if every open covering has an open, uniformly locally finite refinement.. The research aims to provide a thorough exploration of these concepts, contributing to a deeper understanding of their implications within the context of uniform spaces.

Key words: Paracompactness, Strongly paracompact locale, Uniformly paracompact space, Uniform space, Compact spaces, Strongly paracompact locale, R-paracompactness, locales, Uniformly locally finite refinement, Uniform covering.

Contents

Introduction

Historical background

Tukey [33] introduced a novel concept of uniform spaces in 1940, widely considered highly beneficial. Weil's original and renowned formulation of the uniform space [35] generalized the geometric properties of topological groups. Tukey made notable contributions, especially regarding the understanding of completely normal uniform spaces, correctly predicting their importance in mathematics. There exists a connection between full normality and paracompactness, and this concept has been extended to include a full normal frame.

As far as paracompactness goes, Dieudonné [12] introduced it in 1944 as a generalization of the topological property of compactness. He showed that his generalized property, paracompactness, could replace the statement a compact Hausdorff space is normal.

In 1959, Corson [7] introduced a uniform property into paracompactness by incorporating a completeness condition based on a weakly Cauchy filter.

Isbell made a contribution to the theory of uniform spaces, evident in his monograph [18]. He emphasized the importance of distinct terminology for the dual category of frames, calling the objects locales.

Dowker and Strauss [8] introduced the concept of sums in the category of frames **Frm** and frame homomorphisms, showing that any frame family has a sum in **Frm**. They established that sums in **Frm** preserve more properties than products do in the category **Top** of topological spaces and continuous functions. In the presence of a regularity condition, they found that any sum of paracompact frames is paracompact.

Rice in [31] introduced R-paracompact uniform spaces, shown to be complete, implying that spaces metrizable by a non-complete metric are not R-paracompact. Borubaev [3] extended paracompact spaces to B-paracompactness spaces with uniformity conditions. B-paracompact spaces possess desirable properties, such as including all metrizable spaces and all R-paracompact spaces, making the $class(B)$ larger than the $class(R)$. The class of F-paracompact spaces, $class(F)$, originated from Frolik [15] and is significantly larger than both $class(B)$ and $class(R)$.

In the point-free setting, Dube and Naidoo [10] investigated uniformly paracompact uniform frames, showing that these frames, like their spatial counterparts, possess a completeness property. They introduced uniformly para-Lindelöf frames, analogous to uniformly paracompact frames, and modeled their properties along those of uniformly paracompact frames.

Synopsis

This dissertation explores how strongly paracompact and uniformly paracompact spaces, can be understood in the context of locales. The dissertation is divided into four chapters.

Chapter 1 introduces the basic concepts of locales and frames, focusing on their structural properties like partially ordered sets and lattices. It then explores the idea of normality in frames, using examples to illustrate how these theoretical concepts are applied. The chapter also examines the topological properties of locales, particularly discussing how compact regular locales inherently exhibit normality. Throughout, the chapter extends topological theories specifically to locales, demonstrating these ideas with examples, thereby laying the groundwork for a exploration of paracompactness in locales.

Chapter 2 we focus on two properties of strongly paracompact spaces by Qu [29]. The property states that every regular Lindelöf space is a strongly paracompact. The subsequent property establishes that a countably paracompact normal space is strongly paracompact if and only if each increasing open cover of the space possesses a star-countable open refinement. We introduce strong paracompactness into locales and we introduce strong paracompactness into the theory of frames.

In Chapter 3, we explore various aspects of uniform paracompactness. We begin with Kanetov's [20] characterization and then examine uniform paracompactness in locales, following the views of Dube and Naidoo [10]. The concept of strong uniform paracompactness is introduced, showing its equivalence to uniform paracompactness in spaces that are strongly paracompact. This chapter also reveals that uniform paracompactness implies Cauchy completeness and Property P, and it is equivalent to strong Cauchy completeness. Finally, we define uniform countable paracompactness, which we will show to be stronger than countable paracompactness.

Chapter 4 explores the concept of strongly uniformly paracompact spaces, referencing the foundational studies by Zhanakunova [38] and Kanetov [20]. It aims to understand strong uniform paracompactness within uniform spaces by looking at how it relates to open coverings and their refinements. The chapter also studies how this property is maintained when using uniformly perfect mappings and looks at uniform star-finiteness under various conditions. Furthermore, it extends this examination to locales, aiming to define and study the properties of strongly uniformly paracompact locales, thereby broadening the understanding of these complex spaces.

Chapter 1

Some aspects of paracompactness in frames

In this section, we provide a concise introduction to essential background material on frames and locales. Our focus is on presenting definitions, results, and properties relevant to this dissertation. For the reader's convenience, a thorough exploration of the fundament- al concepts related to frames and locales can be obtained from the textbook "Frames and Locales: Topology without points" written by Picado and Pultr [27]. This book is a definitive resource in the field, covering various crucial notions and principles that are essential for understanding the complexities of frames and locales.

1.1 Preliminaries

Definition 1.1.1.

(a) *A partially ordered set* $(P, \leq)$ *is a non-empty set* P *with a partial order* $\leq$*. A meet semi-lattice* $(L, \leq, \wedge, 1)$ *is a partially ordered set* $(L, \leq)$ *in which any finite subset* $S \subseteq L$ *has an infimum, denoted by* $\bigwedge S$*. We have* 1*, called the top element of* L*. On the other hand, a join semi-lattice* $(L, \leq, \vee, 0)$ *is a partially ordered set* $(L, \leq)$ *in which any finite subset* $F \subseteq L$ *has a supremum, denoted by* $\bigvee F$*. We have* 0*, called the bottom element of* L*. Then a lattice* $(L, \leq, \vee, \wedge, 0, 1)$ *is a partially ordered pair* $(L, \leq)$

such that for every finite subset $S \subseteq L$, $\bigwedge S \in L$ and $\bigvee S \in L$. A sublattice of a lattice $(L, \leq, \vee, \wedge)$ is a subset $S \subseteq L$ such that $a \vee b \in S$ and $a \wedge b \in S$, for every $a, b \in S$.

(b) *A lattice $(L, \leq, \vee, \wedge)$ is said to be distributive if*

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \text{ for all } a, b, c \in L.$$

(c) *A subset S of a lattice $(L, \leq, \vee, \wedge)$ is a downset if for all $a \in L$ and $s \in S$, $a \leq s$ implies that $a \in S$. If for all $a \in L$ and $s \in S$, $a \geq s$ implies that $a \in S$, then S is an upset.*

(d) *A lattice $(L, \leq, \vee, \wedge)$ is said to be complete if for every subset $S \subseteq L$, both $\bigwedge S \in L$ and $\bigvee S \in L$.*

(e) *A frame L is a complete lattice in which the following property holds:*

$$a \wedge \left(\bigvee_{\gamma \in \Gamma} S_\gamma \right) = \bigvee_{\gamma \in \Gamma} (a \wedge s_\gamma)$$

for any subset $\{s_\gamma \mid \gamma \in \Gamma\}$ and for all $a \in L$.

Definition 1.1.2. *Let A and B be two frames. A function $f : A \to B$ is called a frame homomorphism if it satisfies the following conditions:*

(a) *Preservation of finite meets: For all finite subsets $S \subseteq A$ and for all $x, y \in A$, we have:*

$$f(x \wedge y) = f(x) \wedge f(y).$$

This captures the preservation of binary meets, which can be extended to finite meets. In particular $f(1) = 1$.

(b) *Preservation of arbitrary joins: For any subset $T \subseteq A$, we have:*

$$f\left(\bigvee T\right) = \bigvee_{t \in T} f(t).$$

This ensures that the join (supremum) of any subset in A is mapped to the join of its image in B. In particular $f(0) = 0$.

It is easy to show that the composition of frame homomorphisms is again a frame homomorphism. The category that arises from frames and frame homomorphisms is denoted by **Frm**. Its opposite category is called *Locale* and is denoted by **Loc**, thus, **Frm**^{op} $=$ **Loc** or **Loc**^{op} $=$ **Frm**.

Example 1.1.3. Example of the frame of the real line $\mathbb{R}$:
Let X be a topological space, and let $\mathcal{O}(X)$ be the poset of open sets of X, partially ordered by inclusion. Then, the frame of X, denoted by **Frm**(X), is the complete lattice given by:

(a) The top element is X and the bottom element is $\emptyset$.

(b) The finite meet (infimum) operation: the finite intersection of sets, denoted by $\bigcap$.

(c) The join (supremum) operation: the union of sets, denoted by $\bigcup$.

In the case of the real line $\mathbb{R}$ with the usual topology, the frame **Frm**$(\mathbb{R})$ is given by:

(a) The top element is $\mathbb{R}$ and the bottom element is $\emptyset$.

(b) The finite meet (infimum) operation: the finite intersection of intervals, denoted by $\bigcap$.

(c) The join (supremum) operation: the union of intervals, denoted by $\bigcup$.

In this example, if we have a collection of open intervals $\{(-1, 0), (0, 1), (2, 3)\}$, their join in Frame$(\mathbb{R})$ is the union of these intervals, which is $(-1, 1)\backslash\{0\} \cup (2, 3)$. Similarly, their meet in Frame$(\mathbb{R})$ is the intersection of these intervals, which is the empty set $\emptyset$, since there is no point that belongs to all three intervals. Note that the definition of Frame(X) captures the topological information of X in terms of its open sets and their relationships, without reference to specific points in X. This is the key idea behind point-free topology. □

Returning to **Top** and **Frm** $=$ **Loc**^{op}, we note that $\mathcal{O}$: **Top** $\to$ **Frm** is a contravariant functor. Given a singleton $\{*\}$, there is a unique topology associated with this set, namely

$\{\emptyset, \{*\}\} = \{0, 1\}$. We shall denote by $\mathfrak{2}$ the frame thus constructed. Our references for frames (= locales) are Isbell [17], Johnstone [19], and Picado and Pultr [27].

Definition 1.1.4. *A point of a frame L is a frame homomorphism $h : L \to \mathfrak{2}$. We shall denote by $\sum L$ the set of all points of L (all frame homomorphisms $h : L \to \mathfrak{2}$). It is known that; for $x \in L$, we set*

$$\textstyle\sum_x = \{h : L \to \mathfrak{2} \mid h(x) = 1\}.$$

Example 1.1.5. Then for all $x, y, x_i \in L$, it holds that

(a) $\sum_0 = \emptyset$ and $\sum_1 = \sum L$

(b) $\sum_{x\wedge y} = \sum_x \wedge \sum_y$

(c)) $\sum_{\bigvee x_i} = \bigcup \sum_{x_i}$

Proof.

(a) We have $\sum_0 = \{h \in \sum L \mid h(0) = 1\}$ by Definition 1.1.4. Assume that $0 \neq 1$. However, for any frame homomorphism $h : L \to \mathfrak{2}$, we have $h(0) = 0$, since 0 is the bottom element in the frame $\mathfrak{2}$ and is mapped to the bottom element of any frame under any frame homomorphism. Thus, there exists no h such that $h(0) = 1$. Therefore, $\sum_0 = \emptyset$.

We also have $\sum_1 = \{h \in \sum L \mid h(1) = 1\}$ by Definition 1.1.4. For any frame homomorphism $h : L \to \mathfrak{2}$, we have $h(1) = 1$, since 1 is the top element in the frame $\mathfrak{2}$ and is mapped to by the top element of any frame under any frame homomorphism. Thus, every $h \in \sum L$ satisfies $h(1) = 1$. Therefore, $\sum_1 = \sum_L$.

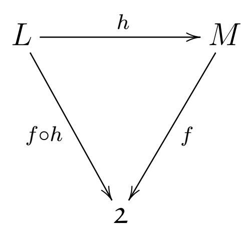
(b) Consider the set $\sum_{x\wedge y} = \{h \in \sum L \mid h(x \wedge y) = 1\}$. By the properties of homomorphisms, $h(x \wedge y) = h(x) \wedge h(y)$. If $h \in \sum_{x\wedge y}$, then $h(x \wedge y) = 1$, which implies both $h(x) = 1$ and $h(y) = 1$. Therefore, $h \in \sum_x$ and $h \in \sum_y$. This implies $\sum_{x\wedge y} \subseteq \sum_x \wedge \sum_y$. Conversely, if $h \in \sum_x \wedge \sum_y$, then both $h(x) = 1$ and $h(y) = 1$. This implies $h(x \wedge y) = 1$, so $h \in \sum_{x\wedge y}$. Thus, $\sum_x \wedge \sum_y \subseteq \sum_{x\wedge y}$. Combining the above, we have $\sum_{x\wedge y} = \sum_x \wedge \sum_y$.

(c) If $h \in \bigcup_{i\in I} \sum_{x_i}$, then there exists some $i_0 \in I$ such that $h \in \sum_{x_{i_0}}$, which means $h(x_{i_0}) = 1$. Also, $1 = h(x_{i_0}) \leq \bigvee_{i\in I} h(x_i) = h(\bigvee_{i\in I} x_i)$, implying that $h(\bigvee_{i\in I} x_i) = 1$ and $h \in \sum_{\bigvee_{i\in I} x_i}$. Hence, $\bigcup_{i\in I} \sum_{x_i} \subseteq \sum_{\bigvee_{i\in I} x_i}$. Conversely, assume $h \in \sum_{\bigvee_{i\in I} x_i}$ and that, contrary to our assumption, $h \notin \bigcup_{i\in I} \sum_{x_i}$. Then, $h : L \to 2$ is a frame homomorphism such that $h(\bigvee_{i\in I} x_i) = 1$. Also, for all $i \in I$, $h(x_i) \neq 1$, which implies $h(x_i) = 0$ for all $i \in I$, since h is a frame homomorphism from L to 2. This leads to a contradiction. Hence, $\sum_{\bigvee_{i\in I} x_i} \subseteq \bigcup_{i\in I} \sum_{x_i}$. Since both $\bigcup_{i\in I} \sum_{x_i} \subseteq \sum_{\bigvee_{i\in I} x_i}$ and $\sum_{\bigvee_{i\in I} x_i} \subseteq \bigcup_{i\in I} \sum_{x_i}$, we conclude that $\bigcup_{i\in I} \sum_{x_i} = \sum_{\bigvee_{i\in I} x_i}$. □

The above Example 1.1.5 shows that the collection $\{\sum_x \mid x \in L\}$ is a base for a topology on $\sum L$, making it a topological space called the *spectrum of* L. Given a frame homomorphism $h : L \to M$, let us define a map

$$\sum h : \sum M \to \sum L$$

by $(\sum h)(f) = f \circ h$ (as in the triangle below:)



Remember, $\sum M = \{f : M \to 2\}$ and $\sum L = \{h : L \to 2\}$. Since $\sum M$ and $\sum L$ are spaces, it makes sense to consider $(\sum h)^{-1} : \sum L \to \sum M$ and say for any $\sum_x \in \sum L$, we define

$$(\sum h)^{-1} \sum_x = \sum_{h(x)} = \{f : M \to 2 \mid f(h(x)) = (f \circ h)(x) = 1\}.$$

This definition is valid because

$$
\begin{aligned}
k \in \left(\sum h\right)^{-1}\left(\sum_x\right) &\iff \left(\sum h\right)(k) \in \sum_x \\
&\iff (k \circ h) \in \sum_x \\
&\iff (k \circ h)(x) = 1 \\
&\iff k \in \sum_{h(x)}
\end{aligned}
$$

This shows that under the definition of $\sum h$ provided above, its inverse is defined by $(\sum h)^{-1}$ so that

$$(\textstyle\sum h)^{-1}(\sum_x) = \sum_{k(x)}$$

and $\sum h$ is a continuous function from the topological space $\sum M$ to the topological space $\sum L$. Moreover, there exists a contravariant functor

$$\sum : \textbf{Frm} \rightarrow \textbf{Top}$$

between the categories **Frm** and **Top**. In fact, the functors $\mathcal{O}$ and $\sum$ are adjoint to each other. Here is a brief insight into the relationship between locales and the categories **Frm**: the category of locales, denoted **Loc** $=$ **Frm**^{op}, is the dual category of the category **Frm** of frames and frame homomorphisms. In other words, **Loc** is the category that encompasses all locales and continuous maps between them. The relationship between locales and the category **Frm** is that locales offer a more algebraic and precise approach to studying topological spaces, and the category of locales serves as a framework for exploring the connections between locales and other mathematical structures. For instance, the contravariant functor Ω : **Top** $\rightarrow$ **Frm**, when restricted to the category **Sob** of sober spaces, acts as a full embedding. This implies that it establishes a one-to-one correspondence between continuous maps between sober spaces and frame homomorphisms between their locales. This correspondence is a powerful tool for investigating the interplay between topology and algebra. Furthermore, every topological space can be viewed as a locale (with some loss of information if the space is not sober), and locales can be utilized to study mathematical concepts in a broader context, even in scenarios where the traditional theory of topology may not be applicable. Overall, the category of locales provides a rich framework for studying the algebraic and topological properties of mathematical structures, and locales offer a powerful tool for understanding the relationships between these structures. Locales are particularly useful for studying sheaf theory. Sheaves are a fundamental tool in mathematics, used to study a wide range of mathematical objects, including algebraic varieties, complex manifolds, and vector bundles. A sheaf can be defined on a topological space or a locale, but in the case of locales, the sheaf theory is particularly well-behaved. The category of locales is also valuable in constructive mathematics, which is a branch of mathematics that imposes restrictions on what constitutes a valid proof. Constructive mathematics is concerned with constructive methods of proof and computation, and in this context, locales provide a suitable framework for studying various mathematical concepts, including algebraic geometry and topology. Another important feature of locales is that they provide a way to study the concept of compactness in a more general setting. In the category of locales, compactness is defined in terms of the existence of certain colimits, rather than in terms of open covers as in the traditional theory of topology. This allows for a more comprehensive study of compactness, which can be applied to a wider range of mathematical structures. Locales provide a more algebraic and precise way to study topological spaces, and the category of locales enables the study of the relationships between locales and other mathematical structures. We recall that given maps $f : X \to Y$ and $g : Y \to X$ between ordered sets, we say that f is the *left adjoint* of g (and so g is the *right adjoint* of f) if they satisfy: for all $x \in X$ and for all $y \in Y$,

$$f(x) \leq y \text{ if and only if } x \leq g(y).$$

In this instance, the maps f and g are called a *Galois connection* between X and Y. In the case of **Frm**, given a frame homomorphism $f : L \to M$, its *right adjoint* exists and is denoted by $f_* : M \to L$ and is uniquely defined by

$$f_*(y) = \bigvee \{x \in L \mid f(x) \leq y\};$$

thus, we also have

$$f(x) \leq y \text{ if and only if } x \leq f_*(y).$$

Many of the concepts that we use in this dissertation arise from topological facts. We use the following example to show how the definitions we use in this treatise can be constructed.

The following is a motivating example.

Example 1.1.6. (For normality in frames). We recall that a space (X, τ) is normal if, whenever A, B are disjoint closed subsets of X, there exist disjoint open sets $U, V \in \tau$ such that $A \subset U$ and $B \subset V$. Now, we observe that $X - A, X - B \in \tau$ so that

$$\begin{aligned}(X-A) \cup (X-B) &= X - (A \cap B) \\ &= X - \emptyset \\ &= X\end{aligned}$$

and that

$$\begin{aligned}A \subseteq U &\implies (X-A) \cup U \supseteq X \\ &\implies (X-A) \cup U = X.\end{aligned}$$

Similarly, we can show that $(X-B) \cup V = X$. Given that $U \cap V = \emptyset$, the following definition 1.1.7 (c) is clear. □

Definition 1.1.7. *Let L be a frame.*

(a) *A pseudocomplement of an element $a \in L$ is the element $a^* \in L$ defined by*

$$a^* = \bigvee \{x \in L \mid x \prec a\}.$$

For any $X \subseteq L$, we denote by $X^* = \{x^* \mid x \in X\}$.

(b) *The rather below relation* $\prec$ *on* L *is defined by* $b \prec a$ *if and only if* $b^* \vee a = 1$*, for any* $a, b \in L$*. The frame* L *is said to be regular if for every* $a \in L$*, it holds that*

$$a = \bigvee\{x \in L \mid x \prec a\}.$$

(c) *A frame* L *is called normal if for* $a_1, a_2 \in L$ *such that* $a_1 \vee a_2 = 1$*, there are* b_1, b_2 *such that* $a_1 \vee b_1 = a_2 \vee b_2 = 1$ *and* $b_1 \wedge b_2 = 0$*.*

(d) *Let* a *and* b *be elements of* L*. We say that* a *is completely below* b *in* L *and write* $a \prec\!\prec b$ *if there are* $a_r \in L$*, where* r *is rational,* $0 \leq r \leq 1$*, such that* $a_0 = a$*,* $a_1 = b$*, and* $a_r \prec a_s$ *for* $r < s$*. Note that the relation* $\prec\!\prec$ *is the largest interpolative relation contained in* $\prec$*.*

(e) *A locale* L *is said to be completely regular if* $a = \bigvee\{x \mid x \prec\!\prec a\}$ *for every* $a \in L$*.*

(f) *A cover of a frame* L *is a subset* $A \subseteq L$ *such that* $\bigvee A = 1$*. A subcover of a cover* A *is a subset* $B \subseteq A$ *such that* $\bigvee B = 1$*.*

(g) *The frame* L *is said to be compact if, whenever* A *is a cover of* L*, there exists a finite subcover F of* L*.*

(h) *The frame* L *is said to be Lindelöf if, whenever* A *is a cover of* L*, there exists a countable subcover F of* L*.*

In Proposition 1.1.8, we show the relationship between compact regular locales and normality.

Proposition 1.1.8. *A compact regular locale is normal.*

Proof. Let L be a compact regular locale. We want to show that L is normal. Let $a \vee b = 1$, where $a \vee b \in L$ and $a, b \in L$. By definition of regularity it follows that,

$$a = \bigvee \{x \mid x \prec a\}$$

and

$$b = \bigvee \{y \mid y \prec b\}.$$

Hence,

$$\bigvee \{x \mid x \prec a\} \vee b = 1 = a \vee \bigvee \{y \mid y \prec b\}$$

and in addition to the compactness there are $x_1, x_2, \ldots \prec a$ and $y_1, y_2, \ldots \prec b$ satisfying

$$\bigvee_{i=1}^{\infty} x_i \vee b = 1 = a \vee \bigvee_{i=1}^{\infty} y_i.$$

Assume that $x_1 \leq x_2 \leq \cdots$ and $y_1 \leq y_2 \leq \cdots$. Let $u_i = x_i \wedge y_i^*$ and $v_i = x_i^* \wedge y_i$. We have

$$a \wedge v_i = a \wedge (x_i^* \wedge y_i) \wedge (a \vee x_i^*) \wedge (a \vee y_i) = a \vee y_i$$

and similarly for $b \vee u_i = b \vee x_i$. Accordingly, let

$$u = \bigvee_{i=1}^{\infty} (a \vee y_i) = a \vee \bigvee_{i=1}^{\infty} y_i = 1$$

and similarly for $b \vee u = 1$. Lastly,

$$u_i \wedge v_j = x_i \wedge y_i^* \wedge y_j \wedge x_i^* = 0.$$

For every i, j (if $i \leq j$, then $x_i \wedge x_j^* = 0$ and otherwise if $i \geq j$, then $y_j \wedge y_i^* = 0$). Hence

$$u \wedge v = \bigvee_{i,j=1}^{\infty} (u_i \wedge v_j) = 0.$$

Therefore, L is normal. □

Remark 1.1.9. We distinguish subframes and sublocales as follows: A *subframe* of a frame L is a subset of a frame which is closed under arbitrary joins and finite meets in that frame. On the other hand, a *sublocale* M of a locale L can be represented in terms of an onto frame homomorphism $h : L \to M$ in the sense that the image of M under the right adjoint $h_* : M \to L$ will represent that sublocale. Given a locale L, denote

$$\uparrow a = \{x \in L \mid x \geq a\} \text{ and } \downarrow b = \{x \in L \mid x \leq b\}.$$

Then the sublocale given by the frame homomorphism $j : L \to \uparrow a$ defined by $j(x) = a \vee x$, for any $a \in L$ is called a *closed sublocale* of L. For detailed reading concerning frames we refer to [27].

Some of the advantages of working with subframes and sublocales in pointfree topology are:

(a) Flexibility: If L is a locale and Y is a sublocale of L, we can study the properties of Y independently of the entire structure of L. This approach allows us to focus on specific aspects of L that interest us, without the need to consider its full complexity. This is particularly useful for isolating and examining certain topological or logical properties within a broader context.

(b) Composability: If L is a locale and Y and Z are sublocales of L, then the meet $Y \wedge Z$ is also a sublocale of L. Furthermore, if there are morphisms of locales (analogous to continuous maps in point-set topology) $f : L \to M$ and $g : M \to N$, their composition $g \circ f : L \to N$ is also a morphism of locales. This property demonstrates how sublocales and their relationships can be constructed and analyzed in a compositional manner, facilitating the study of complex structures through simpler, constituent parts.

(c) Robustness: For locales L and M, and a morphism of locales $f : L \to M$, if Y' is a sublocale of M, then the inverse image of Y' under f is a sublocale of L. Conversely, if X' is a sublocale of L, then the direct image of X' under f can be considered a sublocale of M (noting that direct image constructions in locale theory are more nuanced than in point-set topology). This property underscores the robustness of sublocales under morphisms, allowing for the transfer and preservation of structural properties between locales.

(d) Abstraction: Sublocales offer a high level of abstraction, facilitating the simplification and generalization of concepts in topology. For instance, sheaves can be defined over locales, allowing for a more abstract and general framework for their study. Similarly, the concept of a locale itself generalizes that of topological spaces to include spaces without points, such as function spaces or spaces defined purely by their topology. This level of abstraction is particularly beneficial in fields like algebraic geometry and theoretical computer science, where traditional point-based methods may be inadequate or cumbersome.

Lemma 1.1.10. *Quotients of compact spaces are compact.*

Proof. Let X/A be a quotient compact space X. We want to show that X/A is compact. Let $q : X \to X/A$ be a quotient map. If $\{U_\alpha\}$ is any cover of X/A then $\{q^{-1}(U_\alpha)\}$ is an open cover of X. Since X is compact, there exist a finite subcover $\{q^{-1}(U_{\alpha_i}) \mid i = 1, \ldots, n\}$ such that $\{U_{\alpha_i} \mid i = 1, \ldots, n\}$ is a finite subcover of X/A since $q(q^{-1}(U_\alpha)) = U_\alpha$. Therefore, X/A is compact. □

Proposition 1.1.11.

(a) *A subframe of a compact frame is compact.*

(b) *A closed sublocale of a compact locale is compact.*

Proof.

(a) Let L be a compact frame and K a subframe of L. Suppose $\mathcal{V} = \{V_i\}_{i\in I}$ constitutes an open cover of K within the frame. To construct an open cover $\mathcal{V}'$ of L that extends $\mathcal{V}$, we augment $\mathcal{V}$ with the complement of K in L, thereby defining $\mathcal{V}' = \mathcal{V} \cup \{L \setminus K\}$. This ensures $\mathcal{V}'$ covers L, leveraging the compactness of L to assert the existence of a finite subcover $\mathcal{F} = \{F_j\}_{j\in J} \subset \mathcal{V}'$ that adequately covers L, with J being a finite selection from I, possibly including the index for $L \setminus K$. Notably, the finite subcollection of $\mathcal{V}$ derived from $\mathcal{F}$, excluding $L \setminus K$ if it was included, serves as a finite subcover for K, hence establishing K's compactness. This mathematical discourse underscores that the compactness property of the frame L is inherited by its subframe K, predicated on the extendibility of an open cover $\mathcal{V}$ of K to an open cover $\mathcal{V}'$ of L.

(b) Follows from the fact that if A is cover of $\uparrow a$ then $\{a\} \bigcup A$ is a cover of L. □

1.2 Paracompact spaces and paracompact frames

Definition 1.2.1.

(a) *A subset X of a frame L is said to be locally finite (resp. discrete) witnessed by a cover W if for each $w \in W$ there are only finitely many (resp. there is at most one) $x \in X$ such that $x \wedge w \neq 0$.*

(b) *We say that a cover $\mathcal{U}$ is a refinement of a cover $\mathcal{V}$, and write $\mathcal{U} \leq \mathcal{V}$, if for each $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ such that $U \subseteq V$.*

(c) A frame L is said to be *paracompact* if it is regular and if each of its covers has a locally finite refinement.

Definition 1.2.2. *A Boolean algebra can be defined in terms of sets as an algebraic structure consisting of a set B equipped with two binary operations $\wedge$ (meet) and $\vee$ (join), a unary operation $\neg$ (complement), and two distinguished elements 0 (the bottom or zero element) and 1 (the top or unit element), satisfying the following axioms:*

(a) B *is a distributive lattice with complement.*

(b) 0 *and* 1 *are the bottom and top elements of* B*, respectively.*

(c) *For all* $a \in B$*,* $a \vee \neg a = 1$ *and* $a \wedge \neg a = 0$*.*

(d) *For all* $a, b \in B$*,* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$*.*

The importance of Proposition 1.2.3 in this study is the fact that in almost all variants of paracompactness encountered here are hereditary on closed subspaces or sublocales.

Proposition 1.2.3. *Every closed subspace of a paracompact space* $(X; \tau_X)$ *is paracompact.*

Proof. Suppose $A \subseteq X$ is closed and let $\{U_i\}_{i \in I}$ be an open cover of A. Then each U_i is of the form

$$U_i = A \cap V_i$$

for some open $V_i \in \tau_X$. If we add $\{X \setminus A\}$ to the collection $\{V_i\}_{i \in I}$, we find a cover of X, which by hypothesis has a locally finite refinement, say $\{V_j\}_{j \in J}$. It easily follows that is an open locally finite refinement of the cover $\{U_i\}_{i \in I}$ of A. $\square$

Regular frames that possess σ-locally finite refinements for their covers are paracompact in the sense of the following result. In the classical case, there are interesting properties of local finiteness by Bjork [2].

For the following proposition we will focus on a term called quasi-refinement which is defined as a system of covers whose joins are dense.

Proposition 1.2.4. [28] *If each cover of a regular frame* L *has a* σ*-locally finite refinement, then* L *is paracompact.*

Proof. We need only prove that each cover has a locally finite quasi-refinement. Suppose that U is a cover of L and let $V = \bigcup_{i+1}^{\infty} V_i$ be a refinement such that $V_1 \subseteq V_2 \subseteq \cdots$ is locally finite, and suppose that W_i finitizes V_i. Now, for $y \in V$, let us put $n(y) = \min\{i \mid y \in V_i\}$ and choose anti-reflexive well-ordering R on V such that $i(x) \leq i(y)$ implies that xRy. Putting $W_i = X_i \wedge V_i$, it follows from [28] that $\bigwedge W_i = \bigwedge V_i$ and then $W = \bigvee_i W_i$ is a cover. We set

$$\tilde{y} = y \wedge \neg \bigvee \{x \mid x\, R\, y\} \quad \text{and} \quad W = \{\tilde{y} \mid y \in V\}.$$

Suppose that $a \in L$ is non-zero. Let y be the first in R such that $y \wedge a \neq 0$. Then $a \wedge \tilde{y} = a \wedge y \neq 0$, so that W is a quasi-refinement of U. To complete the proof, we will show that Z finitizes W. Suppose then that $z \in V_i$. Then an $y \in V_i$ exists such that $z \leq y$. Therefore, if $i(x) \geq i$ then $z \leq \{u \mid u\, R\, x\}$ giving rise to $z \wedge \tilde{x} = 0$. Therefore, if $z \wedge \tilde{y} \neq 0$, then the y_i are in V_i. But $\tilde{y}_i \leq y_i$ and $z \leq z_1$ for some $z_1 \in X_i$, so the $\tilde{y}_i$ are only finitely many, whence L is paracompact. □

Definition 1.2.5. *Denote by $\mathbb{R}_+$ the set of all non-negative reals plus $+\infty$. A diameter on a frame L is a mapping $d : L \to \mathbb{R}_+$ such that*

(a) $d(0) = 0$,

(b) $a \leq b \implies d(a) \leq d(b)$,

(c) $a \wedge b \neq 0 \implies d(a \wedge b) \leq d(a) + d(b)$*, and*

(d) *for every $\epsilon > 0$, the set $U_\epsilon^d = \{a \mid d(a) < \epsilon\}$ is a cover of L.*

The diameter d on L is said to be *admissible* if the system of covers $U(d) = \{U_\epsilon^d \mid \epsilon > 0\}$ is admissible. The pair (L, d) is then referred to as a *metric frame*. For metric locales, we also refer to [32].

Definition 1.2.6. Let (X, τ) be a topological space. A point $x \in X$ is called a *cozero* point if there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq 0$ and $f^{-1}(\{0\})$ is a closed subset of X. In other words, x is a *cozero* point if there exists a continuous real-valued function f defined on X such that $f(x) \neq 0$ and $f^{-1}(0)$ is a closed set in X. The set of all cozero points in X is called the cozero set and is denoted by X_f. Thus,

$$x \in X_f \iff \exists f \in C(X, \mathbb{R}),\ f(x) \neq 0,\ \text{and } f^{-1}(\{0\}) \in \tau_{\text{closed}}.$$

Proposition 1.2.7. [34] *Every regular Lindelöf frame is paracompact.*

Proof. Let L be a regular Lindelöf frame, then Coz(L) is a regular σ-frame, and thus paracompact as a σ-frame [34], which generates L. Taking any cover of L, we can get a refinement consisting of only cozero elements. Using the Lindelöf property, we obtain a countable subcover, which is also a cover of Coz(L), and hence there exists a locally finite refinement. □

Remark 1.2.8. *A compact frame is not paracompact in general.* A counterexample is the one-point compactification of an uncountable discrete space, whose open sets lattice is a compact frame but not paracompact. To see this, let X be an uncountable discrete space and let p be a point not in X. Let $Y = X \cup \{p\}$, and let $\mathcal{V}$ be the collection of all open sets in X, together with all sets of the form $Y \setminus C$, where C is a countable subset of X. Then $\mathcal{V}$ is a topology on Y, and it is easy to check that Y is compact and Hausdorff with respect to this topology. Moreover, X is dense in Y.
It can be shown that $\mathcal{V}$ is not a normal topology, and hence Y is not a normal space. In particular, Y is not paracompact. However, since Y is compact and Hausdorff, it is a compact space. Since compactness is a conservative then the lattice of open sets of these space is a compact frame. □

Definition 1.2.9. A *Boolean locale* is a locale in which the lattice of open subsets is a Boolean algebra, i.e., a complemented distributive lattice with a least and greatest element. In other words, a Boolean locale is a locale in which any two open subsets have a disjoint union, and every open subset has a complement.

$RO(X)$ refers to the lattice of regular open sets in a topological space X [32].

Proposition 1.2.10. [32] *Every Boolean locale is paracompact.*

Proof. Let $RO(X)$ be a boolean locale. We want to show $RO(X)$ is a paracompact. Let

$$B = \{b_r \in RO(X) : r \in J\}$$

be a cover of $RO(X)$. We consider the poset

$$S = \{D \subseteq RO(X) : D \text{ refines } B \text{ and } D \text{ is a disjoint}\}.$$

By Zorn lemma, it follows that we have a maximal element V in S whose union is also dense in X. If $X \backslash \overline{UV} \neq \emptyset$, there exists an element U in B and $V \in RO(X)$ such that

$$\emptyset \neq V \subseteq U \bigcap (X \backslash \overline{UV})$$

since UB is dense in X which contradicts with the maximality of V. Thus we have shown that every cover of a boolean locale has a discrete refinement. Therefore, $RO(X)$ is a paracompact. $\square$

Remark 1.2.11. *In general, a metric locale is not paracompact.* In the study of locale theory, the examination of metric properties through the lens of abstract opens rather than point-centric spaces necessitates a reconsideration of classical topological constructs. This remark elucidates the limitations of directly translating the notion of paracompactness, a well-established property in metric space theory, into the abstract framework governed by locales. Consider a locale constructed to reflect the properties of an uncountable set under a discrete metric, wherein each element of the frame, L, symbolizes an isolated open set. This is analogous to the discrete metric condition $d(x, y) = 1$ for $x \neq y$, ensuring that each element is distinct and non-intersecting, formalized as:

$$\forall x, y \in L, x \neq y \Rightarrow x \wedge y = 0,$$

where 0 denotes the bottom element in L, analogous to the empty set in classical topology. The notion of paracompactness in locale theory, akin to its counterpart in metric space theory, necessitates the existence of a locally finite refinement for any given cover in L. The discrete metric locale analogue presents a conceptual challenge to this notion, as the intrinsic separation of elements precludes the formation of a locally finite refinement that is both efficient and comprehensive.This exploration reveals the inherent complexity in transposing the property of paracompactness from conventional metric spaces to the abstract domain of locales. It highlights the discrete metric locale as a pivotal example that underscores the challenges of maintaining classical topological properties within the abstract, point-free framework of locale theory. Such considerations are crucial for advancing our understanding of topological structures through the lens of locale theory, providing a foundation for further exploration and potential reconciliation of these classical concepts within an abstract framework. □

Chapter 2

Some aspects of strongly paracompact locales

The concept of strongly paracompact spaces emerged as a prominent and intricate subject in the field of general topology during the 1950s. Its attribution is dispersed among various mathematicians, this era witnessed significant contributions from esteemed researchers in the domain of strongly paracompact spaces. This chapter provides a comprehensive analysis of the key advancements made by notable scholars, particularly M. Wiscamb [36] and Xin Zhang [39], in their investigations of strongly paracompact spaces. This chapter explores the fundamental ideas presented in their respective works and highlights the ongoing relevance of this topic in contemporary mathematical research, evident through the exploration of properties, characterisations, and variations of strongly paracompact spaces. Moreover, the pursuit of innovative techniques and connections with diverse branches of mathematics continues to drive the progress and significance of this area of study.

M. Wiscamb made substantial advancements in the understanding of strongly paracompact spaces through her seminal paper [36]. Her research looked at the concept of symmetric neighborhood systems, where each point resides in the neighborhood of the other, thereby unveiling essential properties of strongly paracompact spaces. Additionally, Wiscamb explored the notion of star-finiteness in the context of these spaces, proposing that a space may be deemed strongly paracompact if any open cover possesses a refinement that exhibits star-finiteness. Such meticulous investigations significantly enriched the comprehension of strongly paracompact spaces during this epoch.

In his influential paper [39], Xin Zhang made profound strides in advancing the theory of strongly paracompact spaces. A key contribution of his was the introduction of a novel characterisation, based on the space existence of a star-countable open refinement for every increasing open cover. Furthermore, Zhang extended the notion of strong paracompactness to different classes of spaces, demonstrating that countably paracompact normal spaces, perfectly normal spaces, or monotonically normal spaces attain strong paracompactness under specific conditions. Zhang's pioneering research significantly broadened the scope of strongly paracompact spaces and highlighted their relevance in various mathematical contexts.

The theory of strongly paracompact spaces remains an active and thriving area of research in contemporary mathematics. Current scholars continue to investigate the intricate properties and characterisations of these spaces, seeking to unravel the complexities inherent in their structures. Novel techniques are continually being developed to address challenging problems related to strongly paracompact spaces, reflecting the dynamism of ongoing research efforts. Moreover, the exploration of connections with other branches of mathematics remains an integral aspect of current investigations, reinforcing the enduring importance of this subject in modern mathematical discourse.

In conclusion, the emergence of strongly paracompact spaces as a significant topic in general topology during the 1950s resulted in notable contributions from prominent mathematicians such as M. Wiscamb and Xin Zhang. The groundbreaking research of mathematicians like M. Wiscamb and Xin Zhang has significantly advanced our understanding of strongly paracompact spaces and continues to inspire contemporary investigations in the field. The ongoing exploration of properties, characterizations, and variations of strongly paracompact spaces, coupled with the development of innovative techniques and interdisciplinary connections, underscores the enduring relevance and progress of this area of study in modern mathematics.

2.1 On strongly paracompact spaces

In this section, our attention is directed towards two properties of strongly paracompact spaces as posited by Qu [29]. The initial property asserts that every regular Lindelöf space qualifies as strongly paracompact. The subsequent property establishes that a countably paracompact normal space achieves strong paracompactness if and only if each increasing open cover of the space possesses a star-countable open refinement.

Definition 2.1.1. *Let X be a topological space.*

(a) *A cover $\mathcal{U}$ of X is star-finite if the set*

$$\{W \in \mathcal{U} \mid W \cap U \neq \emptyset\}$$

is finite for every $U \in \mathcal{U}$. If $\mathcal{U}$ and $\mathcal{V}$ are covers of X, then $\mathcal{U}$ is a star refinement of $\mathcal{V}$ if, whenever $U \in \mathcal{U}$ then there exists a $V \in \mathcal{V}$ satisfying $\bigcup\{W \in \mathcal{U} \mid W \cap U \neq \emptyset\} \subseteq V$.

(b) *If X is Hausdorff, we say that it is strongly paracompact if every open cover of X has a star-finite open refinement.*

For completeness, we prove the following result from Qu [29]. Whose proof was not provided in [29].

Lemma 2.1.2. *Suppose that $\mathcal{U}$ and $\mathcal{V}$ are covers of a topological space X. If $\mathcal{U}$ and $\mathcal{V}$ are star-finite, then the following statements hold:*

(a) $\mathcal{U} \wedge \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$ *is star-finite.*

(b) $\{\bigcap \Phi \mid \Phi \subseteq \mathcal{U}, |\Phi| < \omega\}$ *is star-finite.*

(c) *For any $U \in \mathcal{U}$, star$(U,\mathcal{U}) = \bigcup\{V \in \mathcal{U} \mid V \cap U \neq \emptyset\}$ is star-finite.*

Proof.

(a) Let $x \in X$. Since $\mathcal{U}$ and $\mathcal{V}$ are star-finite, for x, there exist finite subsets $\mathcal{F}_{\mathcal{U}} \subseteq \mathcal{U}$ and $\mathcal{F}_{\mathcal{V}} \subseteq \mathcal{V}$ such that x is in the star of $\mathcal{F}_{\mathcal{U}}$ and $\mathcal{F}_{\mathcal{V}}$, respectively. Consider $\mathcal{F}_{\mathcal{U}} \wedge \mathcal{F}_{\mathcal{V}} = \{U \cap V \mid U \in \mathcal{F}_{\mathcal{U}}, V \in \mathcal{F}_{\mathcal{V}}\}$. This collection is finite and covers x, thus showing $\mathcal{U} \wedge \mathcal{V}$ is star-finite at x.

(b) Let $x \in X$. For any finite $\Phi \subseteq \mathcal{U}$, the intersection $\bigcap \Phi$ (if non-empty) contains x if x is in the elements of Φ. Since $\mathcal{U}$ is star-finite, there exists a finite $\mathcal{F}_{\mathcal{U}} \subseteq \mathcal{U}$ covering x. For any Φ with $|\Phi| < \omega$ and $\Phi \subseteq \mathcal{U}$, $\bigcap \Phi$ is star-finite because it is covered by the finite intersections of members of $\mathcal{F}_{\mathcal{U}}$ that also intersect with $\bigcap \Phi$.

(c) Consider any $U \in \mathcal{U}$ and let $x \in X$. There exists a finite $\mathcal{F}_{\mathcal{U}} \subseteq \mathcal{U}$ such that x is in the star of $\mathcal{F}_{\mathcal{U}}$. For any $V \in \mathcal{U}$ with $V \cap U \neq \emptyset$, the set of such V's forms a finite collection because $\mathcal{U}$ is star-finite. Thus, star$(U, \mathcal{U})$ is covered by a finite number of sets from $\mathcal{U}$, each intersecting U non-trivially, proving it is star-finite. □

Theorem 2.1.3. *Every strongly paracompact space is paracompact.*

Proof. Let X be a strongly paracompact space, meaning that for any open cover $\mathcal{U}$ of X, there exists a σ-discrete open refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$, where each $\mathcal{V}_n$ is a discrete family of open sets. Since $\mathcal{V}_n$ is discrete, for every point $x \in X$, there exists a neighborhood N_x intersecting at most one set in $\mathcal{V}_n$, ensuring that $\mathcal{V}$ is locally finite at x. Hence, every point in X has a neighborhood that intersects only finitely many sets in $\mathcal{V}$, satisfying the definition of a locally finite refinement and thus proving X is paracompact. This establishes that every strongly paracompact space is paracompact, as a σ-discrete refinement inherently ensures local finiteness, aligning with the requirements for paracompactness. □

Theorem 2.1.4. *A closed subspace of a strongly paracompact space is strongly paracompact.*

Proof. Let X be a strongly paracompact space and let Y be a closed subspace of X. Let $\mathcal{A}$ be an open cover of Y. For each $U \in \mathcal{A}$, let V_U be an open set in X such that $U = V_U \cap Y$.

Then $\mathcal{V} = \{V_U : U \in \mathcal{A}\}$ is an open cover of Y in X. Since X is strongly paracompact, $\mathcal{V}$ has a star-finite open refinement $\mathcal{W} = \{W_i\}_{i \in I}$.
For each $i \in I$, let $U_i = W_i \cap Y$. We claim that $\mathcal{U} = \{U_i\}_{i \in I}$ is a star-finite open refinement of $\mathcal{A}$. To see this, let $U \in \mathcal{A}$. Then there exists $V \in \mathcal{V}$ such that $U = V \cap Y$. Since $\mathcal{W}$ refines $\mathcal{V}$, there exists $i \in I$ such that $V \subseteq W_i$. Then

$$U = (V \cap Y) = (W_i \cap Y) \cap (V \cap Y) = U_i \cap V,$$

which shows that $\mathcal{U}$ is a refinement of $\mathcal{A}$. To see that $\mathcal{U}$ is star-finite, note that $\mathcal{W}$ is star-finite and that for each $i \in I$, $U_i = W_i \cap Y$ is closed in X. Thus, $\mathcal{U}$ is a star-finite open refinement of $\mathcal{A}$, which shows that Y is strongly paracompact. □

Definition 2.1.5. Given any $A, B \in \mathcal{U}$, we define a finite subfamily $\{C_1, C_2, ..., C_n\}$ as a *chain* from A to B if $A = C_1$, $B = C_n$, and $C_i \cap C_{i+1} \neq \emptyset$ for $1 \leq i \leq n-1$. For all $A \in \mathcal{U}$, we denote $B(A) = \{B \in \mathcal{U}$: there exists a chain from A to $B\}$.

To establish Theorem 2.1.8., we require the subsequent findings.

Lemma 2.1.6. [29] *A space is strongly paracompact if and only if every increasing open cover of the space has a star-finite open refinement.* □

Theorem 2.1.7. [24, Mansfield's Theorem] *Every countable open cover of a countably paracompact normal space has a star-finite open refinement.* □

Theorem 2.1.8. [39] *A countably normal space is strongly paracompact if and only if every increasing open cover has a star-countable open refinement.*

Proof. By Definition 2.1.1. *(b)* of a strongly paracompact space, it follows that every increasing open cover of the space has a star-countable open refinement.

Conversely, suppose $\mathcal{V}$ is an increasing open cover of X and let $\mathcal{U}$ be a star-countable open refinement of $\mathcal{V}$. Using the lemma which states that a space is strongly paracompact if and only if every increasing open cover of the space has a star-finite open refinement [39], we prove that $\mathcal{V}$ has a star-finite open refinement.

To start off, we shall introduce the family $\mathcal{U}$ as follows. Consider $\mathcal{U} = \{\mathcal{B}_\alpha : \alpha \in \bigwedge\}$. For each $\mathcal{B}_\alpha$, it is a countable family and $(\bigcup \mathcal{B}_\alpha) \cap (\bigcup \mathcal{B}_\beta) = \emptyset$ for $\alpha \neq \beta$.

We will now provide a proof of this claim, using Definition 2.1.5.

It is easy to know that $\mathcal{B}(A)$ is countable, and, for any $A_1, A_2 \in \mathcal{U}$, $(\bigcup \mathcal{B}(A_1)) \cap (\bigcup \mathcal{B}(A_2)) \neq \emptyset$ if and only if $\mathcal{B}(A_1) = \mathcal{B}(A_2)$. We complete the proof of the claim. For every $\alpha \in \bigwedge$, let $Z_\alpha = \cup \mathcal{B}_\alpha$. By the above claim, we know that the family $\{Z_\alpha : \alpha \in \bigwedge\}$ is an open and closed disjoint family of X. Since X is countably normal, the closed subspace Z_α of X is countably paracompact for every $\alpha \in \bigwedge$. Moreover, it follows from the above claim that the family $\mathcal{B}_\alpha$ is a countable open cover of Z_α. By Lemma 2.1.7., we find a star-finite open family $\mathcal{W}_\alpha$ of the subspace Z_α refining $\mathcal{B}_\alpha$. Since each Z_α is open in X and since $X = \bigcup_{\alpha \in \bigwedge} Z_\alpha$, it follows that the family $\bigcup_{\alpha \in \bigwedge} \mathcal{W}_\alpha$ is an open cover of X. The family $\bigcup_{\alpha \in \bigwedge} \mathcal{W}_\alpha$ is also star-finite since $\{Z_\alpha : \alpha \in \bigwedge\}$ is a disjoint family of X. Furthermore, it is easy to see that $\bigcup_{\alpha \in \bigwedge} \mathcal{W}_\alpha$ is a refinement of $\mathcal{V}$ since $\mathcal{U}$ refines $\mathcal{V}$. By Theorem 2.1.6., the space X is strongly paracompact. $\square$

Theorem 2.1.9. [29, Qu's Theorem] *A topological space X is strongly paracompact if for any monotone increasing open cover of the space there exists a star-finite open refinement.*

Proof. Let X be a topological space which satisfies the condition of Theorem 2.1.9. Consider an infinite open cover $\mathcal{U}$ of X defined as $\mathcal{U} = \{U_\alpha \mid \alpha < k\}$, where $|\mathcal{U}| = k$. The proof proceeds by transfinite induction with respect to $k = |\mathcal{U}|$.
If $k = \omega$, we have $\alpha = 0$ and $V_0 = U_0$. For any $0 < \alpha < \omega$, V_α is defined as the union of $\{U_\beta \mid \beta < \alpha\}$. The set $\mathcal{V} = \{V_\alpha \mid \alpha < \omega\}$ forms a monotone increasing infinite open cover

of X. It is known that there exists a star-finite open refinement $\mathcal{W} = \{W_\beta \mid \beta < k'\}$ of $\mathcal{V}$. Furthermore, there exists a function $f : k' \to \omega$ such that for any $\beta < k'$, W_β is a subset of $V_{f(\beta)}$. The set $\{U_\alpha \cap W_\beta \mid \alpha < f(\beta), \beta < k'\}$ is a star-finite open refinement of $\mathcal{U}$. For any intersection $U_\alpha \cap W_\beta$, given that w is star-finite, the set $\{W \in \mathcal{U} \mid W \cap W_\beta \neq \emptyset\}$ is equivalent to $\{W_{\beta_1}, \ldots, W_{\beta_n}\}$. The cardinality of the union of $\{U_\alpha \cap W_{\beta_i} \mid \alpha < f(\beta_i)\}$ for all $i \leq n$ is less than ω, confirming its star-finite nature.

For the case where $k > \omega$, assume that for any infinite open cover of $\mathcal{U}$ that satisfies the hypothesis of Theorem 2.1.7, if $|\mathcal{U}| < k$, there exists a star-finite open refinement. Now, consider any infinite open cover $\mathcal{U}$ of X that satisfies the condition of Theorem 2.1.7 and where $|\mathcal{U}| = k$. Here, $\alpha = 0$ and $V_0 = U_0$. If $0 < \alpha < k$, V_α is the union of $\{U_\beta \mid \beta < \alpha\}$. The set $\mathcal{V} = \{V_\alpha \mid \alpha < k\}$ is a monotone increasing infinite open cover of X. A star-finite open refinement $\mathcal{W}' = \{W'_\beta; \beta < k'\}$ of $\mathcal{V}$ exists such that there's a function $f \mid k' \to k$ where for any $\beta < k'$, W'_β is a subset of $V_{f(\beta)}$. Based on Lemma 1 of [29], X is paracompact. Consequently, an open cover $\mathcal{U} = \{W_\beta \mid \beta < k'\}$ of $\mathcal{U}'$ exists such that for any $\beta < k'$, $\overline{W}_\beta$ is a subset of W'_β.

For any $\beta < k'$, the set $\{W'_\beta \cap U_\alpha \iff \alpha < f(\beta)\} \bigcup \{X - \overline{W}_\beta\}$ is an open cover of X and the cardinality of this set is $|f(\beta)| < k$. According to the hypothesis of the transfinite induction, there exists a star-finite open refinement w'_β. Set $\mathcal{W}_\beta = \{W \in \mathcal{U}'_\beta \mid W \cap \overline{W}_\beta \neq \emptyset, \}$

For any $x \in \overline{W}_\beta$, $O_{\beta,x} = \bigcap \{W \mid x \in W, W \in \mathcal{W}_\beta\}$. $\mathcal{W}_\beta$ is star-finite, so the cardinality of $\{W \mid x \in W, W \in w_\beta\}$ is less than ω. $O_{\beta,x}$ is an open subset of X. According to Lemma 2.1.2 of [29], $\{O_{\beta,x} \mid x \in W_\beta\}$ is star-finite.

O_x = $\bigcap \{O_{\beta,x} \mid x \in W_\beta\}$. $\mathcal{W}$ is star-finite, so the cardinality of $\{O_{\beta,x} \mid x \in W_\beta, \beta < k'\}$ is less than ω. O_x is an open subset of X.

G_x = $X - \bigcup \{\overline{W}_\beta \mid x \neq W_\beta, \beta < k'\}$. W' is star-finite, so the union of $\{\overline{W}_\beta \mid x \neq \overline{W}_\beta, \beta < k'\}$ is closed, meaning G_x is open.

According to Lemma 2.1.2, {star$(x, \{\overline{W}_\beta \mid \beta < k'\}) \mid x \in X\}$ is star-finite. For any $x \in X$, G_x is a subset of star$(x, \{\overline{W}_\beta \mid \beta < k'\})$. Thus, $\{G_x \mid x \in X\}$ is star-finite.

The set $\{G_x \cap O_x \mid x \in X\}$ is star-finite. For any $x \in X$, if y is not in the union of $\{\overline{W}_\beta \mid x \in \overline{W}_\beta, \beta < k'\}$, then $G_x \cap G_y = \emptyset$. That is, $(G_x \cap O_x) \cap (G_y \cap O_y) = \emptyset$.

Because $\{\overline{W}_\beta \mid \beta < k'\}$ is star-finite, the cardinality of the union of $\{\overline{W}_\beta \mid x \in \overline{W}_\beta, \beta < k'\}$

is less than ω. That is, $\{W_\beta \mid x \in \overline{W}_\beta, \beta < k'\}$ is the set $\{\overline{W}_{\beta_1}, \ldots, \overline{W}_{\beta_n}\}$. For any $1 \leq i \leq n$, $\{O_{\beta_i,y} \mid y \in \overline{W}_{\beta_i}\}$ is finite. Thus, for any $1 \leq i \leq n$, $\{O_{\beta_i,y} \mid O_{\beta_i,x} \cap O_{\beta_i,y} \neq \emptyset; y \in \overline{W}_{\beta_i}\}$ is finite. The set $\{O_y; Ox \cap Oy \neq \emptyset, y \in \overline{W}_{\beta_1} \cup \ldots \cup \overline{W}_{\beta_n}\}$ is finite. The set $\{Gy \mid Gx \cap Gy \neq \emptyset, y \in \overline{W}_{\beta_1} \cup \ldots \cup \overline{W}_{\beta_n}\}$ is finite. The set $\{Oy \cap Gy \mid Ox \cap Gx \cap Oy \cap Gy \neq \emptyset, y \in \overline{W}_{\beta_1} \cup \ldots \cup \overline{W}_{\beta_n}\}$ is finite.
That is, $\{Gx \cap Ox \mid x \in X\}$ is star-finite. It is a open refinement of $\mathcal{U}$. According to the principle of the transfinite induction, for any infinite open cover of $\mathcal{U}$, there exists a star-finite open refinement. That is, X is strongly paracompact. □

Proposition 2.1.10. [39] *A perfectly normal space X is strongly paracompact if and only if every increasing open cover of the space has a star-countable open refinement*

Proof. By the definition of strongly paracompact spaces, it is clear that the condition of necessity it is satisfied. On the other hand, it is a known fact, as mentioned in reference [5], that every perfectly normal space is both countably paracompact and normal. Therefore, a perfectly normal space is sufficient to be strongly paracompact if it has a star-countable open refinement for every increasing open cover of the space, which we deduce from Theorem 3.1 of [39]. □

2.2 On strongly paracompact locales

This section covers two aspects: first, characterisation of countable paracompact frames - in the main according to Dowker [8] and Garcia *et al* [16], and an attempt to define and characterise strong paracompactness in the context of locales.

Definition 2.2.1. [26] *Let L be a frame.*

(a) *A frame L is countably paracompact if every countable cover of L has a locally finite refinement.*

(b) *A subset U of a frame L is star-finite if the set $\{u \in U | u \wedge x \neq 0\}$ is finite.*

(c) *A frame L is strongly paracompact if every cover of L has a star-finite refinement.*

(d) *A frame L is said to be perfectly normal if for each $x \in L$ there exits a countable subset $U \subseteq L$ such that*

$$x = \bigvee U \text{ and } U \prec x,$$

where $U \prec x$ means that $u \prec x$, for all $u \in U$.

(e) *A cover $(a_j)_{j \in J}$ is shrinkable if there exists a cover $(b_j)_{j \in J}$ such that $b_j \prec a_j$ for every $j \in J$.*

Remark 2.2.2. *In relation to $\prec$, it is not difficult to show that a frame L is normal if and only if, whenever $x \vee y = 1$, there exists an $s \in L$ such that $x \vee s = 1$ and $s \prec y$:* For suppose L is normal. If $s \wedge t = 0$ and $x \vee t = 1 = x \vee y$, then it follows that $s \prec y$.
On the other hand, if the condition $x \vee y = 1$ holds, this implies the existence of an element $s \in L$ such that $x \vee s = 1$ and $s \prec y$. This ensures the existence of another element $t \in L$ satisfying

$$s \wedge t = 0 \text{ and } t \vee y = 1.$$

Given that $x \vee s = 1 = t \vee y$ and $s \wedge t = 0$, it is confirmed that L is normal. □

Recall that every paracompact frame is regular and a regular compact frame is normal, it is therefore obvious that the paracompact frame is normal. By [9, Proposition 1], a frame is normal if and only if each point-finite cover is shrinkable.

Proposition 2.2.3. [9] *Each cover of a paracompact normal frame is shrinkable*

Proof. Let L be a paracompact normal frame, and let $(a_\alpha)_{\alpha \in I}$ be an arbitrary cover of L. For each a_α indexed by $\alpha \in I$, there exists a locally finite cover $(b_\lambda)_{\alpha \in I}$ such that $b_\lambda \leq a_\alpha$. Given that L is paracompact, by the discussion above the cover $(b_\lambda)_{\lambda \in I}$ is shrinkable. Consequently, there exists a cover $(c_\delta)_{\delta \in I}$ such that $b_\lambda \vee c^*_\delta = 1$. This implies that $a_\alpha \vee c^*_\alpha = 1$ for all $\alpha \in I$.

Hence, (a_α) is shrinkable, as we have shown that there exists a cover (c_α) such that $a_\alpha \vee c_\alpha^* = 1$.

Therefore, we have established that each cover of a paracompact normal frame is shrinkable. □

The following results mentioned in [8] is due to Dowker from whom our proof is adopted. We also refer to Garcia, Khubiak and Picado [16].

Theorem 2.2.4. *A frame L is countably paracompact if and only if each countable non-decreasing cover is shrinkable.*

Proof. *Necessity:* Suppose that L is countably paracompact. Consider a non-decreasing sequence U in L, denoted as $U = (u_i)_{i \in I}$. There exist covers $V = (v_i)_{i \in I}$ and $W = (\omega_\alpha)_{\alpha \in I}$ such that $v_i \leq (\omega_\alpha)$ and $\omega_\alpha \wedge u_i = 0$ for all but finitely many i. Given that the sequence $(u_i)_{i \in I}$ is non-decreasing, if $i \leq j$ then $v_i \leq u_i \leq u_j$. Thus,

$$\bigvee_{i \leq j} u_i \leq v_j \tag{*}$$

Define $s_j = \bigvee_\alpha \omega_\alpha$ where $\omega_\alpha \wedge v_i = 0$ for all $i > j$. Then,

$$\bigvee_I s_j = \bigvee (\bigvee \omega_\alpha) = 1,$$

since $(\omega_\alpha)_\alpha$ is a cover. Moreover,

$$
\begin{aligned}
s_j \wedge \bigvee_{i>j} v_i &= \bigvee \omega_\alpha \wedge \bigvee_{i>j} v_i \\
&= \bigvee \bigvee (\omega_\alpha \wedge v_i) \\
&= \bigvee 0 \\
&= 0,
\end{aligned}
$$

which implies that

$$\bigvee_{i>j} v_i \leq s_j^*.$$

From (*) above,

$$v_j \vee s_j^* \geq \bigvee_{i\leq j} v_i \vee \bigvee_{i>j} v_i = 1;$$

hence,

$$u_j \vee s_j^* = 1 \text{ and } s_j \leq u_j \text{ for all } j.$$

This establishes that (u_j) is shrinkable.

Sufficiency: Assume that for each countable increasing cover (a_i), there is a cover (b_i) such that $b_i^* \vee a_i = 1$. Define $c_j = a_j \wedge \bigwedge_{i\leq j} b_i^*$. Then, we have $\bigwedge_{j\leq k} c_j = \bigvee_{j\leq k} a_j = a_k$, which can be shown by induction.

The base case is $c_1 = a_1$. For the inductive step, assume that $\bigvee_{j<k-1} c_j = \bigvee_{j\leq k} a_j$. Then, we have that

$$
\begin{aligned}
\bigvee_{j<k-1} c_j &= \Big(\bigvee_{j<k} a_j\Big) \vee c_k, \\
&= \Big(\bigvee_{j<k} a_j\Big) \vee \Big(a_k \wedge \bigwedge_{i<k} b_i^*\Big), \\
&= \Big(\bigvee_{j\leq k} a_j\Big) \wedge \bigwedge_{i<k}\Big(\bigvee_{i<k} a_j\Big) \vee b_i^*, \\
&= \bigvee_{j\leq k} a_i.
\end{aligned}
$$

Hence, (c_j) is a cover and $c_j \leq a_j$. Since (b_i) is a cover and $b_i \wedge c_j \leq b_i \wedge b_i^* = 0$ for all

$i > j$, it follows that (c_j) is locally finite. □

Lemma 2.2.5. Given a frame L, if $b_i \prec c$ for each i, then $\bigcup_i b_i \prec c$: for, if $b_i \prec c$, we pick c_i satisfying $c_i \wedge b_i = 0$ and $c_i \vee c = 1$. That $(\bigvee_i c_i) \vee a = 1$ follows from the fact that $(\bigvee_i c_i) \vee a \geq c_i \vee a = 1$. On the other hand, we have that

$$(\bigvee_i c_i) \wedge (\bigvee_i b_i) = \bigvee_i (c_i \wedge (\bigvee_i b_i)) = \bigvee_i (\bigvee_i (c_i \wedge b_i)) = \bigvee_i 0 = 0$$

□

Theorem 2.2.6. [16, Garcia, Khubiak and Picado] *Each perfectly normal frame is countably paracompact.*

Proof. Given a perfectly normal frame L and a non-decreasing cover (c_n), for each $n \in \mathbb{N}$, there exists a family $\{b_{n,m} : m \in \mathbb{N}\}$ such that $c_n = \bigvee_{m \in \mathbb{N}} b_{n,m}$ and $b_{n,m} \prec c_n$. We define a sequence (a_n) where each $a_n = \bigvee_{i,j \leq n} b_{i,j}$. This sequence (a_n) is non-decreasing and shrinks the cover (c_n). For each $n \in \mathbb{N}$, it follows from Theorem 2.2.4 that $a_n \prec \bigvee_{i \leq n} c_i = c_n$ and

$$\bigvee_{n \in \mathbb{N}} a_n = \bigvee_{n \in \mathbb{N}} \bigvee_{i,j \leq n} b_{i,j} = \bigvee_{n \in \mathbb{N}} c_n = 1.$$

Therefore, each perfectly normal frame is countably paracompact. □

Proposition 2.2.7. *For a fixed $n \in \mathbb{N}$ in a frame L, if $b_i \prec c_n$ for all $i \in I$ in L then $\bigvee_i b_i \prec c_n$.*

Proof. Let L be a frame. Let i range over an index set I and n range over an index set J. Given that for each b_i, where $i \in I$, we have $b_i \prec c_n$ for some $n \in J$. By the definition of the way-below relation, for each b_i, there exists a directed set D_i such that $b_i \leq \bigvee D_i$ and $\bigvee D_i \leq c_n$. Consider the union $D = \bigcup_{i \in I} D_i$, which remains directed due to the properties of frames. For each b_i, $b_i \leq \bigvee D$. Using the completeness of frames, it follows that $\bigvee_{i \in I} b_i \leq \bigvee D$. Combining these inequalities, we deduce that

$$\bigvee_{i \in I} b_i \leq \bigvee D \leq c_n,$$

which implies $\bigvee_{i \in I} b_i \prec c_n$. □

We conclude this chapter by introducing strong paracompactness into the theory of frames. It is shown that some results on strongly paracompact spaces of Zhang [39] and Mansfield [24] are translatable to frames. These results are based on the fact that, star-finiteness of open covers (refinements) are easily translated into the terminology of frames.

Definition 2.2.8. *A collection $\mathcal{U}$ of subsets of a frame L is said to be star countable if every element of $\mathcal{U}$ intersects only countable many elements of $\mathcal{U}$. A frame L is said to be strongly paracompact if every cover of L has a star-finite refinement.*

It is clear that since a star-finite cover of a frame is locally finite, it follows that a strongly paracompact frame is paracompact. Here is the first characterisation of strong paracompactness in frames:

Theorem 2.2.9 (Smirnov's Theorem) (Wiscamb [36]). *In a regular frame L, the following are equivalent:*

(a) *The frame L is strongly paracompact.*

(b) *Every cover of L has a star-countable refinement.*

Proof. We aim to prove the implication $(b) \Rightarrow (a)$ within a specific mathematical context. Assume $\mathcal{W}$ represents an open cover of a locale L, which possesses a star-countable refinement denoted as $\mathcal{U} = \{u_\alpha \mid \alpha \in \mathfrak{A}\}$. Given the countability inherent to $\mathcal{U}$, it is permissible, without loss of generality, to depict $\mathcal{U}$ as consisting of elements $u_{\gamma i}$ for $\gamma \in \Gamma$

and $i = 1, 2, \ldots$, maintaining the condition $u_{\gamma i} \wedge u_{\delta j} = 0$ whenever $i \neq j$. We define u_γ as:

$$u_\gamma = \bigvee_{i=1}^{\infty} u_{\gamma i},$$

confirming that $u_\gamma \in L$. The selection of one $u_{\gamma i}$ for each $\gamma \in \Gamma$ results in a collection $\{u_{\gamma i}\}$, thereby forming a discrete assemblage where a countable number of these elements constitute $\mathcal{U}$. Given the paracompact nature of L, it follows that the closed sublocale $\uparrow u_\gamma$ retains paracompactness, implying $\{u_{\gamma i} \mid i = 1, 2, \ldots\}$ operates as a countable cover for L. This cover, in turn, acquires a star-finite refinement, $\mathcal{W}_\gamma$. By aggregating all such refinements $\mathcal{W}_\gamma$, it becomes evident that the collective refinements serve as a star-finite refinement of $\mathcal{U}$, and by extension, a star-finite refinement of $\mathcal{W}$. Therefore, L demonstrates the characteristic of possessing a star-finite refinement, conclusively establishing its strong paracompactness. $\square$

Theorem 2.2.10. *Every countable cover of a countably paracompact normal frame has a star-finite refinement.*

Proof. Mansfield demonstrated in his article [24] that a normal space is countably paracompact if and only if every open cover of the space, denoted as X, has a countable, open, star-finite refinement. Based on this foundation, it can be deduced that if L represents a countable paracompact and normal frame, then every countable cover of L necessarily has a star-finite refinement. $\square$

Theorem 2.2.11. *A countably paracompact normal frame L is strongly paracompact if and only if every non-decreasing cover of L has a star-finite refinement.*

Proof. *Necessity*: Suppose that L is countably paracompact and normal, and let $\mathcal{U}$ be a non-decreasing cover of L. If L is strongly paracompact, then the cover $\mathcal{U}$ has a star-finite refinement by definition. But a star-finite cover is also star-countable, so $\mathcal{U}$ must be a star-countable refinement.

Sufficiency: Assume the hypothesis of the theorem, assume that L is countably paracompact and normal and take a non-decreasing cover $\mathcal{U}$ of L. Since a star-finite refinement is star-countable, we must have that $\mathcal{U}$ has a star-countable refinement. For this cover $\mathcal{U}$, let $\mathcal{W}$ be its star-countable refinement. We need only show that $\mathcal{U}$ has a star-finite refinement. First, we assume without loss of generality that $\mathcal{W} = (w_i)_{i \in \mathbb{N}}$.

We then set

$$\mathcal{G}_i = \{\uparrow w_i \mid i \in \mathbb{N}\}$$

and note that each $\uparrow w_i$ is a closed sublocale of L, so, it must be a countably paracompact locale for each $i \in I$ as well and satisfies $\bigvee(\uparrow w_i) = 1$. Then there is a refinement $\mathcal{W}_i$ of $\uparrow w_i$ with $\bigvee \mathcal{G}_i = \bigvee\{\uparrow w_i \mid i \in \mathbb{N}\} = 1$, therefore $\mathcal{W}_i$ is also a cover of L. But $\mathcal{W}$ was chosen in such a way that it refines $\mathcal{U}$, so we must have $(w_i)_{i \in \mathbb{N}}$ is also a refinement of $\mathcal{U}$. □

Proposition 2.2.12. *Every perfectly normal frame is strongly paracompact if and only if every shrinking cover has a star-countable refinement.*

Proof. Necessity is clear from the definition of strong paracompactness. For sufficiency, suppose that $\mathcal{U}$ is a non-descreasing cover of a perfectly normal frame L. Then L is countably paracompact and normal. By hypothesis, $\mathcal{U}$ has a star-finite refinement, so the frame L must be strongly paracompact by Zhang's Theorem 2.2.10. □

Remark 2.2.13. If Zhang's contribution to strong paracompactness is anything to go by, it seems that a frame is strongly paracompact if and only if every non-decreasing cover of L has a star-finite refinement. However, we were unable to establish the fact.

Chapter 3

Aspects of uniformly paracompact locales

In this chapter, we are indebted to characterisation of uniform paracompactness according to Kanetov [20] on the one hand; and uniform paracompactness in locales from the point of the view of Dube and Naidoo [10]. To avoid confusion, constistent with the origin of the most of our topological results in this dissertation, we will use covering in topological spaces but use cover when we work in locales.

3.1 Uniformly paracompact spaces

This section is devoted to uniform paracompactness in the sense of Kanetov [20]. It is shown that uniformly perfect mappings reflect uniform paracompactness and that uniform paracompactness is related to the existence of a continuous mapping on a metrizable space in the presence of a finitely additive covering. We define strong uniform paracompactness and show that a space which possess this property is equivalent to it being uniformly paracompact if the underlying space is strongly paracompact.

Definition 3.1.1. [20] *We recall that given a set X, an element $x \in X$, and a subset $V \subset X$, we have:*

(a) $St(\mathcal{U}, x) = \{U \in \mathcal{U} \mid x \in U\}$*, which represents the set of all elements of* $\mathcal{U}$ *that contain* x*;*

(b) $St(\mathcal{U}, V) = \{U \in \mathcal{U} \mid U \cap V \neq \emptyset\}$*, which represents the set of all elements of* $\mathcal{U}$ *that intersect with* V*;*

(c) $\mathcal{U}(V) = \bigcup St(\mathcal{U}, V)$*, which represents the union of all elements in the star of* V *with respect to* $\mathcal{U}$*, for some* $V \subseteq X$*.*

According to Kanetov [20], given a cover $\mathcal{U}$ of a topological space, we define $\mathcal{U}^{\angle}$ as the set

$$\mathcal{U}^{\angle} = \left\{ \bigcup \mathcal{W} \mid \mathcal{W} \subset \mathcal{U} \text{ and } \mathcal{W} \text{ is finite} \right\}.$$

We say that the cover $\mathcal{U}$ is *finitely additive* if $\mathcal{U}^{\angle} = \mathcal{U}$.

A uniform cover is a cover in a nearness space. Then a uniform refinement is a refinement by uniform covers.

Definition 3.1.2. [20] *A uniform space* $(X,\mathcal{U})$*:*

(a) is said to be uniformly paracompact if every finitely additive open cover has a σ*-locally finite uniform refinement.*

(b) is said to be strongly uniformly locally compact if there exists a σ*-locally finite uniform cover that consists of compact subsets.*

The three theorems, namely Theorem 3.1.3, Theorem 3.1.6, and Theorem 3.1.7, below are adapted from Kanetov's paper [20].

Theorem 3.1.3. *A strongly uniformly locally compact uniform space is uniformly paracompact.*

Proof. Suppose that $\mathcal{V}$ is a finitely additive open covering of a strongly uniformly locally compact space $(X,\mathcal{U})$. By definition, we find a σ-locally finite uniform covering $\mathcal{W}$ of X, such that

$$X = \bigcup \mathcal{W} = \bigcup_{W \in \mathcal{W}} W,$$

where each W is compact. Since $X = \bigcup_{V \in \mathcal{V}} V$, for each $W \in \mathcal{W}$ there exists $V \in \mathcal{V}$ such that $W \subseteq V$. Therefore, $\mathcal{W} \leq \mathcal{V}$, implying that $\mathcal{W}$ is a σ-locally finite uniform covering of $(X,\mathcal{U})$. □

Subspaces of uniformly paracompact spaces are not uniformly paracompact, in general. However, we have the following result:

Proposition 3.1.4. *A closed subspace of a uniformly paracompact space is uniformly paracompact.*

Proof. Suppose that $(Y,\mathcal{U}_Y)$ is a closed subspace of a uniformly paracompact space $(X,\mathcal{U})$ and let $\mathcal{V}$ be a finitely additive open covering of $(Y,\mathcal{U}_Y)$, where $\mathcal{U}_Y$ is the induced uniformity on Y. Since Y is closed, it follows that $X - Y$ is open in X, and so the collection

$$\{X - Y\} \cup \{V \mid V \in \mathcal{V}\}$$

is a finitely additive open covering of $(X,\mathcal{U})$. But $(X,\mathcal{U})$ is uniformly paracompact, so there must be a σ-locally finite uniform refinement, say $\mathcal{W} \leq \{X - Y\} \cup \{V \mid V \in \mathcal{V}\}$. It is now routine to show that

$$\mathcal{W}_Y = \{Y \cap W \mid W \in \mathcal{W}\}$$

is a σ-locally finite uniform refinement of $(Y,\mathcal{U}_Y)$; thus, $(Y,\mathcal{U}_Y)$ is uniformly paracompact.□

For our next result, we need the following concept:

Definition 3.1.5. [13] *A continuous function $f : X \to Y$ between topological spaces is* *called perfect if it is closed,surjective and satisfies the property that $f^{-1}(y) = f^{-1}(\{y\})$ is compact in X for each $y \in Y$.*

Theorem 3.1.6. *If $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a uniformly perfect mapping with $(Y,\mathcal{V})$ uniformly paracompact, then $(X,\mathcal{U})$ is also uniformly paracompact.*

Proof. We assume the hypothesis of the theorem and take a finitely additive open covering $\mathcal{W}$ of X. We observe that

$$\mathcal{V}_{f^{-1}} = \{f^{-1}(y) \mid y \in Y\}$$

is a covering of X and that $\mathcal{V}_{f^{-1}} \leq \mathcal{W}$, where $f^{-1}(y) \subseteq W$, for some $W \in \mathcal{W}$.
We claim that

$$\mathcal{W}^f_W = \{(Y - f(X - W)) \mid W \in \mathcal{W}\}$$

is a covering of Y, that is,

$$Y = \bigcup \mathcal{W}^f_W = \bigcup \{(Y - f(X - W)) \mid W \in \mathcal{W}\}$$

That $(Y - f(X - W)) \subseteq Y$ is clear. On the other hand, we find that

$$\begin{aligned} y \in Y &\Rightarrow f^{-1}(y) \in W, \text{ for some } W \in \mathcal{W} \\ &\Rightarrow f^{-1}(y) \notin (X - W), \text{ for this } W \\ &\Rightarrow y \notin f(X - W), \text{ for this } W \\ &\Rightarrow y \in Y - f(X - W), \end{aligned}$$

which shows that

$$Y \subseteq \bigcup_W \{Y - f(X - W)\},$$

hence $\mathcal{W}^f_W$ is an open covering of $(Y,\mathcal{V})$. Now we consider

$$(\mathcal{W}^f_W)^{\angle} = \left\{\bigcup\{(Y - f(X - W))\} \mid (Y - f(X - W)) \subseteq \mathcal{W}^f_W \text{ is finite}\right\} = \mathcal{W}^f_W,$$

that is, it is a finitely additive open covering of $(Y, \mathcal{V})$. Since $(Y, \mathcal{V})$ is uniformly paracompact, it follows that there is a σ-locally finite uniform covering $\mathcal{F}$ of $(Y, \mathcal{V})$ in $\mathcal{V}$. We claim that

$$f^{-1}((\mathcal{W}_W^f)^{\angle}) \leq \mathcal{W},$$

to this end, take

$$V \in f^{-1}((\mathcal{W}_W^f)^{\angle}) = f^{-1}(\mathcal{W}_W^f).$$

Then

$$f(V) = (Y - f(X - W)), \text{for some } W \in \mathcal{W}. \tag{*}$$

For a contradiction, suppose that there is no $W \in \mathcal{W}$ for which $V \subseteq W$; that is, for every $x \in V$ it holds that $x \notin W$ for all $W \in \mathcal{W}$. Then $x \in (X - W)$ whilst

$$f(x) \in f(V) \text{ and } f(x) \in f(X - W)$$

or equivalently

$$f(x) \in f(V) \text{ and } f(x) \notin (Y - f(X - W)),$$

for all $W \in \mathcal{W}$. But this contradicts (*) above, therefore we must have

$$f^{-1}((\mathcal{W}_W^f)^{\angle}) \leq \mathcal{W}.$$

Since

$$\bigcup f^{-1}(\mathcal{F}) = f^{-1}\left(\bigcup \mathcal{F}\right) = f^{-1}(Y) = X,$$

it is immediate that $f^{-1}(\mathcal{F})$ is a σ-locally finite uniform covering of $(X,\mathcal{U})$. To see that $f^{-1}(\mathcal{F}) \leq \mathcal{W}$ observe that

$$\mathcal{F} \leq (\mathcal{W}_W^f)^{\angle} = \mathcal{W}_W^f \Rightarrow f^{-1}(\mathcal{F}) \leq f^{-1}(\mathcal{W}_W^f) \leq \mathcal{W},$$

which concludes the proof that $(X,\mathcal{U})$ is uniformly paracompact. $\square$

Theorem 3.1.7. *A uniform space $(X,\mathcal{U})$ is uniformly paracompact if and only if for every finitely additive open covering $\mathcal{W}$ of $(X,\mathcal{U})$ there exists a uniformly continuous mapping $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ of $(X,\mathcal{U})$ onto a metrizable uniformly paracompact space $(Y,\mathcal{V})$.*

Proof. Suppose $(X,\mathcal{U})$ is a metrizable uniformly paracompact space and let $\mathcal{W}$ be a finitely additive open covering of $(X,\mathcal{U})$. Setting ω to be the identity mapping

$$id_X:(X,\mathcal{U})\to(X,\mathcal{U}),$$

we then have a uniformly continuous ω-mapping $\omega=id_X:(X,\mathcal{U})\to(X,\mathcal{U})$ of $(X,\mathcal{U})$ onto a metrizable uniformly paracompact space $(X,\mathcal{U})$.
Conversely, suppose that ω is a finitely additive covering of a uniform space $(X,\mathcal{U})$ so that $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ is a uniformly ω-continuous mapping of $(X,\mathcal{U})$ onto a metrizable uniformly space $(Y,\mathcal{V})$. For each $y\in Y$, let us assume without loss of generality that N_y is some neighborhood of y in Y. Since $X=\bigcup\mathcal{W}$, there exists a $W\in\mathcal{W}$ such that

$$f^{-1}(N_y)\subseteq W. \tag{**}$$

Now, we set

$$\mathcal{F}=\{N_y\mid y\in Y\}$$

and consider $\mathcal{F}^{\angle}=\{\bigcup N_y\mid N_y\subseteq\mathcal{F}\text{ is finite}\}$. Then $\mathcal{F}^{\angle}$ is a finitely additive open covering of $(Y,\mathcal{V})$. By hypothesis, there exists a σ-locally finite uniform covering $\mathcal{G}\in\mathcal{V}$ of $(Y,\mathcal{V})$ such that

$$\mathcal{G}\leq\mathcal{F}^{\angle}.$$

We claim that $f^{-1}(\mathcal{G})\leq\mathcal{W}$: For, if $U\in f^{-1}(G)$ then $f(U)\in\mathcal{G}\leq\mathcal{F}^{\angle}$; so there exists an $N_V\in\mathcal{F}^{\angle}$ such that $f(U)\subseteq N_V$, hence $U\subseteq f^{-1}(N_V)\subseteq W$, for some $W\in\mathcal{W}$ by (**). Hence $f^{-1}(\mathcal{G})\leq\mathcal{W}$. We shall complete the proof by showing that $f^{-1}(\mathcal{G})$ is a σ-locally finite uniform covering of $(X,\mathcal{U})$. We consider a set $f^{-1}(G)$, for some $G\in\mathcal{G}$ such that

$\{F \in \mathcal{G} \mid G \cap F \neq \emptyset\}$ is finite. Then it is easy to see that the set

$$\{f^{-1}(G) \in f^{-1}(\mathcal{G}) \mid f^{-1}(G) \cap f^{-1}(F) \neq \emptyset\}.$$

is also finite, hence $f^{-1}(\mathcal{G})$ is σ-locally finite as desired, which completes the proof. □

In passing, we recall that if $(X,\mathcal{U})$ is a uniform space and $A \subseteq X$, the induced uniform structure $\mathcal{U}_A$ on A is given by:

$$\mathcal{U}_A = \{U \subseteq (A \times A) \mid U = V \cap (A \times A), \text{ for some } V \in \mathcal{U}\}$$

The induced uniformity structure $\mathcal{U}_A$ on A is the coarsest on A for which the inclusion map $i : (A,\mathcal{U}_A) \to (X,\mathcal{U})$ is uniformly continuous. Uniform paracompactness relates to strong uniform paracompactness as follows:

Theorem 3.1.8. [21] *For a uniform space $(X,\mathcal{U})$, the following are equivalent:*

(a) $(X,\mathcal{U})$ *is strongly uniformly paracompact.*

(b) $(X,\mathcal{U})$ *is uniformly paracompact and the topological space* (X,τ_A) *is strongly paracompact.*

Proof. (a) ⇒ (b). Since a star-finite open covering is σ-locally finite, if $(X,\mathcal{U})$ is strongly uniformly paracompact, then it is uniformly paracompact.

(b) ⇒ (a). Assume the conditions in (b) and suppose that $\mathcal{V}$ is a finitely additive open covering of the uniform space $(X,\mathcal{U})$. By strong paracompactness of (X,τ_A), we pick a star-finite open covering $\mathcal{W}$ of $(X,\mathcal{U})$ such that $\mathcal{W} \leq \mathcal{V}$, and set

$$\mathcal{W}^{\angle} = \left\{\bigcup \mathcal{F} \mid \mathcal{F} \subset \mathcal{W} \text{ is finite}\right\}.$$

Then $\mathcal{W}^{\angle}$ is a finitely additive open covering of $(X,\mathcal{U})$. We claim that $\mathcal{W}^{\angle}$ is a star-finite covering: By its choice, $\mathcal{W}$ is star-finite, so since it is additive, we must have that $\mathcal{W}^{\angle}$ is also star-finite. Since $(X,\mathcal{U})$ is uniformly paracompact, it follows that the additive open covering $\mathcal{W}^{\angle}$ of $(X,\mathcal{U})$ has a σ-locally uniform covering refinement $\mathcal{G}$ of $(X,\mathcal{U})$. Thus, the star-finite uniform covering $\mathcal{W}^{\angle}$ of $(X,\mathcal{U})$ is a refinement of the additive open covering $\mathcal{V}$ of $(X,\mathcal{U})$; hence $(X,\mathcal{U})$ is strongly uniformly paracompact. □

3.2 Uniformly paracompact locales

We devote this section to studying uniform paracompactness in locales according to influential results of Dube and Naidoo [10]. We show that uniform paracompactness implies the following: Cauchy completeness, Property P and that it is equivalent to strong Cauchy completeness. We end the section with the definition of uniform countable paracompactness which we show to be stronger than countable paracompactness.

We recall [11] that a set $C \subset L$ is called a *cover* of a locale L if $\bigvee C = 1$. We denote by $Cov(L)$ the collection of all covers of L. If $C \in Cov(L)$, the *star of* $x \in C$ is the set

$$Cx = \bigvee \{y \in C \mid y \wedge x \neq 0\}.$$

Given covers $C, D \in Cov(L)$, we define $CD = \{Cx \mid x \in D\}$ and say C *star-refines* D *if* $CC \leq D$ (we denote this by $C \leq^* D$). If $\mathfrak{N} \subseteq Cov(L)$, we say x *is uniformly below* y*, written* $x \lhd_{\mathfrak{N}} y$*, if there exists* $C \in Cov(L)$ *such that* $Cx \leq y$.

A collection $\mathfrak{N} \subseteq Cov(L)$ is called a nearness on L if it satisfies the following conditions:

(a) $U, V \in \mathfrak{N} \Rightarrow U \wedge V \in \mathfrak{N}$;

(b) $U \in \mathfrak{N},\ U \leq V \Rightarrow V \in \mathfrak{N}$;

(c) $\mathfrak{N}$ is *admissible*: For each $x \in L$, we have

$$x = \bigvee y \quad (y \lhd_{\mathfrak{N}} x).$$

A nearness $\mathfrak{N}$ on L is called a *uniformity* if for every $V \in \mathfrak{N}$, there exists a $U \in \mathfrak{N}$ such that $U \leq^* V$, that is, $UU \leq V$. If necessary, we will denote a uniformity on L by $\mathfrak{N}_L$. For more on uniform frames, we refer to [11].

Unless explicitly stated otherwise, all our concepts are defined in [10].

Definition 3.2.1

(a) *A subset S of a uniform frame L is uniformly locally finite if there is a uniform cover $U \in Cov(L)$ such that $\{s \in S \mid s \wedge u \neq 0\}$ is finite for each $u \in U$.*

(b) *A uniform frame $(L,\mathcal{U})$ is uniformly paracompact if for every cover U of L there exists a uniformly locally finite cover V such that $V \leq U$.*

(c) *A filter F in L clusters if every cover of L has an element which meets every element of F. It is equivalent to saying F is clustered if and only if $\bigvee\{x^* \mid x \in F\} \neq 1$.*

(d) *A filter F in a uniform frame L is said to be weakly Cauchy complete if every weakly Cauchy filter clusters.*

(e) *A uniform frame L is strongly Cauchy complete if every weakly Cauchy filter clusters.*

Definition 3.2.2. *We define the sec-operator on a filter F of a frame L by*

$$Sec(F) = \{x \in L \mid x \wedge a \neq 0 \text{ for every } a \in F\}.$$

Thus, F is clustered precisely when $Sec(F)$ meets every cover of L.

For our next result (due to Dube and Naidoo [10]) on uniformly paracompact frame, we will need the following:

Theorem 3.2.3. *A uniformly paracompact frame is strongly Cauchy complete.*

Proof. We assume that (L, U) is uniformly paracompact with a weakly Cauchy filter F and $Cov(L)$. The proof will be complete if we can show that F clusters. To this end, by definition, we have that $A^{<\omega} \in \mathfrak{U}$ so that $A^{<\omega} \cap Sec(F) \neq \emptyset$ which implies that for some $B \subseteq_f A$ then for each $x \in A$ it holds that

$$0 \neq \Big(\bigvee B\Big) \wedge x = \bigvee_{b \in B} \Big(b \wedge x\Big).$$

Accordingly, there exists a $b \in B$ such that $b \wedge x \neq 0$, for every $x \in F$. From $b \in B$ and $B \subseteq_f A$, it follows that $b \in A$. This leads us to $A \cap Sec(F) \neq \emptyset$ and therefore F clusters, whence $(L, \mathfrak{U})$ is strongly Cauchy Complete, as was to be shown. □

We recall that if $C \in Cov(L)$, $\hat{C}$ is defined by

$$\hat{C} = \{x \in L \mid x \prec a \text{ for some } a \in C\}.$$

A uniform frame L satisfies property P if for any $C \in Cov(L), C^{<\omega} \in \mathcal{U}L$. The condition $C \in Cov(L), C^{<\omega} \in \mathcal{U}L.$ is referred to as property P.

We say a subset A of a uniform frame L is uniformly locally finite if there is a uniform cover U such that for each $u \in U$, the set $\{a \in A \mid a \wedge u \neq 0\}$ is finite. Such a uniform cover U is then called a witness for (the uniform local finiteness) of A.

Proposition 3.2.4. *A uniform frame is uniformly paracompact if and only if it satisfies property P.*

Proof. Suppose that (L, U) is uniformly paracompact and let $\mathcal{V}$ be a finitely additive open covering of L. We aim to show that L satisfies property P. In order to do this, we need to find a uniformly locally finite covering $\mathcal{W}$ of L such that $\mathcal{W} \leq \mathcal{V}$. Let $W \in \mathcal{V}$ and assume that W is a witness of $\mathcal{W}$. Define $S = \{t \in U \mid t \wedge w \neq 0\}$, where $w \in W$. Then S is finite. Now, we can say that

$$w \wedge \bigvee \mathcal{W} = w \wedge \bigvee S \leq \bigvee S.$$

This proves that $W \leq \mathcal{W}^{<\omega}$, which implies that $\mathcal{W}^{<\omega} \leq \mathcal{V}^{<\omega}$. Since W is a uniform cover, it follows that (L, U) satisfies property P.

Conversely, suppose that $\mathcal{V}$ is a cover of a uniform space (L, U) that satisfies property P. Our goal is to show that (L, U) is uniformly paracompact. By Proposition 7 in [6], (L, U) is uniformly paracompact. Since (L, U) is paracompact, there exists a locally finite cover $\mathcal{W}$ of L such that $\mathcal{W} \leq \mathcal{V}$. Let W be a witness of $\mathcal{W}$. By the definition of property P, $W^{<\omega}$ is a uniform cover. Given $x \in W^{<\omega}$, let $x = x_1 \vee \cdots \vee x_m$. For each $i = 1, \ldots, m$ with $i \in \mathbb{N}$, let $\mathcal{V}_i$ be the finite subset of $\mathcal{W}$ consisting of those members of $\mathcal{W}$ that meet x_i. Then $\mathcal{V}_1 \bigcup \cdots \bigcup \mathcal{V}_m$ is a finite subset of $\mathcal{W}$ such that $x \wedge t = 0$ for $t \in \mathcal{W}/\mathcal{V}$. This proves that each element of the uniform cover $W^{<\omega}$ meets only finitely many elements of $\mathcal{W}$, which implies that $\mathcal{W}$ is uniformly locally finite. Hence, (L, U) is uniformly paracompact. $\square$

Proposition 3.2.5. *A uniform frame is uniformly paracompact if and only if it is strongly Cauchy complete.*

Proof. Suppose that L is a strongly Cauchy complete uniform frame, and let $C \in Cov(L)$. We aim to show that $C^{<\omega}$ is a uniform cover. Let us take a $\hat{C} \in Cov(L)$. If $\bigvee B = 1$ for some $B \subseteq_f \hat{C}$, then $\bigvee C^{<\omega} = 1$, which implies that $C^{<\omega}$ is a uniform cover. On the other hand, if we assume that $\bigvee B \neq 1$ for each $B \subseteq_f \hat{C}$, and define a set $A \subseteq L$ by

$$A = \left\{ \left(\bigvee B \right)^* \mid B \subseteq_f \hat{C} \right\},$$

then A generates a proper filter F in L. Indeed, if B_1 and B_2 are finite sets, then $B_1 \cup B_2$ is also finite and

$$\left(\bigvee B_1 \right)^* \wedge \left(\bigvee B_2 \right)^* = \left(\bigvee (B_1 \cup B_2) \right)^*.$$

For any $c \in \hat{C}$, it holds that $c^* \in F$. Since $c \wedge c^* = 0$, it follows that $Sec(F) \wedge \hat{C} = 0$, implying that F does not cluster. Therefore, it cannot be weakly Cauchy because L is strongly Cauchy complete. Thus, there exists a uniform cover $U \in Cov(L)$ such that $Sec(F) \cap U = \emptyset$. Take $u \in U$. Since $u \notin Sec(F)$, we have $u \wedge x = 0$ for some $x \in F$. So, there exist finitely many $x_1, \ldots, x_n \in hatC$ such that

$$u \wedge (x_1 \vee \ldots \vee x_n)^* = 0.$$

To complete the proof, let $x_i \prec c_i$ for each $i = 1, \ldots, n$. We conclude that

$$u \leq (x_1 \vee \ldots \vee x_n)^{**} \leq c_1 \vee \ldots \vee c_n,$$

which shows that the cover U refines $C^{<\omega}$, and therefore $C^{<\omega}$ is a uniform cover of L. Consequently, L is uniformly paracompact. □

We will define $p\mathcal{N}$ as follows $\{c \in cov(L) \mid u \leq c$ for some finite$u \in \mathcal{N}\}$.

Corollary 3.2.6. *A uniform frame $(L, \mathcal{N})$ is uniformly paracompact if and only if L is paracompact and $\mathrm{Cov}(L) = p\mathcal{N}/\mathcal{N}$.*

Proof. According to the two previous Proposition 3.2.4 and Proposition 3.2.5, it suffices to show that $\mathcal{F}_L = p\mathcal{F}_L = \mathcal{N}$. By paracompactness, we have that $p\mathcal{F}_L/\mathcal{N} \subseteq \mathcal{F}_L$. On the other hand, $\mathcal{F}_L \subseteq Cov(L) = p\mathcal{N}/\mathcal{N} \subseteq p\mathcal{F}_L/\mathcal{N}$.

Conversely, L is paracompact as it is the underlying frame of a uniformly paracompact uniform frame. Let A be a cover of L and define $B = \{x \in L \mid x \prec a$ for some $a \in A\}$. By admissibility, B is a cover of L, and thus, by Proposition 3.2.4, $B^{<\omega}$ is a uniform cover of L. Let $\omega = x_1 \vee \ldots \vee x_m$ be an arbitrary element of $B^{<\omega}$, and let $a_1, \ldots, a_m$ be elements of A with $x_i \prec a_i$ for $i = 1, \ldots, m$. Now, $A_\omega = \{x_1^* \wedge \ldots \wedge x_m^*\} \cup \{a_1, \ldots, a_m\}$ is a finite uniform cover of L, as each of the covers $\{x_i^*, a_i\}$ is uniform and their meet refines A_ω.
Now, consider the cover A_ω constructed above for each $\omega \in B^{<\omega}$. Then $B^{<\omega} \wedge (A_\omega)_{B^{<\omega}} \leq A$. However, the former cover is in $p\mathcal{N}/\mathcal{N}$. This demonstrates that $Cov(L) = p\mathcal{N}/\mathcal{N}$. □

Remark 3.2.7. *A uniformly paracompact uniform frame is complete.*

Proof. Isbell [18, Theorem 3.9] has shown that a uniform frame is complete if and only if its underlying frame is paracompact and its locally fine reflection is fine. Therefore, if $(L, \mathcal{N})$ is a uniformly paracompact uniform frame, then by Corollary 3.2.6, we know that L is paracompact and $U(\gamma, L) \supseteq p\mathcal{N}/\mathcal{N} = Cov(L)$. The result follows directly from Corollary 3.2.6. $\square$

Definition 3.2.8. *A frame homomorphism $h : L \to M$ is called perfect if $h_\star$ preserves directed joins.*

Following [14], we call a frame homomorphism $h : L \to M$ perfect if h_* preserves directed joins. Furthermore, a uniform frame is uniformly paracompact if and only if every directed cover is uniform. This implies that the following result holds easily.

Proposition 3.2.9. [10] *Let $h : L \to M$ be a perfect uniform homomorphism. If L is uniformly paracompact, then M is uniformly paracompact.*

Proof. The proof is technically similar to that of Theorem 4.2.7. $\square$

Definition 3.2.10. A uniform frame is *uniformly countably paracompact* if every countable uniform cover of the underlying frame has a uniformly locally finite refinement.

Lemma 3.2.11. *The underlying frame of a uniformly countably paracompact uniform frame is countably paracompact.*

Proof. The proof follows directly by modifying the proof of the implication $(\Rightarrow)$ in Corollary 3.2.6, which demonstrates that the underlying frame of a uniformly paracompact uniform frame is paracompact. In this case, we simply start with a countable cover instead of an arbitrary cover. □

Recall that a frame L is *completely regular* if $x = \bigvee\{y \in L \mid y \prec\prec x\}$ for each $x \in L$, where $y \prec\prec x$ if and only if $a_0 = y,\ a_1 = x$ and $a_p \prec\ a_q$ if $p < q$ for some binary relation $\prec$ on L. A *compactification* of a frame L is a pair (M, ρ), where M is a compact regular frame and $\rho : M \to L$ is a dense onto frame homomorphism.

A uniformly paracompact frame $(L, \mathfrak{U})$ is completely regular (being paracompact) and thus has a compactification, namely, its Stone-Čech compactification, denoted by βL.

Briefly, we take a completely regular frame L. We say an ideal J of L is *completely regular* if, whenever $x \in J$, there exists a $y \in J$ satisfying $y \prec\prec x$. The Stone-Čech compactification of L is the pair $(\beta L, j_L)$, where βL consists of completely regular ideals of L, and $j_L : \beta L \to L$ mapping $J \mapsto \bigvee J$ is a dense onto frame homomorphism. The homomorphism j_L has a right adjoint $r_L : L \to \beta L$ which is given by

$$r_L(x) = \{y \in L \mid y \prec\prec x\},$$

for any $x \in L$. For further details, we refer to [19].

The notation above is used in the proof of the following relationship between uniform paracompactness and Stone-Čech compactification [10].

Proposition 3.2.12 *If L is a uniform frame, then the following statements are equivalent:*

(a) *L is uniformly paracompact.*

(b) *For any compactification $h : M \to L$ of L, if $c \in M$ is such that $h(c) = 1$, there is a uniform cover U of L such that $h_*(u) \prec\prec c$ for every $u \in U$.*

(c) *For any $c \in \beta L$ such that $j_L(c) = 1$, there is a uniform cover U of L such that $r_L(u) \prec\prec c$ for every $u \in U$.*

Proof. *(a) $\implies$ (b)*: Let $h : M \to L$ be a compactification of L and c an element of M with $h(c) = 1$. Set $A = \{x \in M \mid x \prec\prec c\}$. Then $h[A]$ is a cover of L. By the properties of the completely below relation, we have that $h[A]^{\leq\omega} = h[A]$, so that, by uniform paracompactness, $h[A]$ is a uniform cover of L. We show that it has the claimed property. Let $a \in A$, and pick $b \in L$ with $a \prec\prec b \prec\prec c$. Then $h_*h(a) \leq b \prec\prec c$. Since $h(a)$ is an arbitrary element of $h[A]$, the result follows.

(b) $\implies$ (c): This is trivial.

(c) $\implies$ (a): Let A be a cover of L. Put $c = \bigvee\{r_L(a) \mid a \in A\}$, and observe that

$$j_L(c) = \bigvee\{j_L r_L(a) \mid a \in A\} = 1.$$

Let U be a uniform cover of L with the hypothesized property. Consider any $u \in U$. By hypothesis, $r_L(u) \prec\prec c$, so that $r_L(u)^* \vee \bigvee\{r_L(a) \mid a \in A\} = 1$. By compactness of βL, there are finitely many elements $a_1, \ldots, a_m$ in A such that $r_L(u)^* \vee r_L(a_1) \vee \ldots \vee r_L(a_m) = 1$, which implies $r_L(u) \leq r_L(a_1) \vee \ldots \vee r_L(a_m)$, whence $u \leq a_1 \vee \ldots \vee a_m$. So U refines $A^{<\omega}$, and therefore L is uniformly paracompact. $\square$

Chapter 4

On strong uniform paracompactness

In this chapter, our primary focus is on the exploration of strongly uniformly paracompact spaces, as presented in the seminal work by Zhanakunova *et al* [38] and Kanetov *et al* [20]. *Note:* For the purpose of clarity and consistency, unless explicitly mentioned otherwise, any space denoted by X throughout this chapter is presumed to be a uniform space characterized by a specific uniformity $\mathcal{U}_X$.

4.1 Strongly uniformly paracompact uniform spaces

We study strong uniform paracompactness of uniform spaces according to [23]. It is shown that strong uniform R-paracompactness is equivalent to an additive open covering having an open uniformly star-finite refinement. Uniformly perfect mappings are defined in terms of paracompact and perfect mappings, which we use to show that they preserve strong uniform R-paracompactness. Additionally, we show that uniform star-finiteness is closed under finite intersections, and it is reflected by uniformly continuous mappings. We close the section by showing that strong uniform R-paracompact mappings compose.

Definition 4.1.1. [38] *Let $\mathcal{V}$ be a cover of the uniform space $(X, \mathcal{U})$. The cover $\mathcal{V}$ is said to be:*

(a) *uniformly star-finite if there exists a uniform cover $W \in \mathcal{U}$ such that every $\bigcup St(\mathcal{V}, W)$*

intersects $\mathcal{V}$ in only a finite number of elements of $\mathcal{V}$;

(b) *uniformly locally finite if there exists a uniform cover $W \in \mathcal{U}$ such that every $W \in \mathcal{W}$ intersects $\mathcal{V}$ in only a finite number of elements of $\mathcal{V}$.*

The cover that consists of the union of a countable number of (uniformly) locally finite families is called (uniformly) σ-locally finite. Star-finitely uniformly σ-locally finite covers will be referred to as uniformly σ-star-finite.

Definition 4.1.2. [38] *A uniform space (X, U):*

(a) is called uniformly R-paracompact if every open cover has an open uniformly locally finite refinement.

(b) is called strongly uniformly R-paracompact if every open cover has an open uniformly star-finite refinement.

Consider a uniform space $(X,\mathcal{U})$ where $\mathcal{U}$ is a diagonal uniformity. This space induces a uniform topology $\tau_{\mathcal{U}}$ on X. The topology $\tau_{\mathcal{U}}$ is defined as:

$$\tau_{\mathcal{U}} = \{V \subset X \mid \forall x \in V,\ \exists u \in U \text{ such that } U(x) \subset V\}.$$

Here, $U(x)$ is given by:

$$U(x) = \{y \in X \mid (x, y) \in U\}.$$

Proposition 4.1.4. *If $(X,\mathcal{U})$ is a strongly uniformly R-paracompact space then the topological space $(X, \tau_{\mathcal{U}})$ is strongly uniformly R-paracompact. Conversely, if (X, τ_X) is strongly paracompact then the uniform space $(X,\mathcal{U}_X)$ is strongly uniformly R-paracompact.*

Proof. Suppose that $(X,\mathcal{U})$ is a strongly uniformly R-paracompact space. Let $(X, \tau_{\mathcal{U}})$ be the associated topological space and take any open cover $\mathcal{V}$ of $(X, \tau_{\mathcal{U}})$. Then there exists a uniformly star-finite open covering $\mathcal{W}$ which is a refinement of $\mathcal{V}$.

Let B be an element of $\mathcal{W}$. Then there exists K in $\mathcal{V}$ such that $B \cap K \neq \emptyset$. Then $B \subseteq \mathcal{W}(K)$ and the set $\mathcal{W}(K)$ meets $\mathcal{W}$ only for a finite number of elements of $\mathcal{W}$. Hence, $\mathcal{W}$ is star-finite. Since the space $(X, \tau_{\mathcal{U}})$ is strongly uniformly R-paracompact, the space $(X, \tau_{\mathcal{U}})$ is strongly paracompact.

Conversely, if $(X, \tau_{\mathcal{X}})$ is strongly paracompact, then the set of all open coverings forms a base of universal uniformity $\mathcal{U}_X$ of the space $(X, \tau_{\mathcal{X}})$. Thus, $(X, \mathcal{U}_X)$ is strongly uniformly R-paracompact. □

Consider a covering $\mathcal{U}$ of a space X. The covering is said to be *finitely additive* if $\mathcal{U}^{\angle} = \mathcal{U}$, where

$$\mathcal{U}^{\angle} = \left\{ \bigcup \mathcal{U}_0 : \mathcal{U}_0 \subset \mathcal{U} \text{ and is finite} \right\}.$$

In the next result, we show that strong uniform R-paracompactness is a hereditary property for closed subspaces.

Theorem 4.1.5. *A closed uniform subspace of a strongly uniformly R-paracompact space is also strongly uniformly R-paracompact.*

Proof. Let us assume that the space $(X, \mathcal{U})$ is strongly uniformly R-paracompact. Consider a closed uniform subspace $(A, \mathcal{U}_A)$ with the induced uniformity $\mathcal{U}_A$. Begin with an open covering $\mathcal{V}_A$ of this subspace. There exists an open covering $\mathcal{V}$ of the space $(X, \mathcal{U})$ such that

$$\mathcal{V} \cap \{A\} = \mathcal{V}_A.$$

Next, define $\mathcal{G}$ as the union of $\mathcal{V}$ and the set difference $X - A$, that is, $\mathcal{G} = \mathcal{V} \cup (X - A)$. This forms an open covering of the strongly uniformly R-paracompact space $(X, \mathcal{U})$. By our initial assumption, there exists a uniformly star-finite open covering $\mathcal{W}$ of $(X, \mathcal{U})$ that refines $\mathcal{G}$. Consequently, $\mathcal{W}_A$ refines $\mathcal{V}_A$ because

$$\mathcal{W} \subseteq \mathcal{V} \Rightarrow \mathcal{W}_A = \mathcal{W} \cap A$$

$$\subseteq \mathcal{V} \cap A$$

$$\leq \mathcal{V}_A.$$

Moreover, $\mathcal{W}_A$ is a uniformly star-finite open covering of the closed subspace $(A,\mathcal{U}_A)$, which demonstrates that $(A,\mathcal{U}_A)$ is also strongly uniformly R-paracompact. □

Strong uniform R-paracompactness is hereditary for closed subspaces, in the sense that any closed uniform subspace of a strongly uniformly R-paracompact space will also possess this property.

Definition 4.1.6. [23] *A uniform space* $(X,\mathcal{U})$ *is called a* uniformly R-paracompact space*, if every open covering has an open uniformly locally finite refinement.*

Theorem 4.1.7. *If the uniform space* $(X,\mathcal{U})$ *is uniformly* R*-paracompact and underlying topological space* (X,τ_U) *is strongly paracompact, then the uniform space* $(X,\mathcal{U})$ *is strongly uniformly* R*-paracompact.*

Proof. Suppose that the space $(X,\mathcal{U})$ is uniformly R-paracompact and its underlying topological space (X,τ_U) is strongly paracompact. Let $\mathcal{V}$ be an arbitrary open covering of the space $(X,\mathcal{U})$. We can then find an open star-finite covering $\mathcal{W}$ such that $\mathcal{W} \leq \mathcal{V}$. There exists an open uniformly locally finite cover $\mathcal{S}$ such that $\mathcal{S} \leq \mathcal{V}$. Since $\mathcal{S}$ is uniformly locally finite, there exists a uniform cover $\lambda \in \mathcal{U}$ such that every $L \in \lambda$ intersects $\mathcal{S}$ in only a finite number of elements. That is, the cardinality of the family $St(L,\mathcal{S})$ is finite for each $L \in \lambda$.

For a given $L \in \lambda$ and $\Gamma \in St(L,\mathcal{S})$, since $\mathcal{S} \leq \mathcal{V}$, for any $\Gamma \in \mathcal{S}$ there exists $B \in \mathcal{V}$

such that $\Gamma \subset B$. Due to the star finiteness of the cover $\mathcal{W}$, the cardinality of the family $St(B, \mathcal{V})$ is finite for each $B \in \mathcal{V}$. Moreover, the cardinality of the family $St(\Gamma, \beta)$ is finite for each $\Gamma \in St(L, \gamma)$. Thus, the cardinality of the family $St(\gamma(L), \mathcal{V})$ is finite. Consequently, the cardinality of the family $St(L, \mathcal{V})$ is finite for each $L \in \lambda$. It can be easily verified that $St(\beta(L), \mathcal{V})$ is finite. Since every $\mathcal{V}(L)$ intersects $\mathcal{V}$ in only a finite number of elements, the uniform space $(X,\mathcal{U})$ is strongly uniformly R-paracompact. □

Theorem 4.1.8. *Any strongly uniformly R-paracompact space is strongly uniformly paracompact.*

Proof. Let $(X,\mathcal{U})$ be a strongly uniformly R-paracompact space. Then, the space $(X,\mathcal{U})$ is uniformly paracompact, and its associated topological space $(X, \tau_{\mathcal{U}})$ is strongly paracompact. Therefore, $(X,\mathcal{U})$ is strongly uniformly paracompact. □

Lemma 4.1.9. *A covering $\mathcal{V}$ of a uniform space $(X,\mathcal{U})$ is uniformly star-finite if and only if it is uniformly locally finite and star-finite.*

Proof. Uniformly locally finite follows directly from the definition of uniformly star-finite. Now, let $A \in \mathcal{V}$ be an arbitrary element. There exists $B \in \beta$ such that $A \cap B \neq \emptyset$. Then $A \subset \mathcal{V}(B)$ and the set $\mathcal{V}(B)$ intersects $\mathcal{V}$ in only a finite number of elements of $\mathcal{V}$. Hence, $\mathcal{V}$ is star-finite.

Conversely, let $\mathcal{V}$ be a uniformly locally finite and star-finite covering. There exists a uniform covering $\beta \in \mathcal{U}$ such that every $B \in \beta$ intersects $\mathcal{V}$ in only a finite number of elements of $\mathcal{V}$, i.e., there exists $A_i(B) \in \mathcal{V}$ such that $B \subset \bigcup_{i=1}^{n} A_i(B)$. By virtue of the star finiteness of the cover $\mathcal{V}$, every $A_i(B)$ intersects $\mathcal{V}$ in only a finite number of elements of $\mathcal{V}$. Then $\mathcal{V}(B)$ also intersects $\mathcal{V}$ in only a finite number of elements of $\mathcal{V}$. Thus, the cover $\mathcal{V}$ of the uniform space $(X,\mathcal{U})$ is uniformly star-finite. □

Theorem 4.1.10. *A uniform space $(X,\mathcal{U})$ is strongly R-paracompact if and only if every* *open covering has an uniformly locally finite and star-finite and star-finite refinement.*

Proof. We refer to [3, Lemma 1] which is in Russian.

Definition 4.1.11. *An onto uniformly continuous mapping $f : (X,\mathcal{U} \to (Y,\mathcal{V})$ between uniform spaces is said to be precompact if, whenever $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ and a finite uniform covering $W \in \mathcal{U}$ such that $f^{-1}(V) \wedge W \leq U$. An onto uniformly continuous mapping $f : (X,\mathcal{U} \to (Y,\mathcal{V})$ is called a uniformly perfect if it is both precompact and perfect.*

[3, Lemma 1 and Theorem 2.3.9], page 155 imply the following theorem.

Theorem 4.1.12. *Let $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a perfect mapping of a uniform space $(X,\mathcal{U})$) onto a uniform space (Y,V). If $(Y,\mathcal{V})$) is strongly uniformly R-paracompact space, then the uniform space $(X,\mathcal{U})$ is also strongly uniformly R-paracompact.* $\square$

Since a uniformly perfect mapping is perfect, the following is immediate.

Corollary 4.1.13. *Let $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a uniformly perfect mapping of a uniform space $(X,\mathcal{U})$ onto a uniform space $(Y,\mathcal{V})$. If $(Y,\mathcal{V})$ is strongly uniformly R-paracompact space then the uniform space $(X,\mathcal{U})$ is also strongly uniformly R-paracompact.* $\square$

A uniform space $(X,\mathcal{U})$ is called strongly paracompact if every open covering of $(X,\mathcal{U})$ has a uniformly σ-star-finite open refinement (See [23]). A uniformly continous mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between uniform spaces is strongly uniformly R-paracompact if, whenever α is an open covering of X, there exists an open covering β of Y and a uniformly star-finite open covering γ of X such that $f^{-1}(\beta) \wedge \gamma \leq \alpha$.

Strong uniform R-paracompactness of a uniform space relates to strong uniform R-compactness of uniformly continuous mappings as follows. The proof is adapted from that of kanetov and Baidzhuranova [23].

Theorem 4.1.14. [38] *If $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a uniformly continuous mapping and $(X,\mathcal{U})$ is a strongly uniformly R-paracompact space, then the mapping f is a strongly uniformly R-paracompact mapping.*

Proof. Given a uniformly continuous mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ on a strongly uniformly R-paracompact uniform space $(X,\mathcal{U})$, suppose that α is an open covering of $(X,\mathcal{U})$. By definition, there exists a uniformly star-finite open covering γ of $(X,\mathcal{U})$ such that $\gamma \leq \alpha$. Now suppose β is an open covering of $(Y,\mathcal{V})$. Then we have

$$\bigcup f^{-1}(\beta) = f^{-1}\left(\bigcup \beta\right) = f^{-1}(Y) = X,$$

which shows that $f^{-1}(\beta)$ is an open covering of $(X,\mathcal{U})$. We claim that $f^{-1}(\beta) \cap \gamma \leq \alpha$. For any $B \in \beta$ and $Y \in \gamma$, we find that (since $\gamma \leq \alpha$)

$$f^{-1}(B) \cap Y \subseteq f^{-1}(B) \cap A,$$

for some $A \in \alpha$. Consequently, the mapping f is strongly uniformly R-paracompact, completing the proof. □

Proposition 4.1.15. [38] *If $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is an onto uniformly continuous mapping where $Y = \{y\}$ is strongly uniformly R-paracompact, then $(X,\mathcal{U})$ is strongly uniformly R-paracompact.*

Proof. Let f be a mapping that is strongly uniformly R-paracompact, and let λ be an arbitrary open covering of the space $(X,\mathcal{U})$. Then there exists an open covering β of the space $(Y,\mathcal{V})$ and a uniformly star-finite open covering α of the space $(X,\mathcal{U})$ such that the covering $f^{-1}(\beta) \wedge \alpha$ refines λ. Since $Y = \{y\}$, we have $f^{-1}(\beta) \wedge \alpha = \alpha$. Thus, $(X,\mathcal{U})$ is strongly uniformly R-paracompact. □

Lemma 4.1.16. *If $\mathcal{V}$ and $\mathcal{W}$ is uniformly star-finite covering of the space $(X,\mathcal{U})$, then $\mathcal{V} \wedge \mathcal{W}$*

is uniformly star-finite cover of the uniform space $(X,\mathcal{U})$.

Proof. Suppose that $\mathcal{V}$ and $\mathcal{W}$ are uniformly star-finite covers of the space $(X,\mathcal{U})$. We aim to show that $\mathcal{V}\wedge\mathcal{W}$ is also a uniformly star-finite cover of $(X,\mathcal{U})$.

Given that $\mathcal{V}$ and $\mathcal{W}$ are uniformly star-finite, there exist uniform covers $\mu\in\mathcal{U}$ and $\eta\in\mathcal{U}$ such that

$$\mathcal{V}(M)\subset\bigcup_{i=1}^{n}A_i \quad \text{and} \quad \mathcal{W}(N)\subset\bigcup_{j=1}^{m}B_j$$

for each $M\in\mu$ and $N\in\eta$.

Since $M\cap N\in\mu\wedge\eta$, we have

$$(\mathcal{V}\wedge\mathcal{W})(M\cap N)\subset\mathcal{V}(M)\cap\mathcal{W}(N).$$

It is evident that $\mu\wedge\eta$ is a uniform covering. Therefore, the covering $\mathcal{V}\wedge\mathcal{W}$ is uniformly star-finite. □

Lemma 4.1.17. [38] *Let* $f:(X,\mathfrak{U})\to(Y,\mathfrak{V})$ *be an onto uniformly continuous mapping. If* $\mathfrak{W}$ *is a uniformly star-finite open cover of* $(Y,\mathfrak{V})$*, then* $f^{-1}(\mathfrak{W})$ *is a uniformly star-finite open cover of* $(X,\mathfrak{U})$.

Proof. First, $f(X)=Y$ and $\bigcup\mathfrak{W}Y$ imply that

$$\bigcup f^{-1}(\mathfrak{W})=f^{-1}(\bigcup\mathfrak{W})=f^{-1}(Y)=f^{-1}\circ f(X)=f^{-1}(Y)=X,$$

which shows that $f^{-1}(\mathfrak{W})$ is a covering of the space X. We take a uniformly star-finite covering $\mathfrak{W}$ of Y and pick a cover $\mathfrak{S}\in\mathfrak{V}$ for which the number of elements in $St(\mathfrak{S}(V),\mathfrak{W})$ is finite, for all $W\in\mathfrak{W}$; thus, for each $W\in\mathfrak{W}$, there exists a finite number of elements $W_i\in\mathfrak{W}$ for $i=1,\dots,n$ such that $\mathfrak{S}(W)\subset\bigcup_{i=1}^{n}W_i$. By hypothesis, f is uniformly continuous, so

$$\bigcup f^{-1}(\mathfrak{S}) = f^{-1}\left(\bigcup \mathfrak{S}\right) = f(Y) = X,$$

that is, $f^{-1}(\mathfrak{S})$ is a covering of X. Now, from $\mathfrak{S}(W) \subset \bigcup_{i=1}^{n} W_i$, it follows that $f^{-1}(\mathfrak{S}(W)) \subset \bigcup_{i=1}^{n} f^{-1}(W_i)$, and then $f^{-1}(W_i) \in f^{-1}(\mathfrak{W})$. Accordingly, the covering $\mathfrak{W}$ of X is uniformly star-finite. $\square$

Theorem 4.1.18. [23] *Strongly uniformly R-paracompact mappings are closed under composition.*

Proof. Consider two strongly uniformly R-paracompact mappings: $f : (X, \mathfrak{U}_X) \to (Y, \mathfrak{U}_Y)$ and $g : (Y, \mathfrak{U}_Y) \to (Z, \mathfrak{U}_Z)$. Let α be an open covering for X. We find an open covering β of Y and a uniformly star-finite covering γ such that

$$f^{-1}(\beta \wedge \gamma) \leq \alpha.$$

From the strong uniform R-paracompactness of g and the fact that β is an open covering of Y, we find an open covering λ of Z such that

$$g^{-1}(\lambda) \wedge \eta \leq \beta.$$

Since γ is uniformly star-finite, it follows that $f^{-1}(\eta)$ is also uniformly star-finite by Lemma 4.1.17. By Lemma 4.1.16, we infer that

$$f^{-1}(\eta) \wedge \gamma \text{ is uniformly star-finite.} \qquad (*)$$

We also have the following calculation:

$$\begin{aligned} & g^{-1}(\lambda) \wedge \eta \leq \beta && (**) \\ \Rightarrow & f^{-1}[g^{-1}(\lambda) \wedge \eta] \leq f^{-1}(\beta) \\ \Rightarrow & (g \circ f)^{-1}(\lambda) \wedge f^{-1}(\eta) \leq f^{-1}(\beta) \\ \Rightarrow & (g \circ f)^{-1}(\lambda) \wedge [f^{-1}(\eta) \wedge \gamma] \leq f^{-1}(\beta) \wedge \gamma \\ \Rightarrow & \alpha \leq \alpha, \end{aligned}$$

Thus, $(g \circ f)^{-1}(\lambda) \wedge [f^{-1}(\eta) \wedge \gamma] \leq \alpha$. By $(*)$ and $(**)$, we conclude that the composition $g \circ f$ is strongly uniformly R-paracompact, as desired. □

Theorem 4.1.19. [38] *If $f : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ is strongly R-paraccompact and $(Y, \mathcal{V})$ is strongly uniformly R-paracompact, then the uniform space $(X, \mathfrak{U})$ is strongly uniformly R-paracompact.*

Proof. We assume the hypothesis of the theorem and let $\mathfrak{A}$ be a cover of X. Since f is strongly uniformly R-paracompact, there is an open cover $\mathfrak{K}$ of Y and a uniformly star-finite open cover $\mathfrak{C}$ of X such that

$$f^{-1}(\mathfrak{K}) \cup \mathfrak{C} \leq \mathfrak{A}.$$

On the other hand, Y is strongly uniformly R-paracompact, so we find a uniformly star-finite open cover $\mathfrak{K}_0$ of Y such that $\mathfrak{K}_0 \leq \mathfrak{K}$. We claim that $f^{-1}(\mathfrak{K}_0) \wedge \mathfrak{C} \leq f^{-1}(\mathfrak{K}) \wedge \mathfrak{C}$: for if $K_0 \in \mathfrak{K}_0$ and $C \in \mathfrak{C}$, then

$$f^{-1}(K_0) \cap C \subseteq f^{-1}(K) \cap C,$$

for some $K \in \mathfrak{K}$, which proves our claim. By Lemma 4.1.17, we have that $f^{-1}(\mathfrak{K}_0)$ is uniformly star-finite. By Lemma 4.1.16, we conclude that the meet $f^{-1}(\mathfrak{K}_0) \wedge \mathfrak{C}$ is a uniformly star-finite open cover refining $\mathfrak{A}$, whence $(X, \mathfrak{U})$ is strongly uniformly R-paracompact. □

4.2 Strongly uniformly paracompact uniform frames

This section is devoted to a study of topological results of Kanetov and Zhanakunova [23] in the context of locales. We attempt to define strongly uniformly R-paracompact locales in an obvious way and present some pointfree results related to theirs in the general setting of strong uniform paracompactness. Amongst the results we establish are: in the presence of a uniformly star-finite refinement in an additive cover, a uniform frame is strongly uniformly R-paracompact; surjective proper localic maps and strongly uniformly R-paracompact maps on uniform locales preserve strong uniform R-paracompactness.

Definition 4.2.1. *A cover $\mathfrak{U}$ of a uniform frame $(L, \mathfrak{N}_L)$ will be called:*

(a) Uniformly locally finite if there exists $V \in \mathfrak{N}_L$ such that for each $v \in V$, the set

$$\{u \in U \mid u \wedge v \neq 0\}$$

is finite.

(b) Uniformly star-finite if there exists $V \in \mathfrak{N}_L$ such that for each $v \in V$, the set

$$\left\{u \in U \mid \bigvee(Uu) \wedge v \neq 0\right\}$$

is finite.

(c) If U is finite, we will write U_f. We denote by $U^{\angle}$ the set

$$\left\{\bigvee U_f \mid U_f \subset U\right\}.$$

The cover U is finitely additive *if $U^{\angle} = U$.*

Definition 4.2.2. *A uniform frame $(L, \mathfrak{N}_L)$ is said to be* uniformly strongly paracompact *(we use usp for short) every cover of L has a uniformly σ-star-finite refinement. A uniform frame*

$(L;\mathfrak{N}_L)$ is strongly uniformly R-paracompact *if every cover of L has a uniformly star-finite refinement.*

Recall that *a closed sublocale* of a frame L determined by $c \in L$ is the frame homomorphism

$$\breve{c} : L \to \uparrow c,\ u \mapsto u \vee c.$$

Lemma 4.2.3. *If $(L,\mathfrak{N})$ is a strongly uniformly R-paracompact and $\uparrow c$ is a closed sublocale determined by c, then $\uparrow c$ is also strongly uniformly R-paracompact.* □

Lemma 4.2.4. *If $(L,\mathfrak{U})$ is strongly uniformly R-paracompact, then it is uniformly strongly paracompact.* □

Lemma 4.2.5. *The underlying frame of a strongly uniformly R-paracompact uniform frame is strongly uniformly R-paracompact frame .* □

We observe that if $(L,\mathfrak{N}_L)$ is strongly uniformly R-paracompact and $\mathfrak{A} \in Cov(L)$ is finitely additive, then $\mathfrak{A}$ is has a uniformly star-finite refinement. In the following result, we show that the converse holds as well.

Theorem 4.2.6. *If every finitely additive cover of a uniform frame has a uniformly star-finite refinement, then the uniform frame is strongly uniformly R-paracompact.*

Proof. We start with $\mathfrak{A} \in Cov(L)$ in a uniform frame $(L,\mathfrak{N}_L)$ and observe that $\mathfrak{A}^{\angle}$ is finitely additive. Without loss of generality, we can find a uniformly star-finite $\mathfrak{F} \in Cov(L)$ satisfying $\mathfrak{F} \leq \mathfrak{A}^{\angle}$. Using the finiteness of $\mathfrak{A}^{\angle}$, we consider elements $u \in \mathfrak{A}$ of the form

$$u = u_1 \vee u_2 \vee \ldots \vee u_n,$$

where each $u_i \in \mathfrak{F}$. Now, for each $v \in \mathfrak{A}$, we select a $u \in \mathfrak{A}^{\angle}$ such that $v \leq u$. Then the finite collection

$$\mathfrak{A}_v = \{u_1 \wedge v, u_2 \wedge v, \dots, u_n \wedge v\}$$

satisfies: (a) $\mathfrak{A}_v \leq \mathfrak{A}$: this is straightforward. (b) $\bigvee \mathfrak{A}_v \leq \mathfrak{A}$: this follows from the fact that each $u_i \wedge v \leq v$, for each $u_i \in \mathfrak{F}$, for $i = 1, \dots, n$. □

Recall that a localic map $h : X \to Y$ between locales is said to be *proper* if it is closed and the right adjoint $h_* : \mathcal{O}X \to \mathcal{O}Y$ of the frame morphism $h^* : \mathcal{O}Y \to \mathcal{O}X$ preserves directed joins. This is a pointfree version of the classical notion of what are called *perfect maps* in the category of topological and continous functions(see Johnstone [19])

Theorem 4.2.7. *Let $f : (L, \mathfrak{N}_L) \to (M, \mathfrak{M}_M)$ be a proper localic map between uniform locales. If M is strongly uniformly R-paracompact, then so is L.*

Proof. Take $\mathcal{K} \in \mathrm{Cov}(L)$ and note that $\bigvee \mathcal{K} = 1_L$, hence

$$\bigvee h[\mathcal{K}] = h\left(\bigvee \mathcal{K}\right) = h(1_L) = 1_M.$$

Thus, $h[\mathcal{K}] \in \mathrm{Cov}(M)$. For the localic map $h\colon L \to M$, we have a frame homomorphism $h^*\colon M \to L$ whose associated right adjoint is $h_*\colon L \to M$. Now, since $(M, \mathfrak{M}_M)$ is strongly uniformly R-paracompact, there exists a $V \in \mathfrak{M}_M$ such that for each $v \in V$, the set

$$\left\{h_*(k) \in h_*[K] \mid \bigvee(h_*[K])h_*(k) \wedge v \neq 0, \quad K \in \mathcal{K}\right\}$$

is finite, where h_* is the right adjoint of $h^*\colon M \to L$. Since h_* preserves directed joins because h is a proper map and onto, it follows that the set

$$\left\{k \in K \mid \bigvee(K_k) \wedge f^*(v) \neq 0\right\}$$

is finite, where $f^*(v) \in f^*[V]$ and $f^*[V] \in \mathfrak{N}_L$. That is, $\mathcal{K}$ has a uniformly star-finite refinement. Then $(L, \mathfrak{N}_L)$ is a strongly uniformly R-paracompact locale. □

Recall that a frame homomorphism $h : (M, \mathfrak{N}_M) \to (L, \mathfrak{N}_L)$ is called a *quotient map* if it is surjective (that is, onto) and

$$\mathfrak{N}_L = \{h[V] \mid V \in \mathfrak{N}_M\}.$$

Definition 4.2.8. *A quotient map $h : (M, \mathfrak{N}_M) \to (L, \mathfrak{N}_L)$ between uniform frames is said to be strongly uniformly R-paracompact if, whenever $\mathfrak{U}$ is a cover of L, there exists a cover $\mathfrak{K}$ of M and a uniformly star-finite cover $\mathfrak{H}$ of L such $h[\mathfrak{K}] \wedge \mathfrak{H} \leq \mathfrak{U}$.*

In the following result, we show that uniform star-finiteness of a cover is preserved by a quotient map which establishes the pointfree version of Zhanakunova and Kanetov's result [38].

Lemma 4.2.9. *If $h : (M, \mathfrak{N}_M) \to (L, \mathfrak{N}_L)$ is a quotient map and $\mathfrak{K}$ is a uniformly star-finite cover of M, then $h(\mathfrak{K})$ is a uniformly star-finite cover of L.*

Proof. Indeed, $h[K]$ is a cover because

$$\bigvee h[K] = h\left[\bigvee \mathfrak{K}\right] = h(e_M) = e_L.$$

Since $\mathfrak{K}$ is uniformly star-finite, we pick a cover $W \in \mathfrak{N}_M$ such that for each $w \in W$, the set

$$\left\{m \in \mathfrak{K} \mid \bigvee (Mm) \wedge w \neq 0\right\}$$

is finite. Since $W \in \mathfrak{N}_M$ and h is a quotient, we must have $h[W] \in \mathfrak{N}_L$ and $h(w) \in h[W]$. We claim that the set

$$\left\{h(m) \in h[\mathfrak{M}] \mid \bigvee (h[M]h(m)) \wedge h(w) \neq 0\right\}$$

is finite. This follows from the finiteness of the set related to the uniformly star-finite property of $\mathfrak{M}$. Therefore, $h(\mathfrak{K})$ is uniformly star-finite, as was to be shown. $\square$

In the next result, we show that a strongly uniformly R-paracompact map reflects strong uniform R-paracompactness, which is pointfree analogue of Zhanakunova and Kanetov [38].

Theorem 4.2.10. [23] *If $h : (M, \mathfrak{N}_M) \to (L, \mathfrak{N}_L)$ is strongly uniformly R-paracompact and $(L, \mathfrak{N}_L)$ is strongly uniformly R-paracompact, then M is strongly uniformly R-paracompact.*

Proof. Suppose that h is strongly uniformly R-paracompact with $\mathfrak{H} \in$ Cov(L). We then choose $\mathfrak{K} \in$ Cov(M) and a uniformly star-finite $\mathfrak{J} \in$ Cov(L) satisfying

$$h(\mathfrak{K}) \wedge \mathfrak{J} \leq \mathfrak{H}.$$

Since L is strongly uniformly R-paracompact, by definition, there is a uniformly star-finite cover $\mathfrak{F} \in$ Cov(L) satisfying $\mathfrak{F} \leq \mathfrak{K}$. To complete the proof, we will show that

$$h(\mathfrak{F}) \wedge \mathfrak{J} \leq h(\mathfrak{K}) \wedge \mathfrak{J}.$$

To this end, if $F \in \mathfrak{F}$ and $S \in \mathfrak{J}$, then for some $K \in \mathfrak{K}$ (as $\mathfrak{F} \leq \mathfrak{K}$), it follows that

$$h(F) \wedge S \leq h(K) \wedge S,$$

which establishes our claim. $\square$

CONCLUDING REMARKS:

Our attempt to translate strong uniform R-paracompactness into the pointfree setting paves the way for possible similar attempts of other notions of paracompactness. For instance, there is a rich literature on B-paracompactness, F-paracompactness, and P-paracompactness (see [22], [4] and [15]). Moreover, these variants of paracompactness relate to each other in **Top**. It would be interesting to translate their topological versions into pointfree setting. However, that exercise will be pursued elsewhere.

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