USING GEOGEBRA TO ENHANCE GRADE 11 LEARNERS' CONCEPTION OF TRANSFORMATIONS OF FUNCTIONS

by

MATIYENGA TENDAI DAVID

DISSERTATION

Submitted in fulfilment of the requirements for the degree of

MASTER OF EDUCATION

in

MATHEMATICS EDUCATION

in the

FACULTY OF HUMANITIES

(School of Education)

at the

UNIVERSITY OF LIMPOPO

SUPERVISOR: DR P MUTODI

CO-SUPERVISOR: DR K MAPHUTHA

2023

DEDICATION

To my wife, Jublie, and kids- Atinatsira and Nolan.

ACKNOWLEDGEMENTS

I want to start by sincerely thanking my supervisor and co-supervisor, Dr P Mutodi and Dr K Maphutha, for their tremendous assistance with my research. I highly value the advice they provided during the entire study process. I would not have been successful without their recommendations, words of support and inspiration.

I am also grateful to my Headmaster for granting me the opportunity to carry out my research at his school. Thanks also to my Grade 11 learners for participating in the study.

Finally, my heartfelt honours go to my wife, kids, siblings, relatives and my friends for their contributions and encouragement to get me to this stage of higher learning.

DECLARATION

I declare that "**USING GEOGEBRA TO ENHANCE GRADE 11 LEARNERS' CONCEPTION OF TRANSFORMATIONS OF FUNCTIONS"** is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references and that this work has not been submitted before for any other degree at any other institution.

Tendai David Matiyenga
Full names

26 February 2023
Date

ABSTRACT

The function concept, which is a central idea in mathematics, is difficult for learners to grasp, nonetheless, transformations of functions aid in this understanding. The purpose of the study was to investigate whether the use of GeoGebra
enhances Grade 11 learners' conception of transformations of functions. A qualitative case study design, as envisaged by Merriam (1998), was used to solicit learner interpretations and reasoning when working with transformations of functions. Participants for the study consisted of 28 mixed ability Grade 11 learners who were purposively and conveniently selected at the researcher's workplace in Polokwane, South Africa. The APOS theory was put to use as the theoretical framework for the study and the ACE cycle adopted for instructional design. Indicators for APOS theory conception levels were formulated in a genetic decomposition of transformations of functions and used together with deductive thematic analysis of qualitative data to analyse the collected data from learners' written activities and task-based interviews. The results brought out into the open that Grade 11 learners could not easily recognise transformations on functions and that their mental constructions were based on isolated facts or procedures. The results also conveyed that GeoGebra is an effective tool in supporting conceptual understanding in the learning of transformations of functions. The failure to recognise transformations of functions stemmed from the failure by learners to apply learnt concepts in the solution of new problems and also from giving partial descriptions of transformations of functions. These findings reveal that further research on GeoGebra use is needed so as to explore its capabilities to support retention of knowledge of transformations of functions.

TABLE OF CONTENTS

Contents

DEDICATION i
ACKNOWLEDGEMENTS ii
DECLARATION iii
ABSTRACT iv

TABLE OF CONTENTS *v*

LIST OF BOXES *ix*

LIST OF FIGURES *xi*

LIST OF ABBREVIATIONS AND ACRONYMS *xiii*

CHAPTER 1 1

INTRODUCTION AND BACKGROUND 1

1.1 Introduction 1
1.1.1 Conceptual understanding 1
1.1.2 Information and communication technology 1

1.2 Background to the study 2
1.2.1 Functions and the South African context 2
1.2.2 Technology in mathematics instruction 5

1.3 Rationale for the study 10

1.4 Problem statement 10

1.5 Objectives of the study 12

1.6 Purpose of the study and the research questions 12
1.6.1 Purpose 12
1.6.2 Research questions 12

1.7 Significance of the study 12

1.8 Scope and limitations of the study 13

1.10 Chapter summary 15

CHAPTER 2 *16*

LITERATURE REVIEW *16*

2.1 Introduction 16

2.2 Instruction with GeoGebra 16
2.2.1 Properties and characteristics of GeoGebra 17
2.2.2 Literature on the use of GeoGebra 19

2.3 Transformations of Functions 28
2.3.1 Conceptual difficulties of transforming functions 28

2.4 The role of theory in the study **34**
2.4.1 The APOS theory 35
2.4.2 The constructs of APOS theory 37
2.4.3 Genetic Decomposition of the function Concept 42
2.5 ACE learning cycle 43
2.6 Transformations of functions, APOS theory and GeoGebra 46
2.7 Gap in knowledge **50**
2.8 Chapter summary **50**
CHAPTER 3 ***52***
RESEARCH METHODOLOGY ***52***
3.1 Introduction **52**
3.2 Research paradigm **52**
3.3 Research approach **52**
3.4 Research design **53**
3.5 Sampling **55**
3.6 Data collection **56**
3.6.1 Task-based interviews 56
3.6.2 Participant Observation 58
3.6.3 Homework activity 58
3.7 Data analysis **59**
3.8 Quality Criteria **62**
3.8.1 Transferability 62
3.8.2 Confirmability 62
3.8.3 Dependability 62
3.8.4 Credibility 63
3.9 Ethical considerations **63**
3.9.1 Permission 63
3.9.2 Informed consent 64
3.9.3 Confidentiality and anonymity 64
3.9.4 Assent 64
3.10 Chapter summary **65**
CHAPTER 4 ***66***
FINDINGS AND INTERPRETATION ***66***
4.1 Introduction **66**
4.2 Analysis of classroom observations **66**
4.2.1 The action level 67
Sub-theme 1 68
Sub-theme 2 70
Sub-theme 3 73

Sub-theme 4 76
4.2.2. The process level 76
Sub-theme 1 77
Sub-theme 2 78
4.2.3. The object level 78
Sub-theme 1 78
Sub-theme 2 79
Sub-theme 3 79
Sub-theme 4 80
Sub-theme 5 82
4.2.4. The schema level 83
Sub-theme 1 84
Sub-theme 2 86
4.3 Analysis of homework activity **86**
4.3.1 Analysis of Question 1 87
Sub-theme 1 89
Sub-theme 2 90
Sub-theme 3 94
4.3.2 Analysis of Question 2 94
Sub-theme 1 97
Sub-theme 2 98
Sub-theme 3 100
4.3.3 Analysis of Question 3 100
Sub-theme 1 101
Sub-theme 2 103
Sub-theme 3 105
4.3.4 Analysis of Question 4 105
Sub-theme 1 107
Sub-theme 2 108
4.4 Triangulation of results **108**
4.5 Discussion of the findings **109**
4.6 Chapter summary **112**
CHAPTER 5 ***114***
SUMMARY, RECOMMENDATIONS AND CONCLUSION ***114***
5.1 Introduction **114**
5.2 Summary of the results **114**
5.3 Implications and suggestions **117**
5.4 Conclusion **118**
5.5 Recommendations **119**
5.6 Chapter summary **121**
REFERENCES ***123***
Appendix 1: Homework activity and genetic decomposition **132**

Appendix 2: Capricorn District Department of Basic Education Letter 134

Appendix 3: Headmaster consent and permission letters 135

Appendix 4: Parent/ Guardian consent letter 137

Appendix 5: Assent Form 139

Appendix 6: TREC Approval 140

Appendix 7: Lessons Outline 141

Appendix 8: Lesson Observation Sheet 142

Appendix 9: Translation of graphs worksheet 144

Appendix 10: Deductions from sketch graphs worksheet (Adapted from: The Solution for Revision and Examination Preparation- Grade 11 Mathematics) 145

LIST OF BOXES

Table	Description	Page
4.1	Interview on changing parameters	65
4.2	Interview excerpt with L5	69
4.3	Interview excerpt with L10	71
4.4	Interview excerpt with L5	72
4.5	L1 interview excerpt	74
4.6	L19 interview excerpt	75
4.7	L1 interview excerpt	76
4.8	L13 interview excerpt	77
4.9	L18 interview response	81
4.10	L16 responses to interviews	85
4.11	L14 interview response	87
4.12	L19 interview response	89
4.13	L13 interview response	90
4.14	L22 interview responses to Question 2	91
4.15	L16 interview response	93

4.16 L24 response to interview 94

4.17 L4 response to interview 96

LIST OF FIGURES

Figure	Description	Page
2.1	Changing constant in linear function	16
2.2	Transformed graph	32
4.1	Researcher explaining use of GeoGebra sliders	63
4.2a	Graph of $y = ax + q$ before any transformation	64
4.2b	Transformed graph	65
4.3	Learner noting observations after moving sliders	66
4.4	Observations on changing parameters by L13	67
4.5	Observations by L18 on effects of changing parameters in hyperbola	68
4.6	L18 response to Question 5	68
4.7	L22 response on parameters of a parabola	70
4.8	L10 hyperbola drawn on GeoGebra	70
4.9	Graph disappeared from computer screen	72
4.10	L1 responses to Question 6 and Question 7	73
4.11	L19 response to Question 9	75
4.12	L1 response to Question 9	75
4.13	L13 response to Question 9	77
4.14	Confirming transformed graphs on GeoGebra	78
4.15	L16 response to Question 19.6	79
4.16	L19 response to Question 19.6	80
4.17	L18 response to Question 19.6	80

4.18	L22 response to Question 1	83
4.19	L16 response to Question 1	84
4.20	L14 response to Question 1	86
4.21	L19 response to Question 2	89
4.22	L13 response to Question 2	90
4.23	L22 response to Question 2	91
4.24	L14 response to Question 2	92
4.25	L24 response to Question 3	92
4.26	L4 response to Question 3	94
4.27	L19 response to Question 3	95
4.28	L19 response to Question 4	97
4.29	L13 response to Question 4	98
4.30	L15 response to Question 4	99
4.31	L20 response to Question 4	100
4.32	L26 response to Question 4	101

LIST OF ABBREVIATIONS AND ACRONYMS

ACE: Activity- Classroom discussion- Exercise

APOS: Action, Process, Object, Schema

CAPS: Curriculum Assessment Policy Statement

DBE: Department of Basic Education

DGS: Dynamic Geometry Software

FET: Further Education and Training

ICT: Information and Communication Technology

NCS: National Curriculum Statement

NCT: National Council of Teachers

PMT: Pre-service Mathematics Teachers

QDA: Qualitative Data Analysis

TA: Thematic Analysis

TREC: Turfloop Research Ethics Committee

CHAPTER 1
INTRODUCTION AND BACKGROUND

1.1 Introduction

The resolve of this study was to find out how the use of GeoGebra software, in teaching as well as learning about function transformations, enhances conceptual understanding of learners. The transformations of functions aid learners in understanding the functions concept, which is a central and essential topic in mathematics even though it can be difficult for learners (Lage and Gaisman, 2006).

1.1.1 Conceptual understanding

Conceptual understanding is the interpretation of mathematical operations, concepts and relations, or the coherent and systematic clench of mathematical ideas (National Research Council and Mathematics Learning Study Committee, 2001). The concepts should be sufficiently understood by learners who attain conceptual understanding, and they must be able to use the concepts in productive and sensible ways. These learners are also able to recognise and accept the characteristics that are shared by examples, which improves their comprehension of conjectural ideas. There will be tampering if they acquire traits that are not included in the conjectural idea, which leads to some inaccurate viewpoints. As a result, conceptual understanding is crucial for the successful learning of mathematics, and every mathematics instructor strives to help learners develop their conceptual understanding.

1.1.2 Information and communication technology

Information and communication technology (ICT) integration is praised as the solution, to a certain extent, for effective mathematics education on a global scale. One of the most respected mathematics software programs used in both first and third-world countries to teach mathematics is GeoGebra.

The study's executive summary is presented in this chapter. The researcher outlines the study's background and the motivation for it in this chapter. The problem

statement, the purpose, the research questions and the significance of the study are all given.

1.2 Background to the study

The mathematics curriculum in South Africa's Grades 10 to 12 FET (Further Education and Training) phase is broken down into ten topics (DBE, 2011). The topics covered include Functions; Number Patterns, Sequences and Series; Finance, Growth and Decay; Algebra; Differential Calculus; Probability; Euclidean Geometry and Measurement; Trigonometry, Analytical Geometry and Statistics. At FET, functions are studied in greater depth compared to the GET (General Education and Training) senior phase, covering Grades 7 to 9. The different types and characteristics of functions are highlighted in FET. These include the Linear, Quadratic, Exponential and Hyperbolic functions. Also at FET, functions and transformations of functions are not treated as separate topics like in the GET but are studied as one body of knowledge. Stretches and compressions are the new concepts introduced at FET. The current Curriculum Assessment Policy Statement (CAPS) for GET senior phase covers transformations such as reflections, translations, rotations and enlargements of 2-D shapes. These identified transformations are detailed and make learners comprehensively ready for functions in the FET phase.

1.2.1 Functions and the South African context

According to Burnett (2020), functions can be viewed as one of the objects of central importance in mathematics as they appear at all levels of the mathematics curriculum. Functions range from Patterns in elementary school to Real Analysis in college mathematics. Denbel (2015) similarly observes that knowledge of functions spans across all the content areas and that the function concept amalgamates all mathematics content areas. He adds that understanding the concept of a function is consequently key in high school mathematics. According to Oehrtman, Carlson and Thompson (2008), understanding of functions lays the groundwork for modern mathematics and is critical in the study of science. An enduring understanding of the function concept is also required for mastering calculus, which is a crucial trajectory in the training of future scientists, engineers and mathematicians (Oehrtman et al., 2008).

Undergraduate mathematics is fundamentally based on the idea of functions, modern mathematics is built on this idea and related scientific fields place a high importance on it.

According to Mushipe and Ogbonnaya (2019), one of the keynote topics that South African learners struggle with is functions. They assert that this can be as a result of regularly employed conventional teaching techniques that are not enhancing learner understanding. The chalkboard approach is still widely used in classrooms, which makes it difficult to teach and master basic skills. Both teachers and learners may grow to hate the subject. About 25% of the marks on the Grade 12 NSC Paper 1 examination are allocated to the topic of functions. Not only South Africa faces this difficulty with functions. In Germany, the idea of functions is both a crucial idea and a challenging barrier to overcome (Digel and Roth, 2022). As a result, the teaching and learning of functions has been the subject of a lot of research. The National Council of Mathematics Teachers (NCTM) in the United States of America has consistently urged that the school curriculum should give more importance to the topic of functions because of its crucial place in other topics such as calculus (National Research Council and Mathematics Learning Study Committee, 2001).

Despite these calls, learners still leave college and high school with an incomplete or partial understanding of this crucial idea. A significant number of learners may stop their mathematical studies as a result of their inadequate knowledge of a crucial topic in secondary and undergraduate mathematics. The NCTM (2001) found that the traditional method of applying functions to answer certain issues mainly lacks significance and soundness for learners and has been found to demotivate them. Additionally, according to the NCTM (2001), teaching strategies that foster robust reasoning and complex conceptions may spark learners' interest in mathematics and subsequently raise the proportion of learners who stick with their mathematics studies.

Venter (2020), also inscribed that one of the key mathematical concepts that still lacks a complete understanding in South Africa is the concept of functions. He points out that functions are crucial to learners' readiness to explain how variables change in relation to one another, evaluate changes in parameters, and analyse and understand

graphs. The fact that functions continue to be challenging for learners might be partially ascribed to the challenges encountered when researching the topic's history.

According to Ponte (1992), the origins of infinitesimal calculus can be linked to the birth of the idea of function as an autonomous mathematical system. Descartes (1596–1650) asserted in a way that leaves no doubt that a two-variable equation, represented geometrically by a graph, points to a dependence between variable quantities. Obtaining the tangent to any point on this graph gave rise to the concept of a derivative. One of the first mathematicians to demonstrate how functions may be created was Newton (1642–1727). He used "genita" to denote quantities attained from others using the four basic arithmetic operations, "fluent" to denote independent variables, "relata quantitas" to denote dependent variables. The word "function" was first used in 1673 by Leibniz (1646–1716). He used the term "function" to very broadly refer to the way that geometrical values like subtangents and subnormals depend on the contour of a graph. The terms "constant", "variable" and "parameter" were also pioneered by him. A term to depict quantities that were reliant on one variable by mechanism of an analytical expression became gradually more required with the growth of the analysis of graphs by algebraic methods. Finally, in the interaction between Leibniz and Jean Bernoulli in the period between 1694 and 1698, the word "function" was chosen for that reason.

The functions concept is described as an epistemological difficulty, which means that the problems associated with it have persisted for a long time and are still frequently observed (Venter, 2020). He adds that a learning barrier exists as a result of the learners' limited conceptual knowledge. Prior knowledge cannot be applied by learners as the concept of functions is difficult for them to notice it has different representations, and they fail to connect these as representations of the same mathematical concept. The other point made by Venter (2020) is the absence of research that focuses on theoretical analysis to develop innovative teaching ways for helping learners overcome these problems. Hence this study attempts to use GeoGebra.

1.2.2 Technology in mathematics instruction

According to Zulnaidi and Zamri (2017), high school learners still have a low level of conceptual knowledge, particularly when it comes to the topic of functions. They contend that integrating technology into mathematics education makes it simpler for learners to pick up fundamental concepts. It is suggested that another software product that might be used to involve technology in mathematics instruction is GeoGebra. GeoGebra software uses mathematical symbols and graphs to make it simpler to explain mathematical ideas. It also fosters learner friendly environments since it is a very potent teaching tool with latent capabilities that might be used to assist learners in their mathematical exploration through problem-solving, creativity, analysis, reflection and modelling, among other competencies. Considering technology needs to be incorporated into mathematics teaching, GeoGebra was used in this study to improve learners' conceptual comprehension.

The majority of teachers employ conventional or traditional methods when teaching functions. These methods prioritise routine above actions that would help learners understand the notion deeply, hence the methods have never really proven effective in helping individuals overcome foundational function conceptions (Zulnaidi & Zamri, 2017). Learners will have a tendency to only rely on the learning process they would have gone through in class. As a result, learners' mathematical understanding is lacking in depth and development. Teachers who use traditional methods of instruction hardly ever employ conceptually based strategies like problem solving. They can easily use traditional teaching methods because all the required resources are easily accessible. Textbooks do not really require teachers to change their standard methods of instruction. Years of classroom experience strengthen a certain routine and teaching style, especially when it comes to how mathematics content and tasks are taught. As they do not actually have to struggle to adapt to their teacher's teaching style, learners are also at ease with the traditional learning process. Therefore, there is a need for contemporary tools that allow for the effective delivery and application of functions in a variety of representational and unfamiliar settings. The transformation of functions, whose notion was contemplated in this study, is one example of such a representation.

According to Larson, Edwards and Hostetler (2001), graphs of functions are said to be transformed when they are changed by the addition or subtraction of expressions contained inside the formulas for those functions. These operations are collectively referred to as transformations of functions. The idea of showing connections between relations may be helpful in developing a behaviour-analytic notion of the transformation of mathematical functions in a way that entails a consideration of attributes. For instance, when a learner successfully acquires the ability to connect a particular graph to a certain formula, this involves more than just giving an idea of the relationship between two different stimulus events. In order to relate a graph to a formula, one must relate one set of relations to another set of relations because both the formula and the graph are composed of various stimuli that demonstrate links to one another in certain ways. For instance, the graphs of many functions are connected by the variety of sorts of vertical and horizontal shifts, reflections, compressions and stretches that they can undergo. As an example of an extensive mathematical transformation, the function is shifted to the right when a constant is subtracted from an argument and to the left when a positive number is added.

Although there has been extensive research on learners' struggles with conceptualising functions, Baker, Trigueros and Hemenway (2001) observed that there has been little research on function transformations. They cite research on learners' mastery of transformations of functions by Eisenberg and Dreyfus (1994), Block (2000), and Quiroz (1990), all of which found that transformations of functions is a challenging concept for learners. According to Lage and Gaisman (2006), understanding transformed functions provides learners with new opportunities and new ways to reflect on the idea of functions. Furthermore, understanding complex mathematical topics like calculus requires knowledge of how functions are transformed (Carlson, Madison and West, 2015). Studies on the understanding of transformations of functions by learners are useful not for the reason that it is a topic of discussion in many pre-calculus modules, but that it creates the ideal environment for investigating learners' perceptions of functions (Boz-Yaman and Koyunkaya, 2019). The results of these researches can help teachers when preparing instructional materials to improve learners' mastery of function transformations.

Teaching transformations of functions in South Africa adheres to the aims of the NCS-CAPS-FET curriculum documentation, which places an emphasis on mathematical experiences that accord learners the chance to expand their mathematical reasoning and creative skills in getting ready for the abstract mathematics they will encounter in university courses (DBE, 2011). Thus, it is essential that teachers embrace the following in their planning:

- the analysis of graphical connections and the description of mathematical objects;
- assist learners in developing mental practices that will enhance their ability to reason logically, think critically, think accurately and solve problems in order to make decisions;
- teach students the "how" "when" and "why" in order to promote learning strategies and proofs with a clear understanding for what reason or purpose they are required, and to give students the knowledge and abilities they need to start using mathematical language and terminology in later life for analysis, evaluation, and critiquing of hypotheses;
- higher-level mathematical operational abilities like investigating, analysing, problem-solving, critical thinking, and creativity;
- teach students how to pose, recognise, and solve problems creatively and critically by using their spatial abilities and knowledge of shape and object attributes;
- use of word-based examples, graphs, symbols, tables and diagrams to share or transmit information effectively.

When addressing the topic of function transformations for the benefit of learners in the FET phase, all these aims need to be carefully considered.

The researcher has taught mathematics in South Africa for more than ten years and has observed that learners struggle with transformations of functions in Grade 11. Diagnostic reports by the Department of Basic Education (DBE, 2018, 2019) on mathematics examinations revealed that learners performed poorly in the topic, specifically on questions based on the transformations of functions. Several studies have recommended the inclusion of graphing software in lessons on transformations of functions to enhance learners' conception. For example, Bu and Robert (2011)

incorporated major theoretical propositions, namely, realistic mathematics education, model facilitated learning and instrumental genesis in their study and developed an all-encompassing design structure to conceptualise the inclusion of GeoGebra into mathematics teaching and learning. Charles-Ogan and Ibbo (2018) describe GeoGebra as an innovative mathematical software that joins geometry, arithmetic, algebra and calculus to foster mathematical learning. Integrating arithmetic, algebra and calculus concepts makes using GeoGebra relatable to learners, and as a result, reduces the time required to teach learners the technology, giving time for greater focus on developing skills, procedures and mathematical concepts.

One of the most important learning resources today is technology, which allows learners to explore the world and apply what they have learned about mathematics (Arbain and Shukor, 2015). Instead of focusing solely on theoretical methods, mathematics education should emphasise a diversity of learning strategies that draw on teaching tools that have been proven to encourage learners' interest in mathematics. The evolution of the educational process has been significantly influenced by technology. Education professionals should make the most of modern technological tools like GeoGebra, Geometer's Sketchpad and Mathematica. In addition, Arbain and Shukor (2015) stated that the usage of technology is crucial because it serves a purpose as an educational tool, influences the learning objectives and content, and provides a way to make the teaching and learning process better. The task of teaching learners to cognitively access content has been made simpler by using mathematical software that is readily available on the market. It is the responsibility of the teacher to use the available resources without being forced to spend extra time creating new teaching resources. In terms of confidence, drive and enthusiasm, the learners will have favourable opinions of the GeoGebra software. Additionally, their research revealed that the GeoGebra software has a beneficial effect on learners' achievement. In order for learners to explore mathematics in a larger context and enhance their capacity for critical and creative reasoning, Arbain and Shukor (2015) advised mathematics teachers to use this software.

We are currently engaged in a social interaction that, according to Uddin (2011), is dependent on technology, both for its continued existence and as a requirement to be globally competitive. Teachers are engaged in physical and mental activities with

learners whose existence has been subsumed into the media frenzy of the twenty-first century. They faithfully take in the world through a variety of computing devices, including music players, digital cameras, cellular phones, smart phones, iPods, handheld gaming devices, laptops and iPads, in addition to gaming consoles, computers and TVs at home, as well as high frequency direction finders while on the go. Today's learners are what is known as "cybernated learners". As a result, mathematics education must be reorganised to accommodate the needs of these learners. The old top-down, anti-democratic model of education is making a stunning exit. It is a warning to move away from education based on textbooks, teacher control and paper and pencil to education that is learner centered, allowing for problem solving, inquiry and knowledge construction. It implies a new way of defining knowledge and a redefinition of the educated individual. Future generations will need to create knowledge rather than simply receive it, therefore, schools will need to instill a habit of inquiry in its learners. As a result, a contemporary approach to creating and disseminating the content is required.

It should, however, be noted that Nocar and Zdráhal (2016) investigated whether the use of GeoGebra supports the change from a functional to a structural understanding of functions. Their findings were inconclusive and they recommended that more information was required to make sense of learners' buildout of concepts in relation to the use of GeoGebra. Hence, this study quested to make further investigations into this problem.

The DBE (2011) emphasises the constructivist approach as an intervention for understanding mathematical concepts. Guided by the constructivist perspective, this study applied the Action, Process, Object and Schema (APOS) theory to determine conception levels of Grade 11 learners' understanding of transformations of functions using the GeoGebra software. Arnawa and Nita (2020) describe the APOS theory as a learning theory that guides the learning of mathematics by integrating Information Communications Technology (ICT) and cooperative learning, and pays attention to mental constructions that learners make in an attempt to understand a mathematical concept. This qualitative study was guided by the APOS theory to investigate whether the use of GeoGebra enhances Grade 11 learners' conception of transformations of functions.

1.3 Rationale for the study

Around 35% of the questions in the Mathematics Paper 1 examination assess or are concerned with functions. This shows that learning the function idea will improve learners' mathematical skills. In order to develop approaches to increase learners' level of comprehension of functions, it is crucial to investigate learners' experiences with functions in order to obtain insight into how they conceptualise and interpret them.

Worldwide support has been given to the use of information and communication technologies (ICTs) in the teaching and learning of topics like functions. GeoGebra was selected for this study from a pool of current mathematical ICT tools. GeoGebra combines Geometry, Algebra, and Calculus features into a single tool. The study sought to analyse learners' conceptual understanding of transformations of functions so as to reveal their content knowledge, attitudes and behaviours as they interacted with GeoGebra. The experiences of learners when learning functions in this study can aid educators in developing techniques for teaching functions so that learners become proficient in functions. It is important for learners to be encouraged to have a clear understanding of how functions work. The study was guided by the APOS learning theory as tool for both instruction and research.

1.4 Problem statement

The South African Department of Basic Education's (DBE, 2011) Curriculum and Assessment Policy Statement (CAPS) document says that Grade 11 learners should be competent in investigating the change, which is a result of the change in parameters a, p and q on the graphs of functions. Functions are defined as: $y = mx + c$ or $y = ax + q$, known as the straight line or linear function, $y = f(x) = a(x + p)^2 + q$, being the quadratic or parabolic function after completing the square in $y = f(x) = ax^2 + bx + c$; $y = \frac{a}{x+p} + q$, known as the hyperbola and $y = f(x) = a \cdot b^{x+p} + q$, the exponential function. Understanding the behaviour of these parameters can be achieved by promoting a functional and critical approach to learning using dynamic software such as GeoGebra, rather than mindless and uncritical learning of given constancies (DBE, 2011).

The Department of Basic Education (DBE) Diagnostic Report (2020) recommended that when teaching the hyperbola, teachers must start with the basic graph $y = \frac{a}{x}$ and develop the general hyperbola $y = \frac{a}{x+p} + q$. This would enable learners to understand the effect of the changes in the variables a, p and q on the graph, its asymptotes and axes of symmetry. The report added that teachers need to illustrate how transformations influence the equation of a graph. There should be a good understanding of how the graph changes when the equation changes and vice versa. The (DBE) Diagnostic Reports (2018 and 2019) on Grade 12 mathematics examinations also revealed that Grade 11 functions concepts proved difficult for learners in final examinations. The challenges observed showed that a greater number of learners failed to recognise transformed functions and the effects of parameters in the transformed formulae (Ubah and Bansilal, 2018). Bray and Tangney (2017) blamed a behaviourist perspective to teaching whereupon emphasis is on formalised abstract mathematics. The teacher was viewed as the absolute authority with a primary purpose of transmitting knowledge to learners. McCulloch, Hollebrands, Lee, Harrison and Mutlu (2018) added that many teachers use technology to do simple calculations and for data storage, which does not develop learner understanding or stimulate their interest. According to Daher and Anabousy (2015), high school pupils and university students both have difficulty understanding the idea of functions and how they are transformed.

Daher and Anabousy (2015), continue by claiming that it is possible to search for several research findings on how teaching these topics might be aided by modern technology like GeoGebra. For example, Masri, Ting, Zamzamir and Ma'amor (2017) remark that the inclusion of technology in mathematics education has the potentiality to make some topics more accessible to learners. Stols and Kriek (2011) note that high-powered software such as GeoGebra, empower learners to explore and discover patterns. Taking into account the premium value attached to an appropriate understanding of functions in mathematics, a full and comprehensive conception is rare among high school learners (Venter, 2020). Despite more than fifty years of research into the teaching and learning of functions, Venter (2020) noted that this

issue has persisted. These difficulties made it clear that a study is required to determine whether teaching with GeoGebra improves learners' conception of function transformation.

In order to help learners better understand the transformations of functions' different aspects, which include the original function, direction of a translation and magnitude of a transformation, this study aimed to engage learners with the transformations of functions using Geogebra to enhance their conceptual understanding.

1.5 Objectives of the study

The objectives of this study are:

- To identify the challenges that learners face when learning transformations of functions using GeoGebra.
- To investigate if the use of GeoGebra enhances understanding of transformations of functions in learners.

1.6 Purpose of the study and the research questions

1.6.1 Purpose

The purpose of the study was to investigate whether the use of GeoGebra enhances Grade 11 learners' conception of transformations of functions.

1.6.2 Research questions

This study pursued to answer the following research questions:

- What are the challenges faced by Grade 11 learners when transforming functions?
- How does the use of GeoGebra enhance learners' conceptions of transformations of functions?

1.7 Significance of the study

This study has the potential to chip into the development of effective pedagogical instruction for teaching function transformations. In this study, an input is made as a contribution to the teaching and learning of the topic by using GeoGebra to enhance Grade 11 learners' conception of transformations of functions. Additionally, the study aimed to advance knowledge of the subject and enhance the researcher's practise as a mathematics teacher. This study shed light on the challenges learners in Grade 11 faced when conceptualising function transformations.

In pre-calculus courses at universities, research on families of functions and their transformations, notably translations and reflections, has gained importance (Lage and Gaisman, 2006). Transformations give learners fresh contexts in which to apply and reflect on the idea of function, and they may later prove to be an effective tool in more complex mathematical areas. Research on learners' views of function transformations is crucial not exclusively because it is a topic in many calculus courses, but because it offers a chance to look at students' conceptions of transformations of functions. This research will also enable the examination of the adaptability of learners to functional descriptions. With the goal of attempting to enhance learners' conception of transformations of functions, all this data can be used as an architecture that leads the design of appropriate instructional materials and classroom techniques. In addition, the study has the ability to add to the body of knowledge on the mental constructions that students use to deal with transformations of functions. Literature has shown that many students struggle understanding these transformations, and experts have expressed worry over the paucity of studies in this section of functions. The goal of this work was to add to the body of knowledge regarding transforming functions. Curriculum planners, teachers and learners alike, are the potential beneficiaries of this study.

1.8 Scope and limitations of the study

With a total of 28 students in Grade 11, this study was administered at a small private school in Polokwane. Due to their availability and consent, the 28 students were chosen as a sample for the study. The researcher chose the institution since it was his place of employment and was thus easily accessible. The fact that the researcher is a teacher at the school meant that choosing it did not present any ethical challenges.

The APOS theory and ACE learning cycle were employed in the hope that students would quickly pick up the knowledge and competencies needed to comprehend transformations of functions. The study looked at the mental processes that the students used to interact with the concept of transforming functions while using GeoGebra. The goal of the study was to identify mental constructions related to transforming functions. Effective teaching and learning were made possible by the availability of sufficient human and material resources. Such circumstances, according to the researcher, enable and permit the conduct of a study that produces results meeting the quality criteria for qualitative studies. The study was limited by the fact that the circuit office did not grant the researcher permission to conduct the study because it was not being conducted at a government school. The results also cannot be generalised because the research was a case study.

1.9 Overview of the study

Chapter 1: Introduction and background

The setting of the study is described in this chapter. It provides a description of the background, problem statement, purpose, research questions and significance of the study. By providing details about the problem, the background made the issue that motivated the research clear. The problem statement identified problems that made this investigation necessary. These problems began as a result of other researchers' experiences and in-depth discussions found in literature. The study's intent was stated in the purpose statement. The study's primary phenomenon was to be investigated, according to the research questions. In contemplation of broadening the scope of the investigation, the researcher addressed these questions in accordance with the methodology of qualitative research. Finally, the study's significance helped to clarify why it was necessary. It highlighted the significance and effect the study was expected to have on mathematics education, as well as its contribution to new knowledge and how it will benefit others.

Chapter 2: Literature review and role of theory

This chapter gives a scrutiny of literature related to the use of GeoGebra in teaching transformations of functions and a synopsis of the theoretical framework for the study. The researcher learned about the research topic, what has been done on it, how it has been researched and what the main issues are from the literature review. The

utilisation of the concepts from the literature review helped to bolster up the choice of the research methodology and the argument that this research adds anything new. The role of theory and how it offers the research a strong soul and body was described.

Chapter 3: Research methodology

The technique employed in the study is depicted in this chapter, including the research paradigm, research design, sampling strategy, data collection tools, procedures for data collection, quality criteria and research ethical concerns.

Chapter 4: Findings and interpretation

The techniques for data analysis are described in this segment. The determination of the data analysis is provided in line with the study's design and is then utilised to extrapolate findings from the study that address the research questions.

Chapter 5: Summary, recommendations and conclusion

This installment gives a condensation of the study, recommendations commensurate with the findings of the study and the conclusion in line with the purpose and research questions presented.

1.10 Chapter summary

This chapter gave a general inception to the research. The chapter highlighted a challenging circumstance in the instruction and comprehension of function transformations by learners in Grade 11. It provided the study purpose, research questions, and significance. The chapter gave a preview of the detail that will be covered in each of the study's remaining chapters. The next chapter anchors an exploration of literature applicable to the title of the study together with the theoretical framework that guided the study.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The past chapter discussed the background, the purpose and the research questions that guided the study. This chapter will review literature on instructional use of GeoGebra, transformations of functions along with the theoretical framework that guided the study. The primary goals of the literature review were to explore prior study findings on the worth of GeoGebra in the teaching and learning of function transformations as well as to locate knowledge gaps. APOS theory-based studies were also reviewed in the chapter. The chapter concludes by affirming this study.

2.2 Instruction with GeoGebra

In a connected society, Wassie and Zergaw (2018) note that integrating technology into mathematics instruction and learning can meet the needs and interests of a wide range of students. According to Guncaga and Majherova (2012), modelling, simulation and visualisation are three ways that technology enables teachers and students to communicate long-established theoretical concepts in ways that are engaging and within reach. Technology provides the right sets of circumstances for learners to build up knowledge and explore modern perspectives on problem-solving (Bray and Tangney, 2017). In the past, technical tools such as small writing boards, abaci and tangible manipulatives such as calculators, computers, interactive white boards and the like have played a significant role in the teaching of mathematics (Akcay, 2017). Researchers like Moeller and Reitzes, 2011 and Velichova, 2011, cited in Wassie and Zergaw (2018), have observed that integrating GeoGebra into the mathematics classroom can boost learner contentment at all levels of instruction while enhancing the establishment of task solving skills and nurturing active learning. The authors Baye, Ayele, and Wondimuneh (2021) advise against using GeoGebra exclusively. They argue that it is unlikely that GeoGebra will substantially change learners' conceptual understanding. They refer to a study by Arbain and Shukor (2015) which documented learners' post-test achievement that was strikingly lower than their pre-test performance after treatment using GeoGebra. Baye, Ayele and

Wondimuneh (2021) therefore recommend mathematics teachers to integrate GeoGebra with other worthwhile teaching approaches.

2.2.1 Properties and characteristics of GeoGebra

GeoGebra is a productivity software for teaching and learning mathematics from middle school through to college (Hohenwarter and Preiner, 2007). It is useful for active and problem-based teaching to incite mathematical 'aha moments' and experimentation, both in the classroom and at home. Thomas (2016) states that GeoGebra is a completely free download that can be used as an application on many smartphones and tablets. Many of its functions are built-in, for example, zooming in and out of a chart can be achieved by moving graphics with the mouse on a PC or by making use of the popular pinch-zoom motion on devices which allow the user to interact with them by touching areas on the screen. With GeoGebra, graphs of functions and their transformations can be created quickly and accurately. Using GeoGebra extends to learners the capability to draw graphs and view the corresponding equations (Ogbonnaya & Mushipe, 2020). GeoGebra also helps to explore the relationships between the graph and its algebraic representation (Rocha, 2020).

Its most intriguing feature is the online fraternity of users who frequently add to the free of charge educational resources created by GeoGebra. These resources cover a variety of topics, including coordinate systems, linear equations, triangles, angles, the formula space, transformation, symmetry, circles, Bezier curves, quadratic equations, matrix transformation, polynomial functions, derivatives, integrals, complex numbers, line and space, parametric functions, vector functions and others (Zulnaidi & Zakaria, 2012). Algebra, Geometry, and Calculus can all be taught using GeoGebra as well (Baye et al, 2021). GeoGebra effectively imparts knowledge through presentation, guiding, planning and evaluation with the goal of teaching learners the relevant concepts or abilities. The ability to teach and test inferences in Algebra, Geometry and Calculus is included in GeoGebra software.

GeoGebra offers support for learners in the development of the functional concept. Bayazit, Aksoy and Ilhan (2010) point out that at the initial phase of operational

function design, the so-called function action design, involves the ability to work out manipulations with algebraic expressions by slotting items into an expression and computing their images step by step. Learners would not acquire these types of savvy skills in conventional teaching-learning environments. Traditional approaches partially support the development of a process concept. Nevertheless, GeoGebra reinforces the learner co-optation process (Zulnaidi and Zamri, 2017). A key feature of process design is the aptness to roam between algebraic and graphical representations of a function. At the process design level, learners can link representations by point-to-point mappings, they focus on the critical attributes of a diagram, for example, maximum, minimum, points of intersections, then try to make or show connection between them to the appropriate elements of the algebraic expression, or they do the opposite. The use of GeoGebra allows the teacher-learner community to carry out several activities of this type. For example, the graph of the function $f(x) = 3x - 6$ can easily be made with GeoGebra. Next, learners can be asked to identify the kind of changes they notice on the function graphs when they take the place of the constant term with a different number in the algebraic expression as in Figure 2.1 below where $g(x) = 3x + 2$. Here the -6 has been replaced with $+2$. This has the effect of shifting the graph of $f(x)$ by *8* units upwards. More characteristically, they might be asked to examine the relationships between coefficient and constant terms in the algebraic expression and the points where the graph cuts the x and y axes. Instead, the cursor allows learners to drag the graph vertically up and down and then look for the correspondent variations in the coefficient and constant terms in algebraic form.

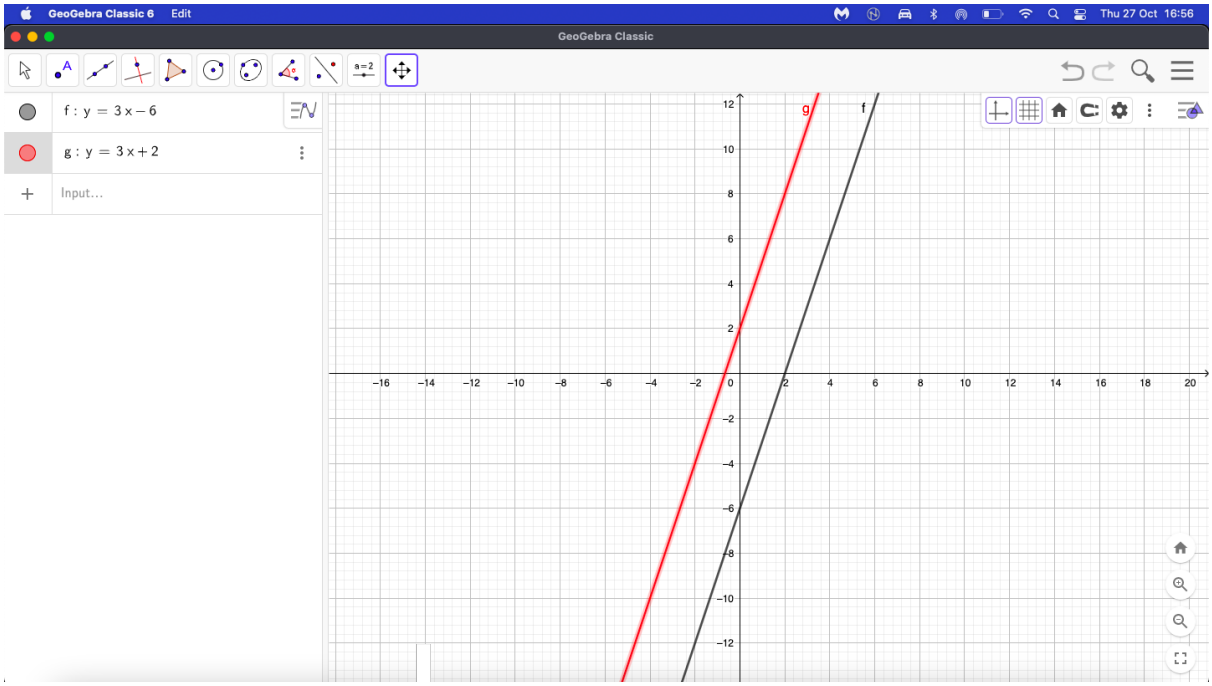


Figure 2.1: Changing constant in linear function

2.2.2 Literature on the use of GeoGebra

Mudaly and Uddin (2016) have done research on the integration of GeoGebra as a pedagogical tool in the teaching and learning of function transformations. They observed that the instruction of mathematics had changed owing to the shift to a digital culture in schools. Learners will not be able to fully participate in the social, economic and cultural life around them if they are unable to navigate this digital environment. The study promoted the idea of investigations as an opportunity to provoke in learners the expertise to devise strategies, to infer from specific cases, to generate conclusions formed on the basis of incomplete information and to try to defend them when learning transformations of functions in 10th grade. The implication was that learners had to think about processes and did not care about getting correct answers. With GeoGebra, one way to overcome misunderstandings and misrepresentations among learners is through virtual manipulation (Mudaly and Uddin, 2016). These virtual manipulations are dynamic and allow learners to control the input. Learners can see for themselves the different transformations and what produces them. Encounters with tangible objects, as opposed to those that exist in thought, permits learners to construct their own meanings.

When learners test their ideas with virtual manipulatives, for example sliders on GeoGebra, they can move without difficulty from practical to analytical thinking. Dragging sliders also aims to conceptualise transformations of functions through changes in function variables, thus visually interacting with once-abstract concepts. The constantly changing character of online manipulatives, coupled with colourful graphics and interchangeable interactivity, can catch and engross learners' attentiveness, motivating them to learn and giving eloquence to their learning. A strong understanding arises when learners deal with the content independently. In the study by Mudaly and Uddin (2016), Geogebra fulfilled its role as a pedagogical tool in teaching function transformations. It ardently involved learners in the exercise of meaning making and they preferred this to simply writing and listening.

In a similar study, Wassie and Zergaw (2018) found that GeoGebra applets have significantly transformed mathematics pedagogy. The teaching of mathematical topics had to be structured to reflect this changing culture. They noticed that learners' choice of how to see, interact with, and engage with what they meet in class is influenced by technology. Compared to static ones, interactive charts of various functions are easier to understand. Additionally, it was discovered that GeoGebra supports learner and teacher-centered approaches. The teacher's already generated materials can be used by the learners, saving time in class. Conversely, the teacher can provide ideas in advance so that learners can independently research problems, identify patterns and confirm hypotheses. It is important to acknowledge that a facilitator is necessary in both situations, and that teachers' roles, content choices and activity plans must all be carefully considered. As GeoGebra use becomes more widespread, additional research through articles, workshops and conferences on its challenges is needed so as to explore its capabilities to support mathematics education.

Another key advantage of GeoGebra is that an internet connection is not required for it to operate (Mthethwa, Bayaga, Boss and Williams, 2020). It can be loaded to a computer or external hard drive and then copied to as many devices as one likes for use by learners. Their study looked into the impact of adopting GeoGebra on learners’ and teacher beliefs. They compared 11th grade learners who used GeoGebra in the learning of Circle Geometry with learners who had learned the same topic through traditional, lecture-based instruction at a rural school in the region of KwaZulu Natal.

The conclusions of the study were that learners who practiced on GeoGebra were outstanding in solving problems and accounting for their statements, while the other learners gave scanty justification for their answers. Furthermore, they found that much of the available literature separates the impact of using GeoGebra on learners and teachers. They claim that few studies have concurrently considered the parallel repercussions on learners and their teachers. In addition, the rural educational setting in which they conducted their study had limited teaching resources, insufficient availability of professional development for teachers and a lessened number of teachers eager to work in that setting. The use of technology in mathematics education is many a time characterised by several challenges like lack of understanding of the course content and a lack of adequate technological resources.

With almost similar observations, Belgheis and Kamalludeen (2018) suggest that the burning challenge for mathematics educators is fusing new technologies and designing new-fashioned learning activities for learners. Their study aimed to examine the intent of mathematics teachers to put GeoGebra into use in the classroom. They found that the convictions and behaviours of teachers in the process of technology integration have been highlighted by several academics. Teachers' beliefs and viewpoints steer most of their teaching ideas and lead to improved choices of learning circumstances in the classroom. They note that GeoGebra, among the most novel teaching tools attracting much consideration from academics and math teachers, has the potential to revolutionise the approach mathematics is taught and learned. Spreadsheets, Dynamic Geometry Software and Computer Algebra System functionality are all merged in this programme. It provides an online and offline environment for learners to synchronously view both a numerical algebraic part and the corresponding geometric properties of an object. Uncovering new patterns, investigating and checking conjectures and manipulating several geometric shapes are among the many pursuits learners can engage in by outlining and producing their own sketches in a dynamic math software application.

The population of the Belgheis and Kamalludeen (2018) study included mathematics teachers who had learned about GeoGebra and its teaching capabilities through a seminar organised either by governmental agencies such as the Ministry of Education in Malaysia as part of regular staff development programs for teachers or non-

governmental bodies such as the GeoGebra Institute of Malaysia. The content material for the workshop was developed and validated by two educational academics and two GeoGebra experts. There were 132 survey replies in all. In terms of gender distribution, there were 56 men and 76 women among the participants. 49 non-users and 83 users of GeoGebra made up the participants. According to the results of a bivariate interdependence analysis, there was a statistically significant positive correlation between respondents' intention to use GeoGebra (IU) and their identified present competencies (PCC) ($r = 0.507$, p .001). The two variables had a positive correlation, which means that as teachers become more confident in their abilities, intention to utilise GeoGebra will rise. This study by Belgheis and Kamalludeen (2018) reported that the lack of adequate training is many a time cited by educators as grounds stopping them from realising the full potential of educational technology and software. This is consistent with McCulloch et al. (2018) who suggested that there is a burning need for secondary school mathematics teachers to develop skills in using Dynamic Geometry Software, such as GeoGebra, so as to buttress the continued integration of dynamic mathematics software into mathematics teaching and to sustain learning.

In another study by Olsson (2017), by giving students a collaborative workspace in which to work and evaluate their activities, GeoGebra was touted as supporting problem solving, inventive mathematical thinking and teamwork. The unguided learners demonstrated greater creativity and problem-solving skills, as well as a profound grasp of their solutions. The activities and didactic circumstances were created to encourage creative mathematical thought and the results indicated that commitment is needed to complete the tasks. In Olsson's (2017) study, the results revealed that learners who resolved a guided task according to instructions that contained a solution method partook only in algorithmic thinking. This shows that learners are predisposed to developing problem-solving and logical thinking abilities as well as a better understanding of the mathematics involved when observations are obtained from GeoGebra. It is essential that learners receive accurate instructions and tasks that structure the activity and articulate goals without disclosing solution approaches in such heuristic settings when learners create solutions. Based on the results of the studies comprehended in this section so far, one deduction is that an

instructional blueprint that encourages learners to create solutions through creative thinking, shored up by GeoGebra, can improve their mathematics learning.

It can also be observed from the study by Olsson's (2017) that when learners have an idea to solve a specific problem, they are required to test their strategy and obtain feedback to make sure or refute ideas and/or propositions. Such pursuits could be shored up by dynamic software. GeoGebra provides features to help learners quickly create mathematical entities that can be viewed as discernible representations such as algebraic expressions and matching graphs. These descriptions are linked in a manner characterised by change, if anything in one representation is changed, the other representations are changed. These are opportunities that could be used to investigate and examine various characteristics and relationships of these entities.

The main goal of Olsson's (2017) paper was to provide some awareness into the blueprint of a didactic circumstance, including the dynamic program GeoGebra, to support learners in mathematical creative thinking and problem-solving as a means of learning. The study also sought to determine whether differences in learning outcomes stemmed from learners' abilities to execute guided or unguided tasks. The main objective of the thesis was to provide some capacity for gaining an accurate and in-depth understanding of the design of a didactic setting that uses GeoGebra to empower learners in using mathematics to think creatively and solve problems. The learners worked in pairs and their interactions were observed and recorded. The data from these recordings was then used to analyse how the learners used GeoGebra while also thinking critically and solving problems. A second didactic scenario was created, which varies only in that it provided the learners with instructions on how to do the task. In order to address the research question, it was determined that a situation where learners use dynamic software to solve problems was appropriate. To confirm which of the guided or unguided activities increased learning most successfully, the data were quantitatively examined. By offering learners access to a collaborative workspace and feedback on their activities, GeoGebra showed that it promotes teamwork, problem-solving and innovative mathematical thinking.

Again from Olsson's (2017), the output from GeoGebra was used as responses to answer questions and to verify or refute the hypothesis. These interplays with GeoGebra were used to drive the construction of responses. The way learners solve problems and reason, and use GeoGebra to do so depends in any case on whether they have been provided with instructions or not. The study further revealed that the learners who unravelled unguided tasks used GeoGebra's capacity to investigate and explore the given task. The unguided learners participated more in creative thinking and problem-solving and expressed a considerable understanding of their solutions. The learners who were able to solve the unguided exercise performed better than the learners who were able to solve the guided task in the post-test.

Olsson (2017) came to the conclusion that it is safe to assume that criteria for educational design will be implemented in order to involve learners in exploiting the aptitude of dynamic software to promote problem-solving and reasoning. The vast majority of research on mathematics education supports the idea that it is critical to increase our understanding of educational technology in order to inspire learners to deepen their comprehension of mathematical concepts and relationships and to develop their critical-thinking abilities and troubleshooting. While traditional teaching demands that teachers give learners sets of rules to utilise to perform tasks, deconcentrating a problem would involve preparing learners to construct a solution in a didactic environment. Here, the teacher creates a problem that the learners have not been told about or for which they have not yet learned a solution.

Despite this, research consistently shows that the characteristics of dynamic software such as GeoGebra, stimulate creativity by pressing users to employ logic and problem-solving skills. The findings of Olsson's (2017) analysis are evidence that, on the one hand, these requirements are satisfied if the endeavour does not include any solution approaches. On the other hand, learners can be guided to use effective strategies if they have a clear understanding of the desired outcome. The findings support the notion that even if they must construct the solutions, learners still can execute effective strategies. The feedback provided by GeoGebra was said to be quick and related to the activities at hand. This is harmonious with the findings of the study in that the learners created what seemed to be specific inputs to GeoGebra, intended to add data to facilitate the creation of the solutions. Successful learners reason more effectively,

which was also consistent with studies by Mthethwa et al, (2020) and Wassie and Zergaw (2018). The solution could be created in conjunction with software, with the outcome of one activity generating data useful in creating the following supporting activity. Executing the planned activities was reported to have improved the learning outcomes for the learners. It has been asserted that straightforward instructions that simply describe how to carry out a task encourage shallow understanding and poor memorization. That is also in line with the characteristics of consistent engagement in learning environments, which is presumably advantageous for the improvement of students' analytical and problem-solving abilities.

According to Shadaan and Leong (2013), using technology to teach and learn has several advantages such as giving learners a broad variety of learning opportunities, raising learner engagement and encouraging inquiry-based learning. They went on to aver that it is crucial for learners to have the means to visualise, construct and analyse the creation of geometric patterns in order to relate them to incidental facts in mathematics instruction, especially geometry. Learners can elaborate and leave an impression using a computer. There are several technology resources available, including calculators, interactive whiteboards, Geometers, Sketchpad and GeoGebra. Their article went into great length on using GeoGebra for learning circles. Their primary objective was to examine the impact of GeoGebra on learners' understanding of circles. The purpose of the study was to find out if this form of learning has replaced the traditional approach and whether learners think using technology to learn is worthwhile. Finding out how much knowledge students had about GeoGebra in learning circles was the secondary goal. The purpose of this study was to address the ensuing research questions: First, how effective is GeoGebra compared to the conventional method at helping learners understand circles? Second, what did learners think about GeoGebra when they were learning about circles?

Twenty-eight learners made up the experimental group in Shadaan and Leong's (2013) study, whereas the control group consisted of twenty-five learners. The effectuality of employing GeoGebra in the teaching of circles was the main focus of the study. An independent sample t-test was administered to see whether or not there were any statistically significant differences between the control and experimental groups' pre-test average scores. The outcome demonstrated that prior to the

experiment, the performance of the learners in the control and experimental groups were equivalent. About 75% of learners reported being able to think critically and creatively throughout the discussion and question-and-answer period. When they attempted to predict, they were also capable of making logical assumptions. When utilising GeoGebra, the learners enjoyed learning mathematics more and were better able to make connections between their prior knowledge and what they were learning now. When utilising the GeoGebra software, some participants indicated they were not really feeling as confident.

The results of Shadaan and Leong (2013) demonstrate that GeoGebra can be put to use as a teaching and learning tool for mathematics and circles because a noticeable increase in conceptual knowledge of circles in the experimental group compared to the control group took place. Usability of GeoGebra software increased student achievement; it was discovered that the tool created a lively classroom where the ideas of cooperative and collaborative learning were clear. This finding is reinforced by Olsson (2017), who has used GeoGebra before and post-performance assessment in a two-group study to master mathematics concepts showed improvement. The aforementioned findings support those of earlier studies in this section that looked at how a technology-rich domain affects learning outcomes.

Technology was named by Shadaan and Leong (2013) among the leading propositions to raise the bar for mathematics education, and they recommended that teachers use it to increase the learning opportunities for their students by selecting or creating mathematical activities that effectively utilise technology. Through exploration and visualisation, technology allowed both educators and learners to go through concepts together. This sparked a more active teacher-learner relationship where everyone consorted as a team member to mentor, assist and support one another in accomplishing the set objectives. According to Shadaan and Leong's (2013) study findings, encouraging teachers to use GeoGebra in mathematics lessons is highly suggested. To finally decide if GeoGebra has an impact on the learning of various mathematical ideas at various levels of learners, this needs to be combined with research to obtain better findings. Zengin, Furkan and Kutluca (2012) and Bakar, Ayub, Luan and Tarzimi (2002) cited in Shadaan and Leong (2013), provide support

for this finding. Overall, GeoGebra is endorsed as a useful tool for helping both educators and learners in mathematics classes meet the tenets of constructivist learning.

Uwurukundo, Maniraho and Tusiime (2020) assert that recent decades have seen a rapid advancement in technology, which includes the uptake and creation of instructional software. Numerous researches on technology-enhanced teaching and learning have been conducted, finding various elements for the gainful integration of such technologies (Reis and Ozdemir, 2010; Akkaya, Tatar and Kaizmanli, 2011 and Gmez-Chacn, 2011). One educational programme that has undergone research to the extent to which it enhances learners' conceptual understanding and attitudes toward mathematics is GeoGebra. This study looked at the integration of GeoGebra and why it was seen to be effective in raising mathematics learner achievement. The authors conducted a study of the literature to gather information about the effectiveness of GeoGebra integration in secondary school mathematics instruction. 23 studies that had no bearing on the integration of GeoGebra and its effectiveness in mathematics instruction and learning were removed. The authors chose to focus on 33 studies after further filtering out studies pertaining to instructors' perceptions toward the efficacy of GeoGebra in mathematics instruction and studies pertaining to teachers' competence in GeoGebra.

When the authors thought about the framework of the research, they kept on 20 publications that discussed GeoGebra integration and its potency in mathematics teaching and learning. These research papers were prepared and issued for public readership between 2010 and 2019. Twelve papers on GeoGebra integration and its effectuality in Geometry were found in the reviewed literature. Three groups of learners were formed and given the same material to learn in an inquiry by Seloraji and Eu (2017) to see if GeoGebra helps ninth-graders with geometric reasoning. Twelve studies on GeoGebra integration along with its usefulness in Geometry were found in the literature review. GeoGebra was used to provide instruction to a class of 24 ninth-graders, and all groups had to take a pre- and post-test. 53 learners in the ninth grade participated in the study by Shadaan and Eu (2013), which looked at the impact of GeoGebra on learners' understanding of learning circles. It was discovered that the experimental group outran the control group in terms of GeoGebra usage. Masri et al,

(2016) looked at how GeoGebra affected circle teaching and learning among Malayan learners in the 10th grade.

2.3 Transformations of Functions

Function transformations include translating a function's points onto new points using a clear rule or procedure (Lewis, 2002). According to Boz-Yaman and Yigit-Koyunkaya (2019), taking the inverse of a function and composing it with other functions has a significant impact on the idea of transforming a function. This essentially means that transforming a function includes moving the entire function or specific function points from their initial positions to new positions or images while following a predetermined rule. The graph of a function may transform or seem to shrink or stretch, shift horizontally or vertically, when certain computations are used. This challenging large idea is reserved until the end of the chapter because it necessitates an overall comprehension of how the other big ideas interact (Lage and Trigueros, 2006).

2.3.1 Conceptual difficulties of transforming functions

According to Gcasamba (2014), studies in mathematics education give ample proof of the significant difficulties learners face when attempting to learn the concept of function. Learners must master the transition from one functional representation to the next in order to get a far-reaching overview of the functions. The representations are namely; algebraic, tabular and graphical. By changing from one description to another, learners discover new features of the functional concept. Additionally, as learners interact with the various representations, they have a plausibility of deciding which description provides more useful information. Linking these representations together will help learners to find general procedures that allow recognising the occurrence of a representation in different forms.

According to Denbel (2015) the idea of function has undergone an intriguing history. In order to describe an object's motion in mathematics, the concept of the function was first created in the 17th century. The present-day notion of a function with reference to

it as a univalent harmony between two sets emerged in the 19th and 20th centuries as a result of new discoveries and a renewed emphasis on rigour. A function from A to B is more precisely described as any subset of the Cartesian product of A and B that has exactly one b such that (a, b) for every aϵA. There has been research on the relationship linking the definition of a function and the mental picture that learners construct of a function (Vinner cited in Denbel, 2015). Sadly, learners still have a limited conception of a function, akin to one out of the 18th century, where function was a dependent variable or an analytic expression with no "sharp points" on its graph, despite the fact that function is frequently defined in schoolbooks in a modern sense.

There are two possible ways for including functions in the curriculum, and both have been applied in a number of countries during the past few decades (Denbel, 2015). First, lessons on types of functions—such as linear, quadratic, reciprocal, exponential and logarithmic—are given to learners. Second, learners are exposed to many classes of functions as distinct cardinal objects, but no overall explanation of the concept is provided. The first perspective may result in learners learning about functions in a very limited way, for as when they are only exposed to interminable functions that are written in algebraic symbols or that have a 2-D graph. A drawback of the second line of action is that learners may initially have such limited exposure to the idea of a function that the explanation may not be very meaningful to them and may not be used to clarify their subsequent work. Learners should experience functions in many representations and draw connections between these to form an opulent concept image, which is crucial to advance a rich meaning for an application of the idea (Thompson, cited in Denbel, 2015). According to Denbel (2015), some of the more complex ways that functions are used to support highly developed mathematical thinking are appended in school curricula in a number of different countries. He also points out that it is evident that technology provides excellent opportunities to support learners learning about functions, and it appears that this will always be a promising area for teachers.

According to the study by Denbel (2015), functions become prominent in all high school curriculum, and this requires continued clarity of intent so that mathematics teachers may make informed decisions about what to teach and why. It is necessary to make decisions about how to apportion this work over the high school years as well

as the relative significance of the formal definition of a function and the applied uses of functions for modelling. While technology would seem to be effective in supporting teaching and learning in the long term, it is not a panacea, so caution must be taken to make certain that it is used effectively, in accordance with the facilities that are available in different frames of reference, and that the objectives for having functions in the curriculum are not unintentionally hampered by assessment practises.

According to Oehrtman, Carlson and Thompson (2008), research has revealed that it is challenging to understand the concept of a function, many brilliant undergraduates, for instance, who received distinctions in calculus, possess a poor comprehension of functions. This bothered them and it prompted them to try to understand why it is that the conceptualisations and reasoning processes required for a thorough and adaptable grasp of functions are more tortuous than is normally thought out.

Learners who only think of functions in terms of symbolic manipulations and procedural operations will not be able to comprehend a general mapping of a set of input values to a set of output values. For modelling function relations in which the function value or output variable changes continuously in response to changes in the input variable, these learners also need conceptual frameworks. It has been suggested that these cognitive abilities are essential for expressing and understanding the dynamic nature of a variety of functional situations. They serve as a foundation for understanding sophisticated mathematical ideas.

It is noteworthy that several early 1990s pro-change calculus texts included a higher conceptual focus on studying functions (Oehrtman et al., 2008). A strong grasp of function develops from the perspective of a machine that processes input and output to a conception that supports inferences about dynamic mathematical knowledge and scientific contexts. This understanding can be developed through future curricular interventions that focus on helping learners achieve this goal. According to research, the predominant method of calculus instruction does not foster the fundamental knowledge and problem-solving skills needed for learners to continue their mathematical development and enroll in courses. According to the authors, the mathematics community is ready for a thorough revision of the pre-calculus and calculus curricula, one that is informed by the mathematicians' prior research as well

as the enormous body of literature on understanding and learning function and core calculus concepts. Additionally, they think that learners would be better able to apply their algebraic knowledge in a situation-appropriate manner when dealing with new issues and tasks if established algebraic methods were more directly tied to conceptual learning.

Why is it so difficult for students to understand the function concept? Oehrtman et al. (2008) make an effort to answer this question. They assert that as learners work through the high school and university mathematics curricula, they regularly come across instructions that ask them to compute answers to a range of questions and solve algebraic equations. The strong emphasis on procedures has proven to be unhelpful for developing fundamental function understandings—ones that allowed valuable explication and use of functions in various representational and novel settings—as there are no accompanying activities to develop a deep understanding of the concept. Even a basic understanding of how functions behave in terms of input and output can be a significant challenge for many learners. For instance, 43% of top college algebra learners attempted to calculate f (x + a) by adding "a" to the end of the formula for f rather than substituting x + a into the function. When asked to elaborate on their reasoning, learners typically provided a learned-by-heart rule to back up their responses. It was obvious that these learners were not using the value of x + a to calculate the function. Another misconception is the idea that because a function's value does not vary, such as y = 5, it is not a function. It can be difficult for learners to view y = 5 as if it were not an example of a function; as an example, when looking at equilibrium solution functions for differential equations like dy/dt = 2y(y – 5). In one research, 25% of the top calculus learners gave the answer y = x as an example, while only 7% of the top learners in a college algebra course were able to provide a suitable example of a function whose entire range is constant. More concerning is the fact that students frequently confuse functions with two expressions that are separated by an equal sign. A solid understanding of more challenging mathematical ideas cannot be provided by such a limited understanding of functions.

The authors continue by stating that it is common for emerging learners to find it difficult to compare and contrast an algebraically denoted function with an equation. This is to be expected given the numerous uses of the equal sign and the fact that

many teachers refer to a formula as an equation. The learner finds it difficult to distinguish between the usage of the equal sign to define a relationship between two different values and the expression of equality of two expressions due to this ambiguous interpretation of the word equation. They assert that their research has shown that learners benefit from being helped to clearly distinguish between equations and functions. The authors also developed lesson plans that encourage learners to think of an equation as a way to compare the values of two functions' outputs and to approach equation solving by finding the input value or values at which these functions' outputs are equal.

According to Oehrtman et al., (2008) many learners are prone to thinking that all functions should be understandable by a single algebraic formula due to a lack of foundational analytical thinking that promotes rationality in learners' function understanding. Thinking in this manner is rigid and can lead to false conclusions, such as the notion that all functions exhibit some sort of "pleasant" behaviour. For instance, many students are prone to claim that a piecewise function is really two different functions or that a function like Dirichlet's is not even a function due to the way it behaves. Similarly, many learners find it difficult to understand the multiple formula representations of the same function that, despite employing quite dissimilar algebraic processes, explain the same function on the cartesian plane. Many learners have also shown a tendency to make assumptions about functions that are not necessary, for as by believing that any "u-shaped" graph is a parabola. Given that specific function types are utilised to incorporate functions into educational institutions' curricula, these tendencies are to be expected. In essence, equating functions with formulas in a definition of an operational notion is quite appropriate and even conforms to the historical ideas of mathematicians like Euler, Bernoulli, Lagrange and d'Alembert. But ultimately, neither Euler nor the mathematical community as a whole found this method of thinking to be effective. The arguments between d'Alembert and Euler regarding the characteristics of a solution to the vibrating string differential equation, as well as Cauchy's and others' attempts to identify the conditions under which a limit of a series of continuous functions is a continuous function, all had a significant influence on how we currently define function. Therefore, using the modern definition of a function as the starting point for the function concept means giving learners the responsibility of finding solutions to problems they are unable to articulate. The

comprehension of reciprocal interactions and various representations of intercorrelations, such as using different coordinate systems, should be emphasised in school mathematics instructional programmes (Oehrtman et al., 2008). They also recommended giving students a variety of situations in which they might deal with different function types and different representations of the same functions. Then, the curricula of tertiary institutions might expand on this base. This would motivate learners to have a more solid and adaptable understanding of functions, one that does not lead them to unintentionally equate functions with formulas.

Another typical difficulty for learners is to distinguish between discernible characteristics of a physical situation and corresponding intuitive characteristics of the graph of a function that describes reality (Oehrtman et al., 2008). When using functions as representations of actual events, it is also typical for learners to notice geographic features mirrored in the function's graph, such as the elevation of a road cutting through steep terrain, the curves of a racetrack, or the shape of a tanker carrying liquid. Even for learners who have a solid understanding of functions, the obvious prominence of these physical characteristics frequently induces uncertainty. A variety of errors can be attributed to confusing visual elements of the situation with the shape of a graph.

According to Borba and Confrey (1996), function transformations have historically been taught in relative isolation from the unmistakable transformation concepts in geometry and with a persuasive emphasis on algebraic symbolism. The typical method for handling function transformations in most conventional textbooks involves changing a function's coefficients and taking into account the ensuing changes in the graph. For instance, how does the transformed function change from the original function if $f(x) = x^2 + 5$ is transformed to $g(x) = 2x^2 + 5$ (see Figure 2.2). Learners are expected to respond by noting that the graph narrows inwards as the coefficient of x goes up from *1* to *2*. This is what Borba and Confrey (1996) called a template method. They describe issues with learning, such as a propensity to memorise rules without knowing how they came to be. The technologies used in Borba and Confrey's

(1996) study transformed the graphs indirectly either through the use of simple command codes or algebraic symbolism.

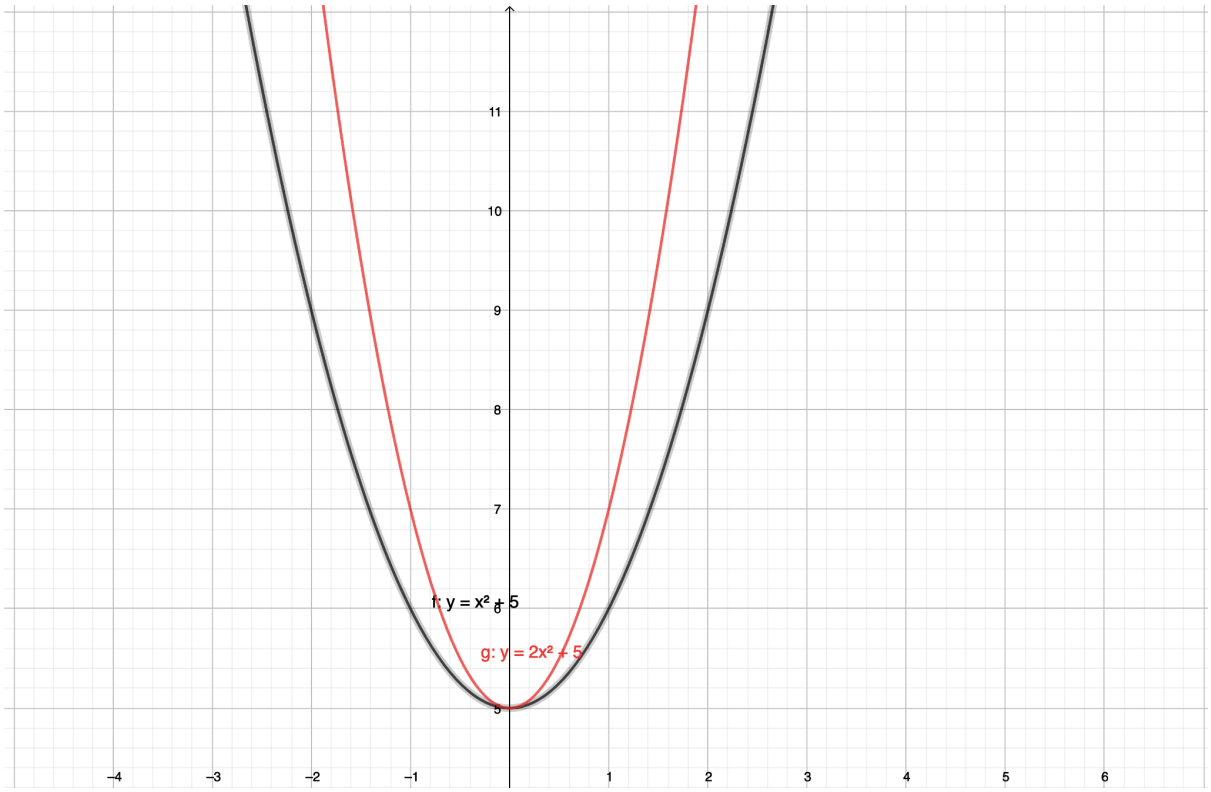


Figure 2.2: Transformed graph

2.4 The role of theory in the study

The theoretical framework used in this study was the APOS theory. Oktaç (2022) remarks that APOS theory puts forward a means to describe the learning of mathematical concepts. She further says that the visible history of the APOS theory antedates the 1980s, when Ed Dubinsky, a mathematician popular for his input into functional analysis, began to share the ideas of a cognitive approach that would subsequently come to be recognised as APOS, an acronym drawn up from the initials of its individual components. Following years of study of Piaget's theory of knowledge and theory of cognitive growth in children, and the naturalisation of ideas such as reflective abstraction to undergraduate mathematics, Dubinsky established the tenets of this theoretical framework.

Tatira (2021) claims that the APOS theory enables mathematics education researchers or teachers to recognise the similarities and differences among learners

who learn by carefully following external instructions (Action conception), those who put forth mental effort by internalising actions (Process conception), and those who are able to apply concepts to solve new problems and connect current concepts to other concepts (Object conception). Despite the fact that the action, process, and object mental structures provide a broad explanation of how a specific person produces distinct transformations, a mathematical concept involves several actions, processes, and objects that need to be intelligently organised. This presumption is known as a schema, which is an individual's collection of actions, processes and objects, as well as additional schemas connected to the current schema by a common principle.

The APOS Theory can be used as a teaching method or for data analysis in a study (Dubinsky and McDonald, 2001; Trigueros and Martinez-Planell, 2010 cited in Şefik, Uzun and Şenol, 2021). As a teaching method, the theory is used to design activities for the teaching of a mathematical concept and to examine the mental structures of learners after the implementation of these activities. For data analysis, the theory is used to examine the structures formed in the minds of learners after the application or teaching of a mathematical concept. Şefik et al., (2021) further note that, even though the APOS theory is mostly used as a teaching method and for data analysis, it can also be used for both purposes. In this study, the APOS theory was used both as a teaching method and for data analysis.

2.4.1 The APOS theory

Dubinsky and McDonald (2001) provide a detailed outline of the APOS theory, and after that, an example of a Mathematical Discourse and Structures lesson on quantifying learning. The theory makes testable conjectures that if a particular set of actions, processes, objects and patterns are constructed in a certain way by a learner, they are expected to accomplish a feat using learnt mathematical ideas in given situations. Theoretical analysis proposes, organised as a genetic decomposition, a string of mental constructs that a learner could construct so as to know the mathematical concept being studied. APOS theory can direct them to pedagogical programs that lead to a notable boost in the learners' uptake of complex or abstract mathematical concepts and in their use of these concepts to prove theorems, give examples and solve problems.

Theoretical analysis, according to Daher and Anabousy (2015), helps predict the learners' cognitive structures that are required to understand a mathematical concept. It encourages the development and delivery of instruction. The development of schema during learning activities is influenced by action, process and object. Opportunities for data gathering and analysis are created by the execution of instruction. This data analysis will bring about the findings of theory-based research. In an effort to understand Piaget's theories on reflective abstraction and reconstruct them in the context of high school and college-level mathematics, the theory employs interpretive methods for research and is based on a specific theoretical perspective.

Utilising an initial epistemological investigation from Dubinsky, Dautermann, Leron and Zazkis (1994), the authors determine the extent to which the APOS theory explains students' cognitive constructions of the concepts of cosets, normality and quotient groups, assess the effectiveness of instructional treatments designed to strengthen students' mental constructions, and compare the performance of students receiving this teaching treatment with those completing a traditional course. The authors report on the collective works of a group of researchers who have adopted APOS as the basis for research studies of college students' cognitive development and understandings of mathematical concepts in order to demonstrate that when a group of researchers adheres to a specific framework and theoretical perspective over time, considerable progress can be made both in understanding how learners learn mathematics and in building pedagogy on that understanding. This genesis must be put into practise by the students themselves. Students should plan ahead, exercise critical thinking and actively make sense of the material they are learning. This approach highlights how students can actively engage in creating their own knowledge. The greatest possible learning achievement is made possible by effective learning procedures. As a consequence, the researcher feels compelled to investigate how learners who use GeoGebra conceptualise function transformations using the APOS theory.

The APOS theory explains how individuals develop mathematical knowledge by constructing cognitive actions, processes and objects and ordering them into schemas

so as to muster situations and solve problems (Dubinsky and McDonald, 2001). The learner will therefore be at the centre of instruction when using a constructivist perspective. According to Nagle, Martínez-Planell and Moore-Russo (2019), the APOS theory enables interrogation of learners' cognitive development with regards to different mathematical conceptualisations. In essence, it can be considered as a way to back research or as a means to formulate instruction.

2.4.2 The constructs of APOS theory

Action level

Action is the first step in learning new concepts, claim Boz-Yaman and Yigit (2019). A learner will be unable to explain their actions because of the narrow interpretation of a truth they have memorised (Nagle et al., 2019). Learners are capable of performing operations on functions and variables step-by-step at the action level, according to Tumenbayar, Amarzaya, and Navchaa (2021). These processes, which are dependent on previously understood knowledge or outside indications, can be implemented in the context of analytical or pictorial depiction. The learners can only identify similarities between a function and its transformations in terms of a global feature of the graph, and they can only distinguish differences between a function and its transformations in terms of the syntax of the rule that defines the function. An activity will get interiorised into a process when it is repeated and the learner is able to reflect on it. In this study, a learner that could perform operations on GeoGebra and on functions and variables step by step, and these operations being applied either in the analytical or graphical representation context, relying on memorised facts or external signs was said to be operating at action level.

Process level

Learners are expected to describe transformation steps without actually performing them at the process stage. According to Tumenbayar et al., (2021), at the process stage, learners can describe changes in the basic functions as a result of applying transformations without having to carry out each step of the transformation or shift

the graph of a function one step at a time. They are able to describe the changes brought about by the transformation through examining the graph of the transformed function. In order to determine the function to which a certain set of transformations was applied, learners can also reverse the procedure. However, they struggle to coordinate information from several representational contexts and to translate information from one representational context to another. In this study, a learner able to describe changes in the basic functions as a consequence of the application of the transformation without the need to perform each step of the transformation and to reverse the process to identify the function on which a set of transformations was applied, was said to be operating at process level.

Object level

It is possible to shorten and modify new processes to create new objects, and then subsequently turn those newly created objects back into processes (Boz-Yaman and Yigit, 2019). Following that, learners will arrange actions, processes, objects and preceding schemas into a novel schema that precisely suits recently acquired knowledge (Boz-Yaman and Yigit, 2019). Tumenbayar et al., (2021) claim that learners who have an object conception of transformations are able to coordinate the properties of transformed functions in light of potential changes to the original function. At this level, learners may identify the basic function upon which any transformed function is constructed, they can de-encapsulate any transformed function into the process involved in its construction, and can compare several transformed functions in terms of their properties in any representational context. In this study, a learner able to reflect on all transformation processes and was able to think of them as a whole, in any representation or context was said to be operating at the object level.

Schema level

Schemas are conceptualisations that are created through the connection of multiple acts, processes, objects and other previously generated schema (Daher and Anabousi, 2015). According to Borji, Alamolhodaei and Radmehr (2018), the goal of mathematics instruction should be to help learners make use of their own cognitive structures while also fostering the development of new, more sophisticated cognitive

structures. The function schema, the transformation of functions object, and actions and procedures on transformed functions to ascertain their attributes or to categorise them, for instance into rigid or non-rigid transformations, would be some examples of a transformation of functions schema. Not only university students, but even high school learners find the idea of functions and their transformations difficult. Numerous research findings on how cutting-edge tools like GeoGebra can teach these ideas can be found. In this study, a learner that could form conceptions by considering the interconnection of several actions, processes, objects and other previously constructed schema was said to be operating at the schema level.

According to Bansilal, Mkhwanazi and Brijlall (2014), the APOS theory sees an individual's understanding of a concept as changing from an externally driven entity to a wholeness upon which other transformations can act. Thus, one would start by conceptualising each notion, such as the parabola, in terms of plotting the points and creating a graph from a quadratic function's equation. Then, after reflecting on the stages, one might be able to create a graph by using properties like the turning point and the x- and y-intercepts. These small steps can be seen as components of the broader quadratic function schema, and when they are encapsulated into objects, they can be used to generate a quadratic function graph. However, a learner may be hindered from progressing to higher levels of understanding the quadratic function if prior non-encapsulations of the required procedures are not met. When a formula, such as the one to solve a quadratic equation, results in the writer solving an equation that does not involve any rearranging of the equation or requires only the input of numbers, this suggests an action conception since the stimulus, in this case the formula, enabled the writer to solve the equation. Bansilal et al., (2014) use this as an empirical example for an action level. One may think of this formula as an external instruction.

On one hand, a learner would have demonstrated a process conception of solving quadratic equations if the teacher sees that they can rewrite the given equation into standard form, if necessary, and that they can calculate the solution using the quadratic formula if the coefficients are variables. On the other hand, if the teacher sees evidence in a written response of a conclusion on the nature of roots by interpreting and drawing conclusions from the properties suggested by the

expression under the square root sign in the quadratic formula, we can draw the conclusion that the person has demonstrated an object conception of the quadratic equation because they are aware of the function and significance of the individual components in the formula. All these ideas would give us empirical information on an individual's quadratic equations schema.

When grounded in a theoretical framework, research on mathematics teaching and learning is supported in a number of ways (Dubinsky and McDonald, 2001). In accordance with Oktaç, Trigueros, and Romo (2019), theoretical frameworks in mathematics education may differ in regards to the study objectives, goals, types of outcomes, situations in which they can be implemented or research questions that they may aid in addressing, among other things. They contest some of these elements with regard to the specific goals of a theory, which can include understanding a pedagogical reality or a teaching-related objective that may be connected to some rationality. A crucial aspect of the theory, known as genetic decomposition, is the creation of intricate models utilising the theory's ideas in order to achieve desired results from a scientific standpoint. This genetic decomposition, which is based on literary analysis, historical information, or the researchers' own teaching experience, can first be interpreted as assumptions about how individuals learn particular mathematical concepts.

According to Tumenbayar, Amarzaya and Navchaa (2021), a genetic decomposition is a description of how an individual's mind could construct a mathematical concept. This is distinct from a mathematical formulation of the concept, which deals with the idea's location within the context of mathematics. By developing and employing specific mental structures, which are regarded in the APOS theory as stages in the learning of mathematical concepts, individuals are able to make meaning of mathematical notions. The APOS theory served as the theoretical framework for this study.

Oktaç et al. (2019) chose to study learners' mental constructions and their potential interconnection to teaching. The authors created a questionnaire using a genetic decomposition and an interview, with one of the questions specifically intended to address this problem. The results of this study led the authors to the conclusion that

learners who demonstrated an Action or a Process perspective of linear transformations gave different sorts of responses to this question. According to the findings of this study, problem situations that might be useful include asking students to apply a particular linear transformation to various functions and attempt to explain how doing so affected both those functions and the entire domain, as well as repeating the previous question for various linear transformations as well as various domains and ranges. The ability to design instructional activities and didactic tactics with the understanding of genetic decomposition can also help teachers rethink the necessity of creating spaces for students to reflect on their actions and create opportunities for interiorisation.

Although the majority of studies employing the APOS theory focus on learners' understanding of mathematical concepts, this framework may also be used to examine the teacher's role and how it links to students' constructs as well as to consider instructors' professional expertise. Even if the theory is not used in the design of the lesson plans, it is still possible to determine whether or not the reflection opportunities provided by the instructor, the in-class activities, and the teachers' interactions with students favor or do not favor the constructions described by a genetic decomposition. According to Molina and Oktaç cited in Oktaç et al. (2019), some learners in specific situations tended to associate transformations with particular functions rather than the functions spanning the entire domain.

Oktaç et al., (2019) go on to say that due to the influence of some textbook examples where a function is given along with its graph after a single transformation, learners may mistakenly associate functions with single transformations. They chose to analyse this tendency while taking into account learners' conceptions and their potential relationship to instruction because one of the purposes of the APOS theory is the understanding of mathematical phenomena, and since this tendency had not been substantially explored in the literature. Concepts have to deal with a dialectic interaction between the two and are not found in either people or mathematical problems (Dubinsky cited in Oktaç et al., 2019). The results, however, cannot be directly related to a current didactical challenge because students from three quite diverse universities participated in the study.

The three elements of the APOS theory, theoretical analysis, instructional treatment, and data collection and analysis were utilised (Asiala et al., 1996). Theoretical analysis helped to foresee the mental structures required to understand the topic. According to Maharaj (2013), theoretical analysis depends on the researcher's understanding of the relevant idea and the APOS theory. Theoretical analysis will guide the creation and delivery of instruction for a certain idea. It will then be used to gather and analyse data, which could lead to a revision of the initial theoretical analysis of the given mathematical concept.

2.4.3 Genetic Decomposition of the function Concept

According to Sefik et al. (2021), an APOS theory-based study should include the following cyclical content: theoretical analysis of a mathematical concept, development and implementation of instructional practices based on this theoretical analysis, and data collection and analysis to evaluate and organize both the initial theoretical analysis and teaching. This cyclical information is provided up until there is sufficient understanding of the epistemology of a mathematical concept and helpful learning strategies. Generally speaking, the goal of theoretical analysis is to show how a mathematical concept is genetically broken down (Dubinsky and McDonald, 2001).

The most crucial element of the APOS theory is genetic decomposition (refer Appendix 1), a model for teaching that consists of mechanisms and mental structures that students can construct to comprehend a mathematical topic (Dubinsky and McDonald, 2001). The teaching of mathematical concepts is governed by this genetic decomposition (Baye, Ayele and Wondimuneh, 2021). Genetic decomposition can be constructed within the context of the APOS theory depending on the researcher's experience, teaching the concept, mathematical proficiency, knowledge of the concept's literature and knowledge of the concept's historical development (Arnon et al., 2014). In the APOS theory, genetic decomposition serves to both illustrate what the individual's past knowledge of the concept should be and to explain how a concept can occur in the mind of an individual. Genetic decomposition can identify the causes of the variations in learners' mathematical abilities in this situation (Arnon et al., 2014). In conclusion,

genetic decomposition describes mental structures and procedures while the APOS theory is centered on the question of how a mathematical concept is constructed in the mind of the individual.

A genetic decomposition of the concept of transformations of functions was demonstrated by Baker, Trigueros and Hemenway (2001). They suggested that learners are capable of performing transformations on functions in an analytical context by changing their values one at a time and sketching the graph at the action level. These students, who at best have a static understanding of transformation, rely on concrete visualization to infer features of transformed functions. In the case of basic functions, either in a graphical or analytical setting, learners were able to synchronize the acts of evaluating the function at different places and the change in the attributes that follows from the application of the transformation. These students had a more dynamic sense of transformation and could, but only very briefly, imagine steps in between the initial and final states based on the benchmark functions' features. The process of applying a transformation to any function was grasped by learners at the object level, and they could predict the characteristics of the modified function in advance. Without regard to how the provided information was represented, they were able to recognize a more complex function as the result of applying modifications to a simpler one. The connections between various actions, processes, objects and other previously created schema lead to the creation of schema ideas. A function transformation schema would include the function transformation object, the function transformation object schema, and actions and procedures on transformed functions to identify their characteristics or categorize them, such as into rigid or non-rigid transformations. The genetic decomposition used in this study is shown in Appendix 1.

2.5 ACE learning cycle

The teaching strategy used in this study was the Activity- Class Discussion-Exercise (ACE) cycle (Maharaj, 2010). Activity, classroom discussion, and exercises completed outside of class make up the three components of the ACE educational approach, which is based on the APOS theory (Maharaj, 2013). The activities that are intended to encourage learners' development of the mental structures required by an APOS

analysis make up the first stage of the cycle. Here, the instructor helps students reflect on the activities and how they relate to the mathematical ideas being learned. Classroom discussions are used by learners to achieve this. They talk about their findings and listen to the teacher or other students explain the mathematical implications of the tasks they will be completing. The questions in the homework are quite typical. They supplement the information learned through classroom activities and conversations. Students then use this knowledge to address issues associated with the topic being studied.

The ACE learning cycle is a recurring circle comprising three components; Computer-based activities, Classroom discussions, and Exercises completed outside of class (Arnon, Cottrill, Dubinsky, Oktaç, Fuentes, Trigueros and Weller, 2014). The authors assert that numerous research studies have documented the use of the ACE learning cycle and its efficacy in assisting students in building conceptual models of mathematical ideas in their minds. The first stage of the cycle involves activities, where students collaborate in groups on tasks aimed at assisting them in creating the mental structures suggested by the genetic decomposition. Instead of focusing on getting the right answers, the challenges were designed to encourage introspective abstraction. The tutorial-style activities students completed were intended to aid them in building mental models. All of the small groups of students participated in the class discussion, which was the second phase of the ACE learning cycle. The teacher facilitated the discussions. The discussions focused on the paper-and-pencil assignments that students produced based on the lab activities finished in the Activities phase. The third cycle's homework exercises were made up of typical issues that were meant to reinforce the computer activities and discussions that had already taken place in class. The activities supported the ongoing growth of the conceptual frameworks proposed by the genetic breakdown. They aid students in putting what they have learnt into practice and in thinking about related mathematical concepts.

Comparing the constructs envisioned in the genetic decomposition, which were updated when thought essential, was part of the data collection and analysis process. The underlying premise is that a person's ability to respond to situations that are perceived to be mathematical problems and their solutions by reflecting on them in a social context and creating or reconstructing mathematical structures to use in dealing

with situations is what constitutes an individual's mathematical knowledge. According to Maharaj (2013), a person does not immediately learn mathematical concepts. Instead he or she uses mental models to comprehend a mathematical idea. If a person possesses the mental structures necessary for a specific mathematical notion, learning is made easier. Learning the concept is nearly impossible if the necessary mental structures are missing.

The APOS theory and the ACE teaching cycle have several ramifications. First, they suggest that teaching practices should focus on assisting students in developing the right mental structures. According to Amawa, Yerizon, Sri and Putra (2019), one method of assisting students in creating their own mathematical knowledge is by giving them worksheets with learning exercises that must be completed before the student may eventually understand some mathematical concepts or theorems. Worksheets can help students learn maths more effectively. The writers also mention other methods that can be applied when creating a worksheet. They identify the APOS theory as a method of instruction designed specifically for high school or college mathematics instruction that incorporates computer technology, cooperative learning and a focus on the mental processes involved in comprehending mathematical ideas and theorems. Dubinsky and Mcdonald (2001) assert that students' understanding of mathematical concepts is shaped by their production of those concepts and their interactions with others. The second goal is to direct students in using these frameworks to build their comprehension of mathematical ideas. According to the APOS hypothesis, these mental structures are the actions, processes, objects, and schemas.

According to research based on this idea, it is necessary to identify the most likely mental structures for a given topic before creating learning activities that will help those mental structures form (Maharaj cited in Siyepu, 2013). Instead of providing a statistical comparison of students' arithmetic results, implementing the APOS theory aims to show the character of learners' comprehension. The focus of the mathematical knowledge expectation in this study is on performing an APOS analysis of a mathematical concept like function transformations. This analysis' output is referred to as a genetic decomposition (Maharaj cited in Siyepu, 2013). A concept's genetic decomposition is defined as a collection of mental constructions that may explain how

the concept can emerge in a person's mind. It postulates the specific activities, procedures and things that assist in creating a mental model for navigating a specific mathematical scenario.

2.6 Transformations of functions, APOS theory and GeoGebra

Chimhande, Naidoo and Stols (2017) conducted a study on South African Grade 11 learners' levels of mastery of concepts associated with functions. Six learners participated in the study and their reasoning and interpretations when engaged with function connected concepts were sought. Task-based individual interviews which were open-ended were conducted. The interview questions were categorised into APOS theory conception levels which were justified by aligning the demands of each question with the indicators of each conception level. The study mentioned that the DBE through the CAPS document expected Grade 11 learners to be functioning at the object level before instruction and at the schema level thereafter. The findings of their research indicated that a sufficiently great number of learners were functioning at the action and process levels after instruction. Their conception was portrayed by vague descriptions of function associated concepts and cramming of steps for working between algebraic and graphical representations. Harel and Soto (2017) mentioned that for the process level to be conceptualised into the object level, more research was needed and this would benefit pedagogical practice.

Daher and Anabousy (2015) also conducted a study on learners in Grade 9's conception of function transformations using GeoGebra and observed that GeoGebra helps learners to understand transformations of functions. They used APOS theory to analyse learners' understanding and conception of transformations of functions. In like manner, Triantafyllou and Timcenko (2014), conducted their research on challenges and opportunities of using technology in mathematics education. They found that technology could disconcert learners since they generally struggled in mathematics because of a paucity of elemental skill and conceptual understanding or due to a lack of ability to perform deductive reasoning. Furthermore, Bhagat and Chang (2015) noted that learners with a low understanding of mathematics were not in a position to take advantage of tools such as GeoGebra as they felt obliged to use them. From reading these past studies that are mentioned in this chapter, the researcher feels that the benefits of integrating GeoGebra in the teaching of transformations far outweigh

the disadvantages that have been raised by authors such as Triantafyllou and Timcenko (2014) and Bhagat and Chang (2015).

Chen (2020) examined indicators of Dynamic Geometry Software (DGS) take-up by resorting to the use of GeoGebra. The research incorporated elemental predictors of the technology acceptance model like ease of use, perceived usefulness and attitude towards use by learners. Additional features on GeoGebra had an extraordinary effect on learners' behavioural intentions to use it. Salinas-Hernandez, et al., (2018) carried out a study on the use and integration of GeoGebra into classrooms. Participants for that study consisted of a qualified and experienced teacher that used GeoGebra with Grade 11 learners. Findings in that study showed that teachers had difficulties in integrating GeoGebra in their classes despite their experience.

Boz-Yaman and Yigit-Koyunkaya (2019), posit that teaching functions and the concepts that go along with them is a key component of mathematics instruction. So, among other big principles like families of functions, covariation and rate of change, and multiple representation of functions, combining functions and transformations of functions is crucial for learning transformations. The properties of addition, subtraction, multiplication, division, taking the inverse, and composition of other functions serve as the foundation for this concept. When these calculations are used, the graph of a function may change, appear to stretch or shrink, move horizontally or vertically, or undergo other transformations. When teaching, this intricate major idea is saved for the final segment. This is so that it can integrate well-established basic understanding with other major ideas. More than anything else, Boz-Yaman and Yigit-Koyunkaya's (2019) investigation sought to highlight concept images and descriptions of function transformations and examine pre-service mathematics teachers' (PMTs) cognitive constructions about those transformations from an APOS theory perspective. The authors analysed how the participants responded to the definition of function and the transformation of function. The purpose of the interview questions was to identify the PMTs' concept descriptions of function and function transformation as well as concept images, if any were drawn or revealed in response to the questions. Burcu, for instance, described the transformation of a function as shifting the function on the x-y plane when asked about it during an interview. This demonstrated that her concept definition for transforming functions was shifts on the cartesian plane.

The results of the aforementioned investigation showed that each respondent had a unique concept definition and concept image for every family of functions. Nobody on the PMTs could adequately explain what a function transformation was. This finding demonstrates that respondents' knowledge of function transformations, inverses, compositions and their relationships was limited. In explaining the transformation of a function, no responders mentioned shrinking or stretching. A lack of conceptual discussion about function transformation in the mathematics classroom may lead a weak interpretation to gradually emerge. In mathematics classes, transformations of functions were typically not the main topic of discussion.

Zazkis, Liljedahl and Gadowsky (2003), state that throughout the past few decades, research on mathematics education has focused heavily on the learning, teaching or conceptualization of functions. According to a range of scholarly studies on functions, the transformations of functions are not given much attention. They looked at the justifications offered by participants for the perceived incoherency caused by a horizontal translation and proposed a pedagogical approach for functional transformations that covered the puzzling nature of graphs behaving in unexpected ways. The authors suggested a pedagogical method for teaching function transformations and discussed what they believed to be the underlying cause of the observed inconsistency in parabola translation. The findings showed that knowledge of this transformation and the ability to defend it, did not necessarily follow from the ability to correctly predict the direction of horizontal translation. Pre-service and in-service instructors did not significantly differ in how they interpreted the translation. Because translations were the subject of their research, Zazkis et al., (2003) concentrated on this issue. They recommended strengthening understanding of the other transformations using a similar strategy. Recognizing the challenges of visualising a horizontal translation as opposed to a vertical one, they claimed that there is significantly more work involved in visually responding to the transformation of f(x) to f(x + k) than when looking at the transformation of f to f(x) + k. The authors wanted to explore if this problem could be avoided or at least lessened, so they first looked at how teachers and learners dealt with it.

The students offered their consent to take part in the study and had been referred by the participating teachers. They were all judged to have abilities that were above average. The participants were first asked to sketch the two graphs on the same x-y plane and to explain how they did it. They were instructed to compare their graphs with those on a graphing calculator. After it was determined whether the sketch was correct or incorrect, the respondents were asked to comment on where the graph of $y = (x-3)^2$ was located in relation to the graph of $y = x^2$, narrating as necessary any discrepancies between the sketch and the graph displayed on the calculator or discrepancies between the intuitive presupposition and the known result. The participants were instructed to discuss the graph of $y = x^2 - 3$ and compare it to $y = (x-3)^2$, if the question did not spontaneously arise.

Interestingly, both teachers and learners understood that the shape of $y = (x-3)^2$ is a parabola and that it is identical to the canonical parabola $y = x^2$. However, it was implied that there was no need for further explanation because the graph of $y = (x-3)^2$ is merely a horizontal translation of $y = x^2$. The direction of this horizontal translation was then given special consideration by the authors. Prior to the study being conducted, the responding learners had studied parabola transformations. In the interview, it was shown that five out of ten students (or 50 percent) had incorrectly predicted the physical location of the equation $y = (x-3)^2$.

Although everyone agreed that the observed location of the transformed graph was counterintuitive, the students made no sincere attempt to explain the phenomenon. The results of this study demonstrated that students often cling to contradictory and divergent notions while maintaining their intuitive beliefs. The findings have consequences for mathematics education generally, and transformations of functions in particular, as conception problems and technology challenges continue to be barriers to learners' progress.

It is clear from the studies presented so far in this chapter that it is possible to infer the cognitive constructions of learners from responses based on successful description of activities using GeoGebra and paper and pencil exercises. According to the aforementioned studies, GeoGebra can be considered a potent cognitive mechanism

that links learners' physical computer uses as well as their paper and pencil activity with their mental activities in a systematic manner. Additionally, combining effective teaching methods with cutting-edge technology may be required for learners to assimilate concepts like transforming functions and other mathematical concepts in a better way. These were a few of the justifications or explanations for using GeoGebra in this study.

2.7 Gap in knowledge

This review of literature was carried out to appreciate more, the integration of GeoGebra and why it was believed to enhance learners' conception of function transformations. Although not exhaustive, the reviewed studies provide a clear picture of how GeoGebra was incorporated and what it accomplished to enhance learners' conception of function transformations. According to the reviewed studies, learners can explore concepts in GeoGebra on their own and grasp them with the least amount of teacher aid by working alone or with their peers. Additionally, it has been seen that learners become more involved as they use GeoGebra to learn.

The researcher further notes consideration must be given to the type of research that South African mathematic scholars undertake in order to understand the nature of learners' acquisition of functions. In South Africa, particularly in rural regions, there does not seem to be much research on how students learn functions. This chapter has demonstrated the critical need for mathematics education research, particularly in rural areas where there is a deficiency of literature addressing learners' knowledge of functions.

2.8 Chapter summary

This chapter gave a summary of the literature, including an understanding of GeoGebra instruction, an understanding of function transformations, and an awareness of the significance of instructional design in the teaching of function transformations. The debate makes it clear that new information should be added into

current related works to make it easier to grasp how functions transform. A detailed explanation of the APOS theory and how it helps learners learn function transformations was provided. It also considered how the ACE cycle will be implemented in this study. The research approach is outlined in the next chapter.

CHAPTER 3

RESEARCH METHODOLOGY

3.1 Introduction

This chapter describes the techniques used to gather and analyse the data required to answer the research questions. The study methodology includes a description of the research paradigm, research approach, research design, sample methodologies, data generation instruments, data analysis techniques, quality criteria and ethical considerations.

3.2 Research paradigm

According to Kumatongo and Muzata (2021), paradigms are broad philosophies or points of view. This can be construed to mean that paradigms are a collection of shared assumptions and principles that scientists use to frame and approach problems. Positivism, interpretivism, post-positivism, critical theory, constructivism are some categories for research paradigms. Constructivism is the paradigm that brought this study to light. This was mainly because constructivism, also known as social constructivism, is founded on the idea that people attempt to understand their surroundings by creating subjective interpretations of their experiences (Creswell, 2002). The aim of a constructivist approach is to interpret the meanings that participants assign to the situation around them and to rely as much as possible on the participants' perspectives of the topic being studied.

3.3 Research approach

This study used a qualitative approach. According to Creswell (2002), a qualitative approach is an open model that naturally arises in a scenario and gives the researcher the ability to gather a great volume of data from active engagement in the actual experience. The emphasis is on meaningful social activity and a thorough understanding of how meaning is created in daily life and the real world (VanWynsberghe and Khan, 2007). It assumes that every reality has a number of possible entrances. This study used a qualitative methodology to examine detailed descriptions of processes to better understand how GeoGebra could be used to

enhance Grade 11 learners' understanding of function transformations (Merriam, 1998). Since this study is concerned with the meanings that learners formed when transforming functions, qualitative research is also appropriate. Additionally, the researcher chose qualitative research because it is a useful technique for exposing participants' mental constructs during a teaching or learning process. According to Merriam (1998), the goal of qualitative research is to develop a thorough grasp of particular phenomena and go behind the surface of a situation to offer a rich context for comprehending the phenomena being studied.

3.4 Research design

A research design, according to Yazan (2015), is the plan that a researcher chooses to follow when conducting research in order to provide accurate and reliable results. The researcher chose a case study design above other research methodologies to accomplish the goals of this study. This study was driven by Merriam's (1998) case study design. A case study is an in-depth qualitative interpretation and analysis of a bounded situation, such as a social unit or a person (Merriam, 1998). This interpretation leads to a more comprehensive understanding of a certain aspect of a situation and yields useful information that could not be obtained in any other way. This suggests that the case study design is a way to understand a situation where the researcher is more interested in the process of inquiry than the research's findings (Brown, 2008). The 28 students in Grade 11 who were being taught mathematics by the researcher were the bounded circumstance in this study. The study specifically focused on enhancing learners' conception transformations of functions using GeoGebra. The case study in the qualitative paradigm focuses on a specific reality that is pertinent to the phenomena being investigated.

Delimiting the study's target, or the case, is the primary distinguishing factor of case study research (Merriam, 1998). According to Merriam (1998), the case can be thought of as a unit or thing with well-defined borders that the researcher can contain and decide what will be investigated within. There may be a cap on the number of people who can be interviewed, a time limit for observations, or a specific instance of an issue or concern. Consequently, a mathematics class for students in Grade 11 served as the study's case. Merriam (1998) further says that a case has fences or borders. The

boundaries could be set by the maximum number of interviewees, a time limit for observations, or the existence of a specific problem, worry or hypothesis (Brown, 2008). The 28 students in Grade 11 at the chosen school who were using GeoGebra to learn function transformations formed the study's boundaries.

The three essential elements of Merriam (1998)'s case study were met by this investigation. It was particularistic, to start with. According to Merriam, being particularistic means being in tune with the case's particular focus (Brown, 2008). This study investigated if teaching using GeoGebra to Grade 11 learners improves their conception of function transformations. It was also descriptive, secondly. A descriptive case study refers to the extensive anthropology description and is comprehensive and very literal in its portrayal of the research's findings (Brown, 2008). The descriptive case delivers information from a wide range of sources and opinions in a number of ways, illuminating the complexity of the situation. The goal of this study was to provide a thorough analysis of whether teaching Grade 11 learners with GeoGebra improves their conceptual understanding of function transformation. This was achieved from the application of the APOS theory, ACE cycle and genetic decomposition in teaching function transformations using Geogebra. The study also employed heuristics. A heuristic case study can throw light on the phenomenon, enabling the reader to deepen their understanding, find new context, or validate what is already known (Brown, 2008). The problem statement, the study's background, what happened and why are all provided in this study in order to shed light on the challenges Grade 11 learners encountered when learning function transformations.

Merriam discussed three different types of case studies: descriptive, interpretive, and evaluative (Conway, Pellegrino, and West, 1998). By concentrating on case studies' purpose, she discovered what distinguishes these case studies from others. Descriptive case studies thoroughly define the unit of analysis without attempting to analyze or assess it. In an interpretive case study, the researcher gathers as much information as they can on the issue in order to analyze, interpret, or theorize the occurrence. Finally, even though these case studies include explanations and descriptions, the ultimate goal of an evaluative case study is to arrive at a conclusion. The evaluative case study is ideal for assessing a school or an educational initiative. This study is an interpretive case study since it gathered a lot of data about the

challenges with transforming functions in the literature review in Chapter 2. An extensive theoretical thematic analysis was also undertaken in Chapter 4.

3.5 Sampling

The type of data the researcher sought to collect and which participants and documents were most suitable to do so guided the sampling strategy used in this study. The case study design (Merriam, 1998) that was used has two levels of sampling built in. The first level was choosing the case to research, and the second was choosing a sample of the case's participants. According to Njie and Asimiran (2014), the depth and quality of the data to be acquired, as well as its richness in revealing clear viewpoints, are deemed to be more relevant than the sample size, which should not be a significant issue. The researcher concentrated on finding the most accurate and comprehensive data possible. In order to choose the participants from whom the researcher could obtain detailed data about the phenomena, convenience sampling was used. In a convenience sampling, participants are chosen based on their accessibility (Ellinger, Watkins and Marsick, 2005). 28 conveniently chosen Grade 11 students who were being taught by the researcher at a private secondary school in Polokwane, Limpopo Province, made up the sample for this study. 17 of the participants were female, and 11 were male. Two of the males in the class were Indian, while the remaining participants in the whole class were Black. They all attended the same school where I taught as a researcher, so using this sample was indeed convenient. It allowed the researcher easy access to one of the school's classrooms and provided the ability to schedule interviews with the participants. The participants were from relatively wealthy homes that offered strong educational assistance to their children. Approximately 5 of the 28 pupils were boarders.To minimise bias in the study, the researcher’s contribution to all aspects of the research were made known to prospective readers. The researcher explicitly discussed the meanings of key words that were used in the study and the social interactions with participants in their constructions of knowledge throughout the various stages of the research.

3.6 Data collection

The APOS theory and Merriam (1998)'s ideas guided data collection. Transformations of functions were taught using GeoGebra. Each learner had GeoGebra downloaded on their electronic gadget. The chapter was composed of three lessons (**Appendix 7**), where each lesson was an hour long. The first lesson covered translations of different functions or graphs. Learners were using sliders on GeoGebra, writing down their observations on the effects of changing the different parameters in the standard equations for a straight line, parabola, hyperbola and exponential functions. The standard equations used were $y = ax + q$ or $y = mx + c$ for a straight line; $y = ax^2 + bx + c$ or $y = a(x + p)^2 + q$ for a parabola, $y = \frac{a}{x+p} + q$ for hyperbola and $y = a \cdot b^{x+p} + q$ for the exponential function. The second lesson was about deductions from sketch graphs, with learners encouraged to confirm their answers on GeoGebra. In the third lesson we had a consolidation of all concepts and learners got a printed copy of the homework exercise that was used to eventually collect data for this study in line with the ACE cycle. All lessons were designed to enable the use of GeoGebra in knowledge construction and development of learners' APOS levels. Data obtained was interpreted according to the APOS theoretical framework. Data was collected using participant observation, documents in the form of scripts for daily work and a homework activity given to learners and task based interviews.

3.6.1 Task-based interviews

Task-based interviews offer a chance to evaluate learners' conceptual understanding, but they may also present chances to deepen that understanding (Assad, 2015). In a task-based interview, the interviewer engages the interviewee while he or she is working on a task or has submitted a task, and the task has been marked. A task that has been carefully selected is essential to creating this environment. The interview process can be pre-planned and structured, with the interviewer's questions, or it can be semi-structured, with the interviewer deciding how best to respond to the learners' mathematical reasoning. The interviewer might encourage participants to self-correct when they make mistakes or to broaden or generalize a topic by asking them questions. The interview helps the participants

extend their conceptual grasp of the topic by encouraging them to reflect on their own approaches and ways of thinking mathematically.

Task-based interviews were used in this study in conjunction with written work from daily work and homework, and lesson observations. Using the interviews, it was possible to learn more about how learners build the necessary knowledge by investigating how they comprehend how functions are transformed. The task-based interviews were conducted primarily to examine the learners' mental structures created when transforming functions. Task-based interviews also give researchers the opportunity to learn about the strategies utilized by learners and the challenges they face. The researcher made sure to start the interview session by asking the simpler questions in order to get detailed descriptions of the work covered. He also made an attempt to let the responders know what the study's goals were and that they were welcome to ask any questions. In order to acquire authentic data, the interviewer must build a connection with the participants (Denzin and Lincoln, 2011). In order to maintain the flow and better understand how the learners created their knowledge while transforming functions, the researcher carefully probed the learners' responses, followed up on learners' thoughts, and posed open-ended questions flexibly. Task-based interviews must include probes because they provide respondents the ability to elaborate on their answers and offer additional solutions, addressing concerns about the responses' thoroughness (Denzin and Lincoln, 2011). This can be done by asking for clarification on a concept that is not well-expressed or by asking the same question again. Other questions posed elicited the kind of information that the researcher believed is helpful in understanding the participants' level of APOS conception. When conducting task-based interviews, Patton (1990) promoted the use of the "how" and "why" questions, but he also stressed that it is preferable to start with the "what" questions because he thinks the "how" and "why" questions are harder to understand.

Following a review of interviewee's work on GeoGebra, written daily work and homework, interviews were conducted with some of the participants. They were questioned about some of their answers to particular questions. The researcher identified the areas of challenge and the reasons why the essential mental constructions suggested by the genetic decomposition were not successfully made.

Every time a question was presented, the researcher would give the participants time to formulate their thoughts and expound on their answers. Additionally, prompts were utilized to explain any questions that some of the participants did not seem to understand.

3.6.2 Participant Observation

All social and behavioral sciences research is based on participant observation because it gives researchers the chance to collect real-time data from social interactions that are occurring in the real world (Adler and Adler, 1994). They also argue that using participant observation as the primary research data collection tool allows the researcher to apply immediate insight, which may produce data that is more reliable and real. As a participant observer, as defined by Creswell (2002), the researcher in this study observed and questioned the participants in the learning process. The researcher observed how learners engaged with the technology and their peers while teaching transformations of functions in accordance with the ACE cycle (refer Appendix 8). The researcher took note of the APOS theory's action and process levels that the learners demonstrated. For transcriptional purposes during data analysis, videotaping of every interview done with learners during lessons was done. The recordings additionally gave the researcher the chance to revisit the interview sessions and the chance to watch the learners' emotions and other body language while they engaged in the in-class activities. The goal of this data collection strategy was, among other things, to characterise the learners' responses to cues given to them regarding genetic decomposition, note actions as they interacted with the GeoGebra, and identify any difficulties and successes the learners experienced during task completion.

3.6.3 Homework activity

Using genetic decomposition of functions, a homework activity on function transformations was created (refer Appendix 1). The written activity, that each learner completed on their own at home covered the Process, Object, and Schema levels. Written homework scripts by learners served as part of the study's

documents. After the entire section on transforming functions had been taught, the homework task was given to learners. This fits with the ACE cycle, in which the E for exercise is the third step in the cycle. The purpose of the homework was to assist in answering the study's first research question. The first research question focused on the challenges that learners encounter when transforming functions.

3.7 Data analysis

According to Khokhar, Pathan, Raheem and Abbasi (2020), qualitative data analysis (QDA) is a method for establishing themes and linkages by recognising statements or phrases. The purpose for doing QDA, is to examine and analyse the symbolic and meaningful content of the data. A fundamental component of qualitative research is QDA, which is the analytical process of dissecting the data to identify themes. Theories and fresh knowledge are inevitably born from themes. Thematic analysis (TA) is a prominent analysis technique in QDA. According to Maguire and Delahunt (2017), TA is the process of finding themes or patterns in qualitative data. They contend that rather than being a methodology, TA is a method. As a result, unlike many qualitative techniques, TA is not constrained by a certain theoretical or epistemological stance. Bearing in mind the diversity of work in learning and teaching, this makes it a very flexible method.

As suggested by Merriam (1998), data collection and analysis were done simultaneously and iteratively in this study. The data analysis was influenced by the APOS theory and theoretical thematic analysis of qualitative data. Thematic analysis is a technique for identifying, analysing and interpreting patterns of meaning in qualitative data (Clarke, Braun and Hayfield, 2015). A variety of academic frameworks and even research paradigms can use thematic analysis. It is possible to use it to find patterns in data that relate to individuals' actual experiences, opinions, behaviors and practices. It also aims to understand what people say, feel and do. To capture both overt and covert interpretations, it can be applied to both deductive (theory-driven) and inductive (data-driven) analysis. According to Clarke et al. (2015), there are two types of theoretical analysis: bottom up or inductive, which is driven by the data itself, and top down or deductive, which is driven by specific research questions or the theoretical framework, the APOS theory in this study.

The deductive technique, according to Kiger and Varpio (2020), employs an existing theory, framework or other researcher-driven focus to find themes of interest. The deductive technique is effective for focusing on a single feature of the data or a specific result that could be best illuminated or understood in the context of an already-existing theory or frame. The inductive approach tends to provide a more expansive study of the full body of data. While any methodology is valid, it is crucial to specify it in a study so that readers may correctly comprehend and contextualize results. Given that it used the APOS theory as its lens, it was wise for this study to select the deductive TA based on the notions outlined above. The Clarke et al. (2015) definition of TA was used in this investigation. The researcher considered TA to be an effective and relevant approach to apply in this study since it tried to explore a variety of learners' experiences, attitudes and behaviors as they dealt with transformations of functions.

Six steps make up the (TA), namely; learning the data, making codes, searching for themes, analysing the themes, labelling or defining the themes and writing the report or write-up are all steps in the process. The researcher made notes as he marked the learners' daily and homework activities so that he could be familiar with the study's data. The researcher repeatedly viewed videos that were recorded during lessons. While transcribing, he repeatedly listened to the audio recordings of the task-based interviews. The codes were predetermined from the acronym APOS, with indicators from the genetic decomposition being fitted to these codes after the researcher had familiarised himself with learners' written or verbal responses. These responses assisted the researcher to come up with themes that were reviewed and refined so as to produce the write-up.

According to Clarke et al. (2015), TA is intended to be a recursive process rather than a linear one. This means that if new information or new themes emerge that call for more examination, earlier steps may urge the researcher to go back and revisit them. Using the ACE cycle, the researcher evaluated how learner perceptions of transformations of functions had changed as a result of the GeoGebra-based instruction. The researcher examined learners' performance on a mathematical activity and whether they were successful or unsuccessful using the predicted mental structures. The researcher created a genetic decomposition of transformations of

functions and utilised it to create lesson plans (refer Appendix 1). When a learner effectively applied formulas and was able to adhere to instructions that were delivered in a clear and understandable manner, these were action level indicators. A learner was able to explain aspects of specified function transformations in part throughout the process stage. A learner was fully aware of the process at the object stage. The learner was able to provide logical justifications for intricate function transformations at the schema level.

The interview data for this study were gathered through task-based interviews conducted in class and with participants whose answers to the homework assignment the researcher desired to be more clear. The interviews were audio recorded and transcribed to examine how learners conceptualised transforming functions and how they fitted into genetic decomposition indicators to evaluate them using codes predetermined by the APOS theory. The emphasis was on the experiences that learners had when transforming functions, as well as how much they interacted with and used GeoGebra. In order to further analyse the data and link it to APOS theory components, the researcher additionally used observation notes from lessons taken as learners completed worksheets and GeoGebra tasks.

In order to establish the final themes and respond to the research questions, the findings from the analysis were triangulated by combining the findings from the three data collection methods—classroom observations, task-based interviews and written scripts for daily work and homework. Triangulation is the process of combining two or more data sources, for example, researchers, methodological approaches, theoretical viewpoints or analytical methodologies into a single study (Thurmond, 2001). These combinations lead to data triangulation, investigator triangulation, methodological triangulation, theoretical triangulation or analytical triangulation. Data triangulation was used in this study. Time, space and people are the three different categories of data sources in data triangulation. This implies that data sources can differ depending on the dates the data were acquired, the location or environment, and the people who provided the data (Thurmond, 2001). As an example for this study, the task-based interviews and lesson observations took place at various periods in conjunction with the homework and homework interviews. The location or venue for the task-based interviews and lesson observations was different from where the homework was

assigned. The researcher's designated classroom served as the setting for both lessons and task-based interviews. Additionally, several learners contributed to the data collection by writing, gesturing, or speaking.

3.8 Quality Criteria

According to Lincoln and Guba (1985), the essential criteria for quality in qualitative research should be transferability, confirmability, dependability and credibility.

3.8.1 Transferability

Fit or similarity between settings directly influences transferability, claim Lincoln and Guba (1985). This means that the transferability of research findings to other settings is determined by how similar the research environment where the phenomenon being studied occurs and the settings in which the results are expected to be transferable are. To ensure transferability, this study included thorough explanations of the research process, including transcripts from interviews, observations and the written responses from homework activities..

3.8.2 Confirmability

Confirmability, according to Baxter and Eyles (1997), is the likelihood that other researchers will be able to confirm or verify the study's findings. It is focused on proving that the data and interpretations of the findings are obviously drawn from the data and are not the product of the researcher's imagination. The researcher promised to give the audit trail so that we could always go back and double-check the findings of the task-based interviews and written responses. This was done to assure the study's confirmability.

3.8.3 Dependability

The dependability issue should be addressed by explicitly describing the study's processes so that future researchers can replicate the work, even if they don't necessarily get the same results (Shenton, 2004). To make sure that this study was reliable, in-depth methodological descriptions were given, along with interview

transcripts, document analyses of the learners' written responses and a full explanation of the coding technique used.

3.8.4 Credibility

Credibility, in the words of Merriam (2002), deals with the issue of "How consistent are the findings with reality"? According to Lincoln and Guba (1985), establishing credibility is one of the most crucial elements in determining trustworthiness. Prolonged involvement, constant observation, triangulation, and member checking are methods to establish credibility (Korstjens and Moser, 2018). The researcher invested enough time in this study to get to know the learners who had the potential to supply valuable data. The participants were taught by the researcher since Grade 10. Due to the two years he spent teaching the participants, the researcher was able to identify the characteristics and elements that were most relevant to the problem at hand. Data triangulation was made possible by the use of many data sources, including classroom observations, task-based interviews, and written scripts for daily work and homework. When the study's findings were being reviewed in the chapter that follows this one, the researcher combined the recognized themes from the three data gathering instruments, which is how data triangulation was accomplished in this study. The researcher's supervisors received feedback on the data, analytical categories, interpretations and results, which ensured member check. The data were enhanced by the member check, particularly because the researcher and supervisors viewed the data differently.

3.9 Ethical considerations

Permission, informed consent, confidentiality, anonymity and assent were observed in this study (McMillan and Schumacher, 2010).

3.9.1 Permission

The University of Limpopo Turfloop Research Ethics Committee (TREC) granted permission for this study to be carried out (refer Appendix 6). Additionally, the principal gave the researcher permission to conduct the study at the specified school (refer Appendix 3). In a letter to the parents, the researcher outlined the goals of the study and requested that their kids participate (refer Appendix 4).

3.9.2 Informed consent

The participants in this study were informed of the study's goals and the potential consequences of their participation. They were informed of the data collection procedures and given time to ask any queries they might have had. Their participation in this study was voluntary and free from coercion or undue influence because it was made clear that they had the capacity and right to withdraw from it at any time.

3.9.3 Confidentiality and anonymity

In this study, participant confidentiality was always maintained. The researcher utilized pseudonyms to protect participant privacy and ensure that their names and the name of their school were kept secret. The researcher used pseudonyms while gathering data. The participants were coded using the tags "L1," "L2," and so on, where the order had no bearing on anonymity. This allowed for the arrangement of the data while preventing a connection between the responses and the original participants in publications. At every stage of the study, data was collected, stored securely, and handled with confidentiality. Names, titles, ages, and genders of participants were not mentioned. It is important to mention that when a participant is anonymous, you are unaware of their identity; when a participant is confidential, you are aware of their identity but omit that information from your study report. Both represent significant ethical issues.

3.9.4 Assent

The study acknowledged that learners are still regarded as children because they are under the age of 18, which implies that they are not old enough or mature enough to give informed consent to participate in research (Department of Health, 2015). The adoption of a participatory rights agenda that sought the learners' consent to participate in the study (refer Appendix 5). With the participants' assent, task-based interviews and instruction were taped and recorded on video in this study. The assent forms were provided by the researcher so that students may indicate their willingness to participate in the study. This was done prior to the topic being taught and the interviews with the participants. The learners could understand the language used to

write these assent forms. Participants' identities were secured and they were made aware that the data was being collected solely for research purposes during data presentation and discussion.

3.10 Chapter summary

The techniques and tools utilised in the study were the chapter's main focus, and they were thoroughly described. An introduction of the research methodology and a description of the research design employed in the study were included in the chapter's opening. The researcher went over participant eligibility and sampling methods. The chapter went into great detail about the methods and instruments used to collect data. Additionally, the researcher took into account matters relating to ethical issues in line with qualitative research methodologies as well as standards for the findings' quality. The analysis and interpretation of the data gathered will be presented in the following chapter.

CHAPTER 4

FINDINGS AND INTERPRETATION

4.1 Introduction

The techniques for gathering the data required to provide the research questions with answers were covered in the previous chapter. The approach took into account the study's design, sampling procedures, data gathering and analysis methods, quality criteria and ethical consideration. In respect to the two research questions, this chapter presents and examines the study's findings and their interpretation:

- What are the challenges faced by Grade 11 learners when transforming functions?
- How does the use of GeoGebra enhance learners' conceptions of transformations of functions?

Participant observation, task-based interviews, documents in the form of written daily work scripts, and homework exercise scripts given to learners were all used to gather data. The analysis of the data was influenced by the APOS theory and deductive thematic analysis of qualitative data. Predetermined codes and themes identified throughout the data analysis are presented along with the results. In Polokwane, the study was carried out at a small private school where the researcher was working. The learners had access to GeoGebra even after school because they were utilising their own laptops.

4.2 Analysis of classroom observations

The researcher developed lesson plans (refer **Appendix 7**) and a classroom observation sheet (refer **Appendix 8**) on which he made notes of activities by the learners in the teaching and learning of transformations of functions. These activities included learners' participation, interest and performance during lessons.

The predetermined codes for the three lessons were action, process, object and schema levels as explained in the genetic decomposition of transformations of functions (refer **Appendix 1**). Descriptions of learners' work on GeoGebra and on the worksheets (refer **Appendix 9** and **Appendix 10**) were fitted to the appropriate APOS predetermined code and emerging sub-themes were identified by the researcher.

4.2.1 The action level

A learner working at the action level can conduct operations on GeoGebra, as well as on functions and variables. Based on prior information or external signals, these actions may be applied to either analytical or pictorial representations.

In the first lesson, learners were performing step by step operations on the functions by entering the standard equations: $y = ax + q$, $y = a(x + p)^2 + q$, $y = \frac{a}{x+p} + q$ and $y = a \cdot b^{x+p} + q$ on GeoGebra and moving parameter sliders. They were required to note their observations or describe changes in the basic functions. They could visualize the transformations taking place after moving sliders for each of the parameters on GeoGebra as shown in Figure 4.1 below.

Figure 4.1: Researcher explaining use of GeoGebra sliders for a linear function to a learner

The researcher gave out a worksheet on translations of graphs using GeoGebra. Learners were working on their laptops and writing observations in their exercise books. The researcher was moving around the class, checking progress and assisting learners where necessary.

Sub-theme 1

The sub-theme deduced from observing the lesson was that learners were able to *describe the changes on the graph of the basic function after moving sliders on GeoGebra.*

The learners answered the first five questions on the worksheet (refer **Appendix 9**) by writing their observations after moving parameters a, p and q for the different standard graphs: $y = ax + q$, $y = a(x + p)^2 + q$, $y = \frac{a}{x+p} + q$ and $y = a \cdot b^{x+p} + q$.

After entering the equation for the straight line, $y = ax + q$, GeoGebra will return an image with $a = 1$ and $q = 1$ as the one in Figure 4.2a below:

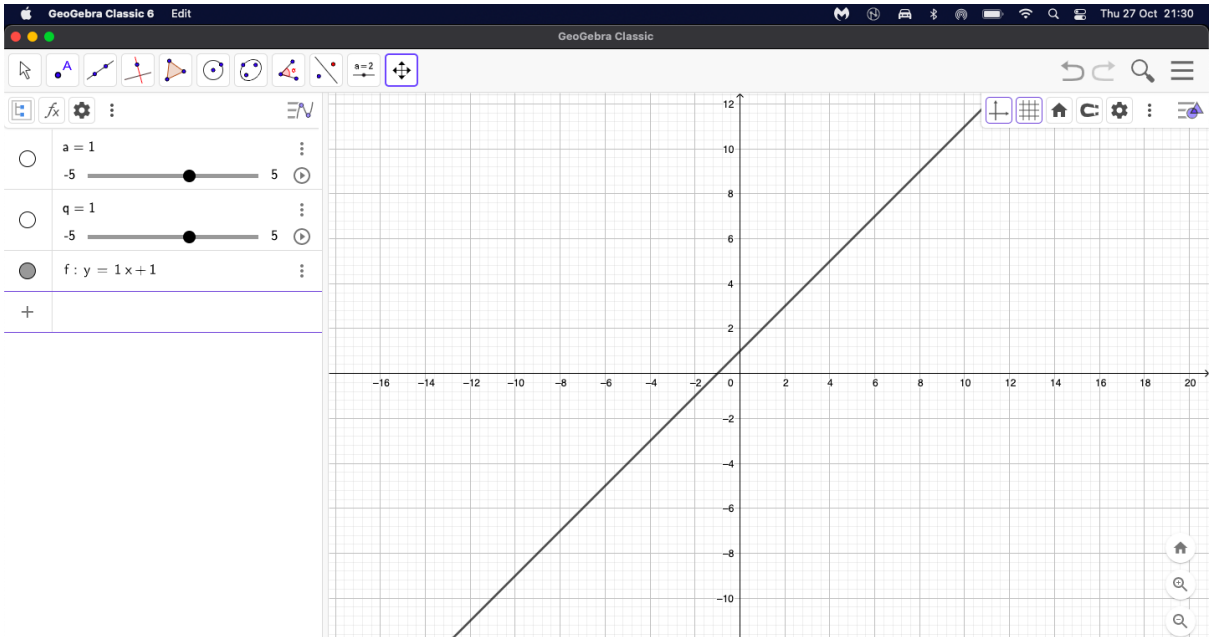


Figure 4.2a: Graph of $y = ax + q$ before any transformation

Moving the sliders for a and q on the top left corner, below the tools bar, will show the effects of changing these parameters on the graph of the straight line. For example, after moving the sliders for a and q to $a = 2{,}3$ and $q = -2{,}5$, the transformed graph will be as in Figure 4.2b below. The steepness and the y –intercept can be seen to have changed.

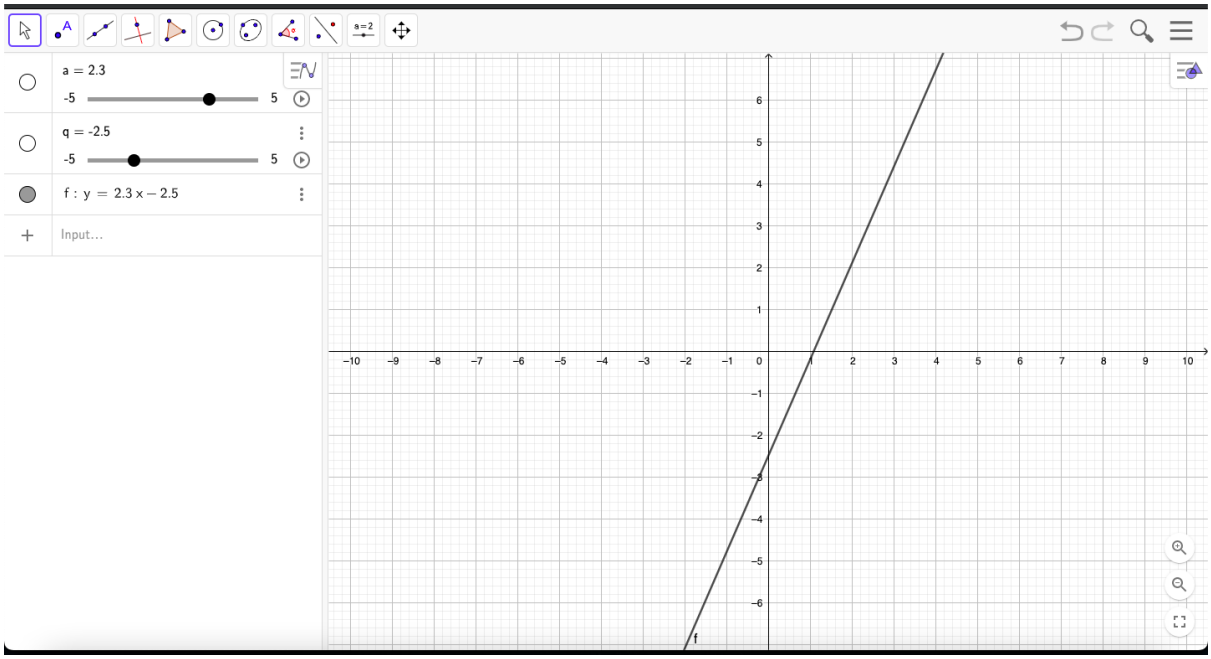


Figure 4.2b: Transformed graph

A task-based interview excerpt with L3 after she had drawn the linear function $y = ax + q$ is presented in Box 4.1 below:

Researcher: *What are the effects of moving the slider for* a*?*
L3*:* a *affects the gradient*
Researcher: *How do we explain gradient*
L3: *change in* x *and* y
Researcher: *What does that mean?*
L3: *no, it's change in* y *over change in* x
Researcher: *do you see the equation also changes as you move the slider for* a*?*
L3: *yes, coefficient of* x *is changing*

Box 4.1: L3 interview on changing parameters for $y = ax + q$

The short exchange in Box 4.1 above was how the lesson progressed after learners had drawn the graph of the straight line on GeoGebra. In the exchange, L3 could not explain the gradient of a linear function. She instead gave how it should be calculated, initially giving an incorrect way, change in x and y. The researcher was expecting her to mention that gradient is a measure of how steep a line is. All learners were actively participating and managed to perform operations on GeoGebra as anticipated by the genetic decomposition.

Sub-theme 2

The sub-theme deduced here was that *learners were able to visualise properties of basic graphs and their transformations.*

The first five questions in the worksheet were answered generally well by the learners (refer **Appendix 9**) as can be seen in Figure 4.3 below.

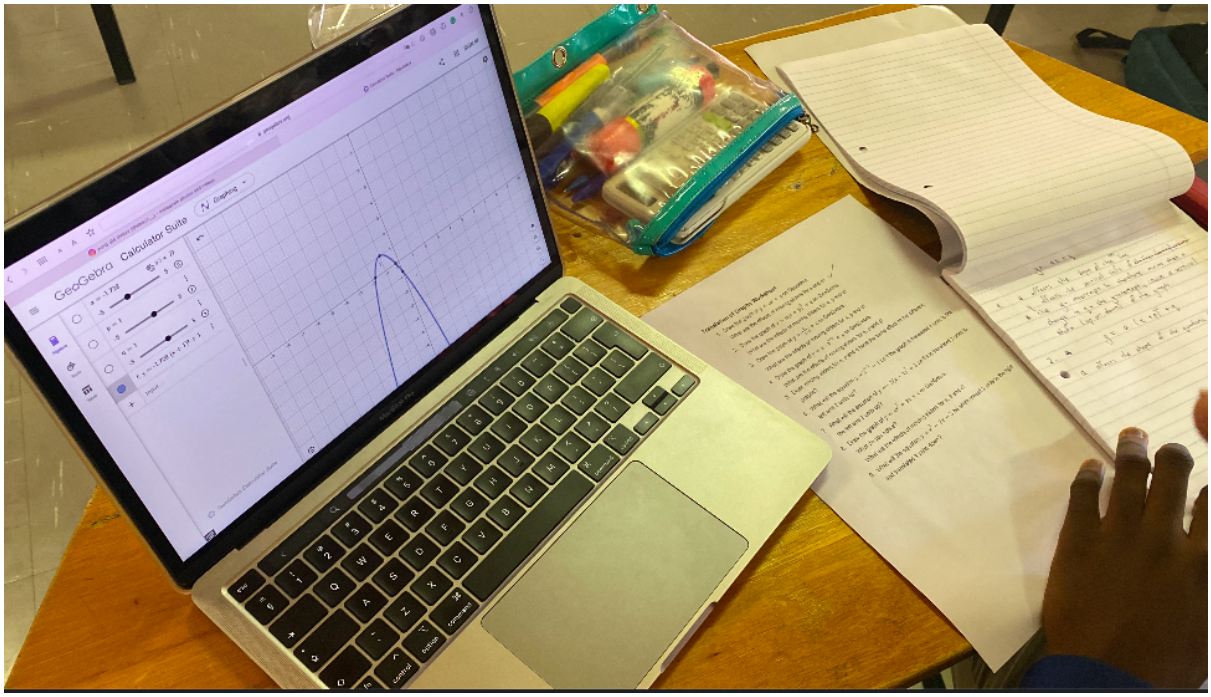


Figure 4.3: Learner noting observations after moving sliders for a parabola

As an example, L13 found Questions 1-5 generally easy as can be seen in Figure 4.4 below. The only issue observed by the researcher was on Question 3 where the learner was not specific on the effect of changing a in a hyperbola. The researcher expected L13 to mention that if a is positive, the graph will be in quadrants *1* and *3*, and in quadrants *2* and *4* when a is negative.

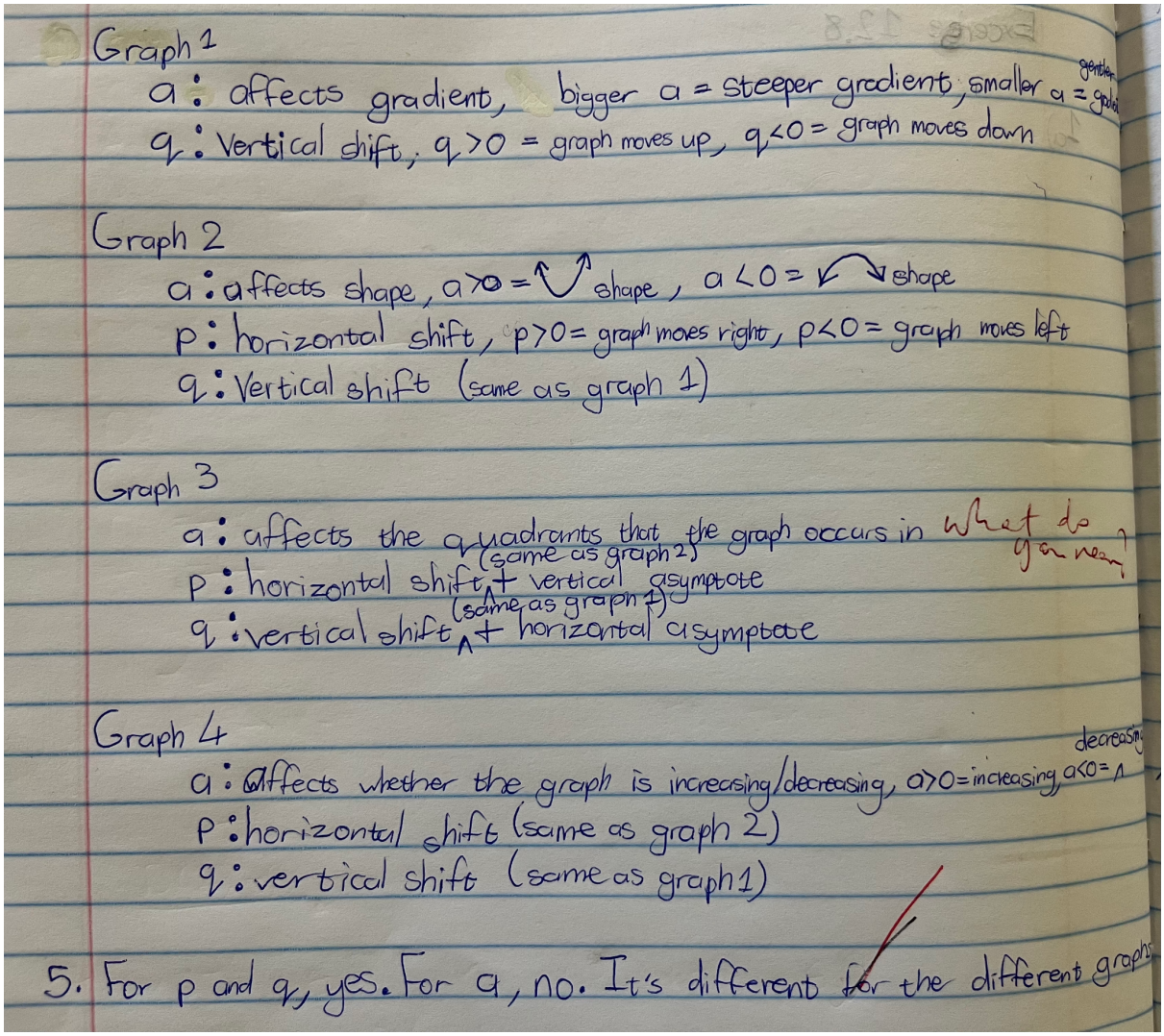


Figure 4.4: Observations on changing parameters by L13.

There was also another case where a learner did not write her observations as expected by the researcher. For example, in Question 3, L18 was expected by the researcher to mention the effects of increasing or decreasing the sliders for p and q on Geogebra but wrote about asymptotes instead, as can be seen on Figure 4.5 below:

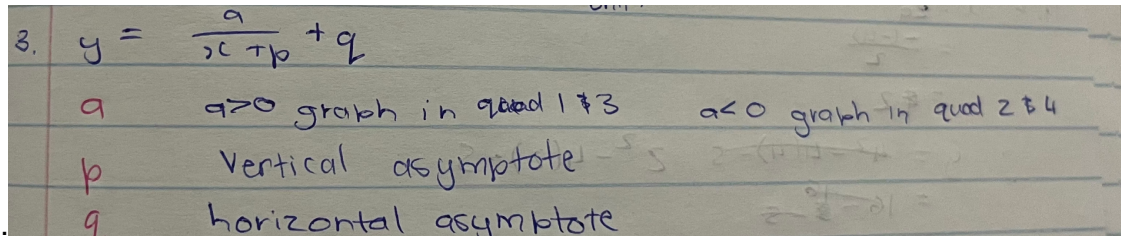


Figure 4.5: Observations by L18 on effects of changing parameters of a hyperbola

L18 did not understand the demands of Question 3.

In Question 5, L18 did not consider the individual parameters a, p and q as seen in Figure 4.6 below:

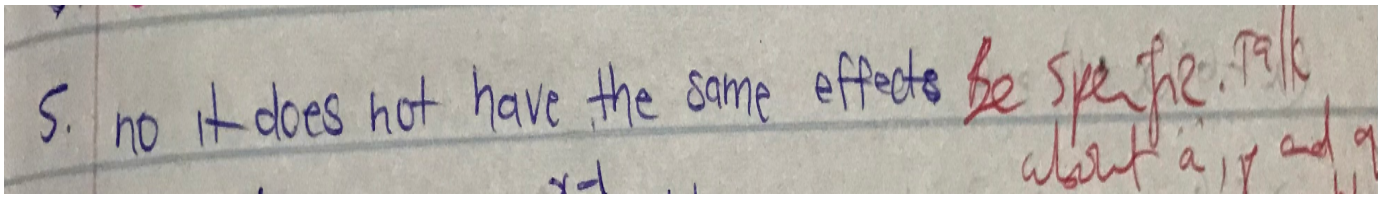


Figure 4.6: L18 response to Question 5

L18 was expected to discuss how each parameter a, p and q affects the different functions when its slider was moved. L18 did not understand the demands of Question 5. On GeoGebra it can be seen that moving sliders for a in the different functions does not have the same effect, but moving sliders for p and q will have the same effect in the different functions.

Interesting encounters on moving sliders as in Figure 4.7, for the parabola occurred in the lesson.

Some learners were able to move the sliders, however they were unable to describe the effect of the parameter which caused the shift. For example L5 was able to draw the parabola on GeoGebra but describing effects of changing parameters was not handled well as can be seen in the interview excerpt in Box 4.2 below:

Researcher: *If you change a what happens?*

L5: *Affects the horizontal shift*

Researcher: *Is your graph shifting horizontally there?*

L5: *vertical*

Researcher: *Is it shifting*

L5*: Yes*

Researcher: *We are talking about a*

L5: *Gradient*

Researcher: *Gradient for a parabola?*

Researcher: *speak aloud, tell us what you are seeing (on computer screen)*

L5: *Am trying*

Researcher: *I know you are trying. But tell me about what is happening on the screen*

Box 4.2: Interview excerpt with L5

The researcher had to remind L5 about how parabolas are described in the lower grades, u-shape and n-shape. After this reminder, that was when she began to see what was happening when moving the slider for a.

About 80% of the learners were able to write their observations on changing the parameters a, p and q for a parabola as can be seen in the written response by L22 in Figure 4.7 below:

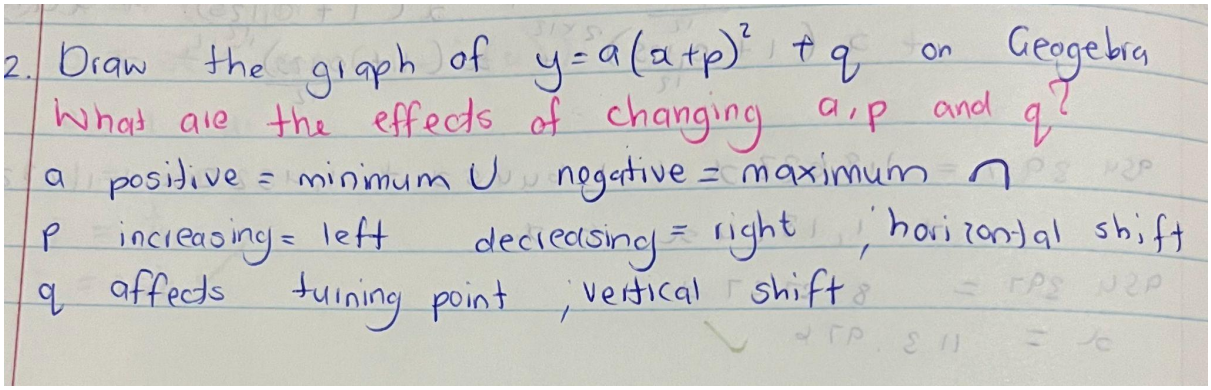


Figure 4.7: L22 response on parameters of a parabola

Sub-theme 3

The sub- theme deduced from L13, L18, L5 and L22 here was that *learners could give descriptions of change of shape and orientation of function and its transformed function.*

L10 was working confidently on GeoGebra, moving sliders for a, p and q for a hyperbola as shown in Figure 4.8 below:

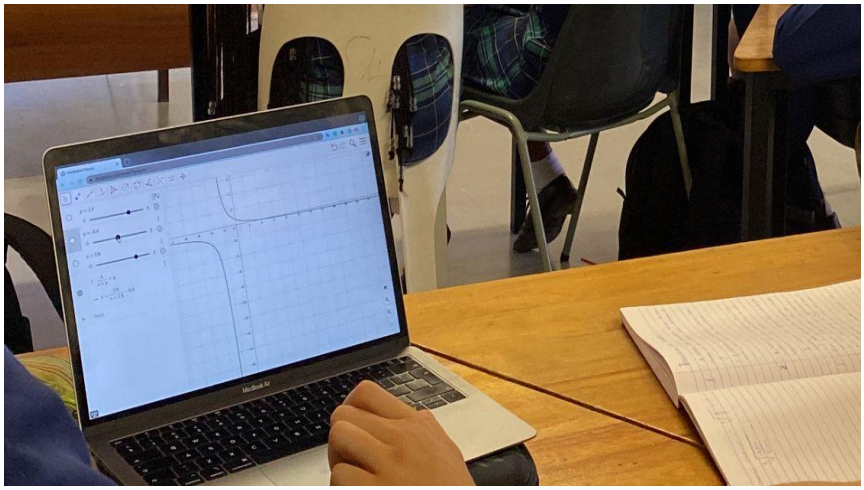


Figure 4.8: L10 hyperbola drawn on GeoGebra

L10 ‘s task-based interview excerpt with the researcher is given in Box 4.3: below:

Researcher: *What are you moving now?*

L10: a

Researcher: *What do you notice?*

L10: *the quadrants change*

Researcher: *change slider for* p*. What do you notice?*

L10: *affects the horizontal shifts*

Researcher: *How?*

L10: *the* x *values change*

Researcher: *Where are you seeing that?*

L10: *(points on graph as he is moving slider for* p*)*

Researcher: *where else can you see that? Do you see you can also see it here on the equation.*

L10: *ooh okay*

Box 4.3: Interview excerpt for L10

L10 managed to express himself well on the effects of changing parameters in a hyperbola. Generally, learners’ responses to changing parameters for the hyperbola were satisfactory.

For the exponential function, the researcher went to great lengths explaining what was happening with the sliders, as the graphs were at times disappearing from computer screens. The researcher had the following exchange with L5, about Figure 4.9 and Box 4.4 below:

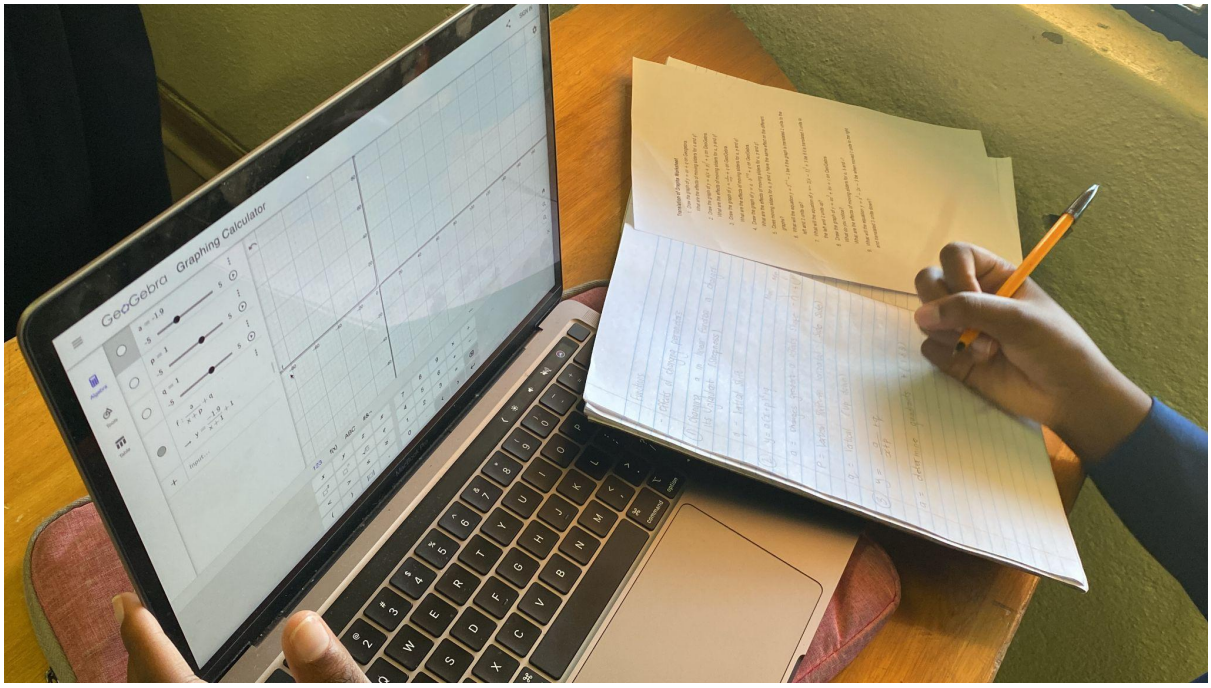


Figure 4.9: Graph disappeared from computer screen

Researcher: *When a is changed, what changes do you see on the graph*
L5: *I don't get it*
Researcher: *(reminded L5 about reflections on the x-axis). Also, can you see what happens with b? Can we talk about b on the exponential. Change b with $a \neq 0$. When $b = 1$, b greater than zero and b between 0 and 1. What do you notice?*
L5*: Just see horizontal line when $b = 1$.*
Researcher: *Good. Write about these observations*

Box 4.4: Interview excerpt with L5

Sub-theme 4

The sub-theme to be deduced from L10 and L5 was that *repetition (ACE Cycle) of actions on GeoGebra leads learners to internalising actions.*

The researcher concluded that the majority, approximately 90% of learners were operating at the action level of the APOS theory. This was because after drawing the basic graphs on GeoGebra, a few struggled to give coherent observations on the effects of changing the parameters. Lesson 1 ended before learners had finished worksheet 1. The researcher asked learners to go home and do the remaining questions on the worksheet as homework.

4.2.2. The process level

The first lesson ended when most learners were done with Questions 1-5. The researcher then asked the learners to do the remaining questions as homework. Questions 6 and 7 on the worksheet (refer **Appendix 9**), were focused specifically on the process component of the APOS theory. This was due to the expectation that a learner at the process level would be able to describe changes in the fundamental functions as a result of applying the transformation without having to carry out each stage of the transformation. Additionally, it was required of the learner to be able to reverse the procedure and recognise the function to which a particular set of transformations was applied. A learner at process level or further can also figure out transformations without necessarily having to use GeoGebra.

Majority of learners were comfortable answering Questions 6 and 7. Learners were generally able to apply their observations from the previous lesson while working on GeoGebra as can be seen in L1's work on Figure 4.10 below.

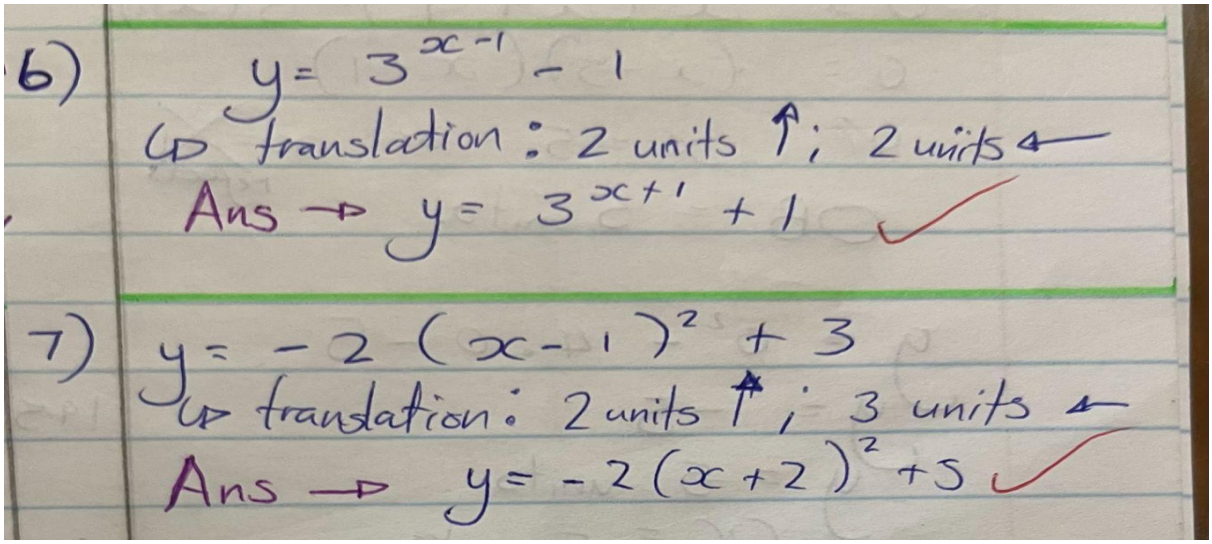


Figure 4.10: L1 responses to Questions 6-7

In Question 6, L1 simply figured out that for the vertical shift she could just add *2* to -1 to get *1*. For the horizontal shift of *2* units to the left, she just added *2* to $x - 1$, thereby overcoming the inconsistency presented by the horizontal translation. Question 7 was answered in similar fashion.

Sub-theme 1

From Questions 6 and 7, the sub- theme deduced was that *learners were able to predict solutions.*

The researcher asked L1 to explain how she got the answer to Question 6. Box 4.5 details her responses:

Researcher: *Tell me how you got to the answer for Question 6*

L1: *Very easy sir. Since it said 2 units to the left, i know from GeoGebra that I must add 2 to p*

Researcher: *How about the vertical shift?*

L1: *it affects q, so i added 2 to -1*

Researcher: *Well done!*

Box 4.5: L1 interview excerpt

The confidence with which L1 responded to the researcher's questions showed that she was ready to progress to the object level of the APOS theory.

Sub-theme 2

The sub-theme deduced here was that there were *learners able to work mentally, having internalised actions*

4.2.3. The object level

If a learner could reflect on all processes and was able to think of them as a whole, in any representation or context, the researcher was coding him or her as operating at the object level. Learners at object level can proceed to answer questions of the object standard without the use of GeoGebra. The whole class struggled with Questions 8 and 9, even though they were free to use GeoGebra. Majority did not even attempt these two questions. The researcher then concluded that the learners in the class were not yet eligible for the object level.

After drawing the graph of $y = ax^2 + bx + c$ on GeoGebra, the researcher was expecting L19 to notice that this graph is the same as the one for $y = a(x + p)^2 + q$. As was noted previously, even after getting the graph correct on GeoGebra, writing down the observations was still a challenge. Again, L19 mentions, in Figure 4.11 below, that moving the slider for a will change the shape of the graph.

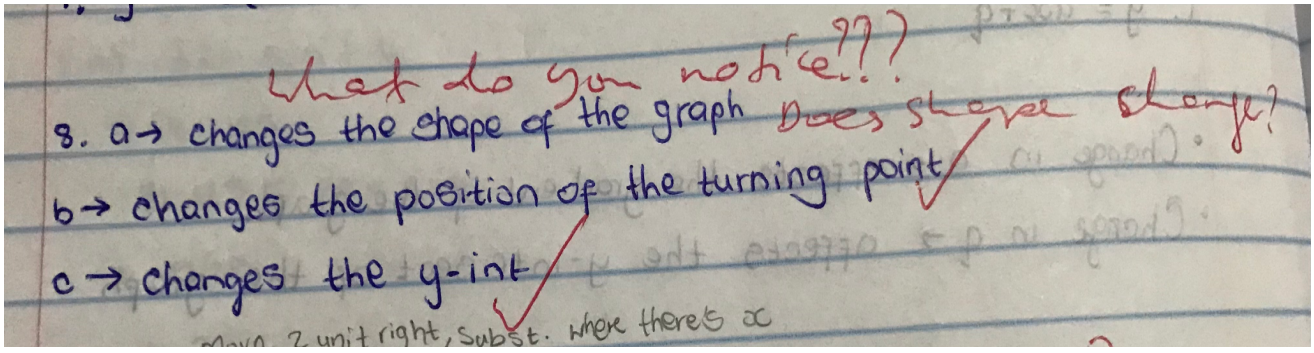


Figure 4.11: L19 response to Question 8

Sub-theme 1

The sub- theme deduced here was that *learners were giving partial descriptions of transformations.*

In the task-based interview with the researcher, L19 had this to say, Box 4.6:

Researcher: After drawing the graph on GeoGebra, what did you notice?
L19: I noticed that a changes the shape of the graph
Researcher: No. Before moving the sliders, what did you notice
L19: I notice that eeh….eehhh
Researcher: I expected you to mention that the graph is a parabola, like we had with $y = a(x + p)^2 + q$
L19: Ooooh okay

Box 4.6: L19 Interview excerpt

The researcher noted that L19 did not realise that the way the equation was given in Question 8 is not the same as the $y = a(x + p)^2 + q$ that was being used in class when they were working on GeoGebra. It is therefore important for learners to appreciate and be confident with working with the different representations of functions.

Sub-theme 2
The sub- theme to be deduced here was *failure to apply learnt concepts in the solution of new problems.*

In answering Question 9, L1, in Figure 4.12 below, correctly applied the vertical translation by subtracting *3*. The horizontal translation was completely misunderstood.

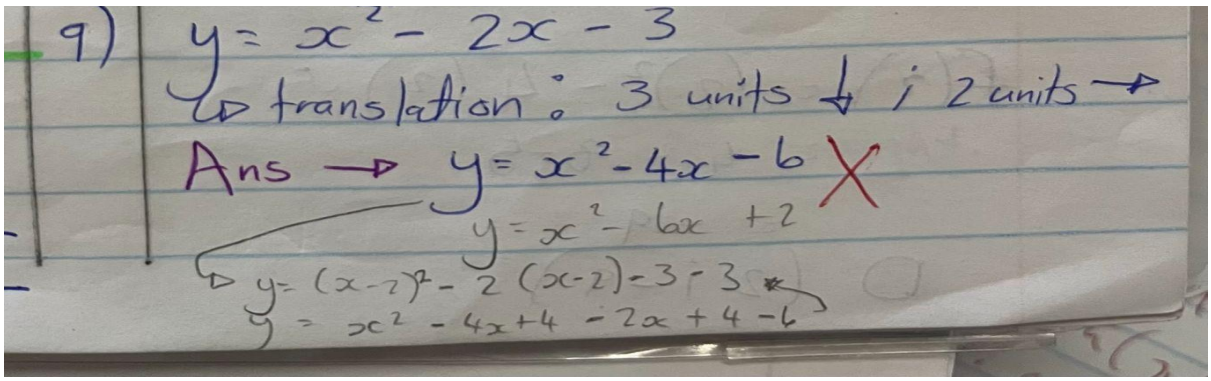


Figure 4.12: L1 response to Q9

Sub-theme 3
The sub- theme deduced here was that L1 was *unaware of the process in its totality.*

In the task-based interview with the researcher, L1 said the following, Box: 4.7:

Researcher: *How did you get this answer?*
L1: *Translated 3 units down means i have to subtract 3 from this -3 (pointing at -3 in the original equation)*
Researcher: *How about the horizontal shift?*
L1: *We subtract 2 from x since it's shifting to the right*
Researcher: *So where did you subtract 2*
L1: *(points at -2x) Here sir*
Researcher: *So what is -2x-2*
L1: *ooh i see my mistake.*
Researcher: *Was that your only mistake*
L1: *Maybe*
Researcher: *Did you use GeoGebra to confirm your answer?*
L1: *Ooohhh i forgot!*
Researcher: *It is always good to check your answers to see if they make sense, right*
L1: *Yes sir*

Box 4.7: L1 interview on Question 9

Sub-theme 4

The sub-theme deduced from L1's task- based interview was that her *algebraic skills were poor.*

L13 began the attempt to Question 9 reasonably well. He completed the square in the original expression but instead of subtracting *2* inside the bracket with $(x - 1)$ for the horizontal shift, he added the *2*. The vertical shift was done well. This is shown in Figure 4.13 below.

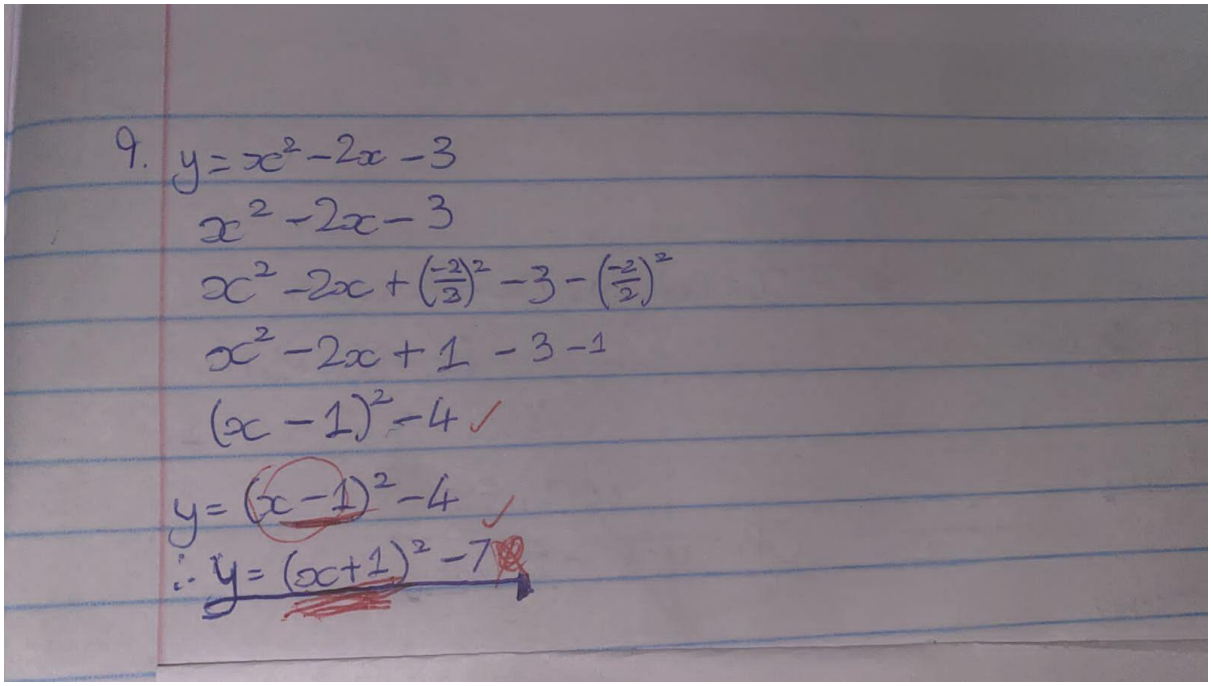


Figure 4.13: L13 response to Question 9

The researcher had the following exchange with L13, Box 4.8 below:

Researcher: *You struggled with Question 9. What was the challenge?*
L13: *It was beyond my abilities*
Researcher: *But you were supposed to apply your experiences with GeoGebra from the previous lesson. We are having* $y = x^2 - 2x - 3$*. You started the question well, had no problem completing the square*
L13: *it was at the end, i added instead of subtracting*
Researcher: *you want the graph to go two units to the right and three units down. So if it is two units to the right, you must put* $x - 2$ *wherever you see x and* $y + 3$ *wherever you see* y
L3*: whhhattt!!! (had not imagined this)*

Box 4.8: L13 Interview excerpt for Question 9

L13 quickly realised his mistake in the interview. The researcher also made him aware of the other approach he could have used. For the horizontal translation he could have just replaced x with $x - 2$ and y with $y + 3$ and simplified. The

researcher coded L13 at the process stage of the genetic decomposition as he had demonstrated an understanding of the process involved.

Sub-theme 5

The sub- theme deduced here was that the learner *struggled with horizontal translations*.

The original graph and the resulting graph for Question 9 were then drawn on GeoGebra by the teacher so that learners could see the transformations that had taken place. Figure 4.14 illustrates this:

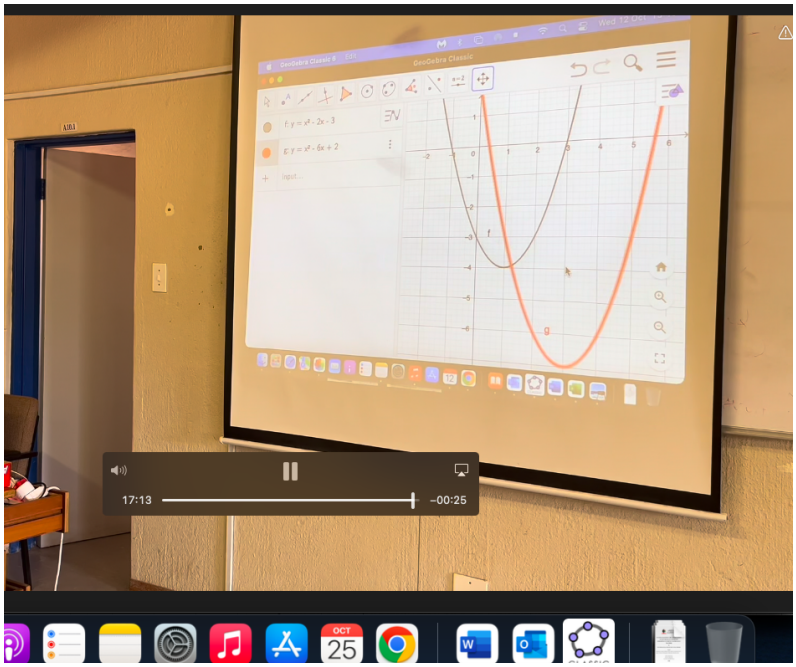


Figure 4.14: Confirming transformed graphs on GeoGebra

The researcher had to probe learners on what they had observed when using GeoGebra and a graph was being translated horizontally and vertically. The researcher had to go through these questions with the learners at the beginning of the following day's lesson. Drawing graphs for Questions 8 and 9 on GeoGebra had a huge impact on learners as they could confirm the final answer on GeoGebra.

Lesson 2 ended with learners working on the worksheet on deductions from sketch graphs. They had to do the remaining questions on the worksheet as homework. The analysis done at the end of the lesson still placed most learners at the action level of the APOS theory.

4.2.4. The schema level

No learners were able to operate at the schema level from the three lessons that were conducted on transformations of functions. Learning at the schema level was shown by a learner's capacity to create conceptions by considering the relationships between diverse activities, processes, objects and other previously produced schema. In the third lesson, learners continued with working on the worksheet on deductions from sketch graphs (refer **Appendix 10**) and received the handout for the homework from the researcher, which is discussed in the next section.
The researcher will focus on Question 19.6 as it is on transformations of functions, the focus of this study. Three learners attempted this question.

L16 had a response for Question 19.6 as shown in Figure 4.15 below:

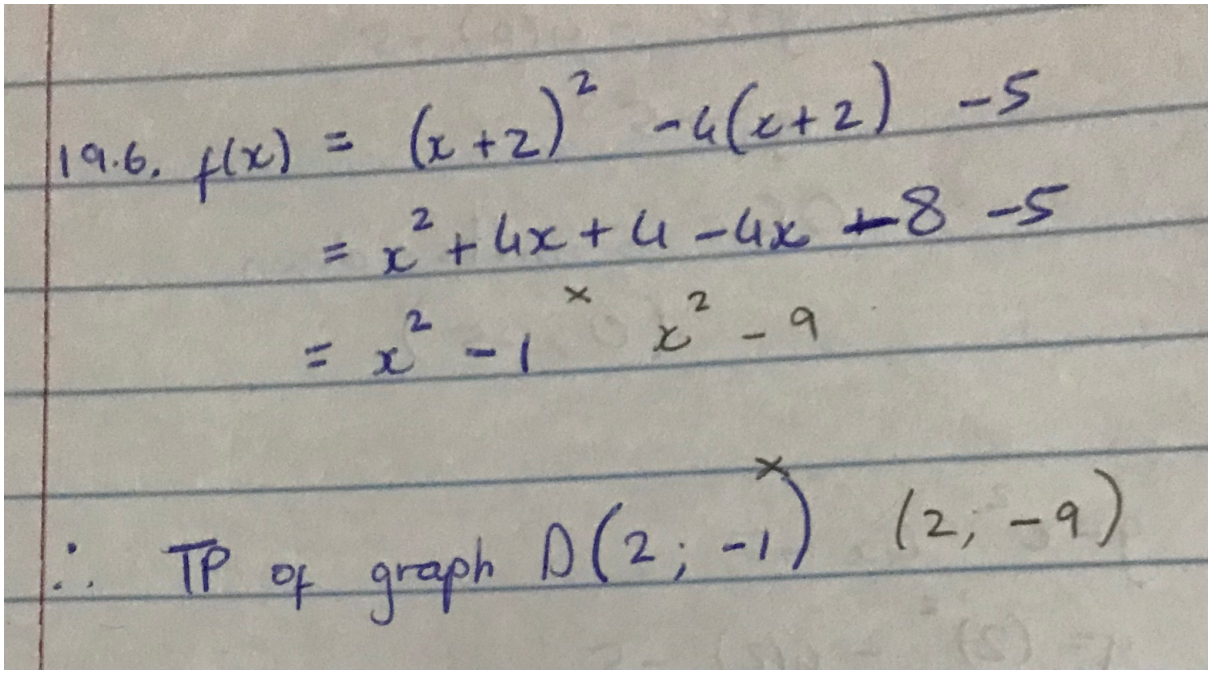


Figure 4.15: L16 response to Question 19.6

L16 was supposed to write $f(x+2)$ on the left hand side. She substituted and expanded correctly on the right hand side but could not get the correct answer for *4 – 8 – 5*. Having arrived at $x^2 - 1$, one would expect the learner to get the coordinates of the turning point, for her wrong answer, correct since the turning point for an equation of this type was done in Grade 10. L16 did not describe the translation as was required

by the question. This confirmed Venter (2020)'s observation that most learners cannot apply previously owned knowledge.

Sub-theme 1

The sub- theme deduced from L16 was that the *learner failed to apply learnt concepts in the solution to new problems.*

L19 had the response for Question 19.6 as shown in Figure 4.16:

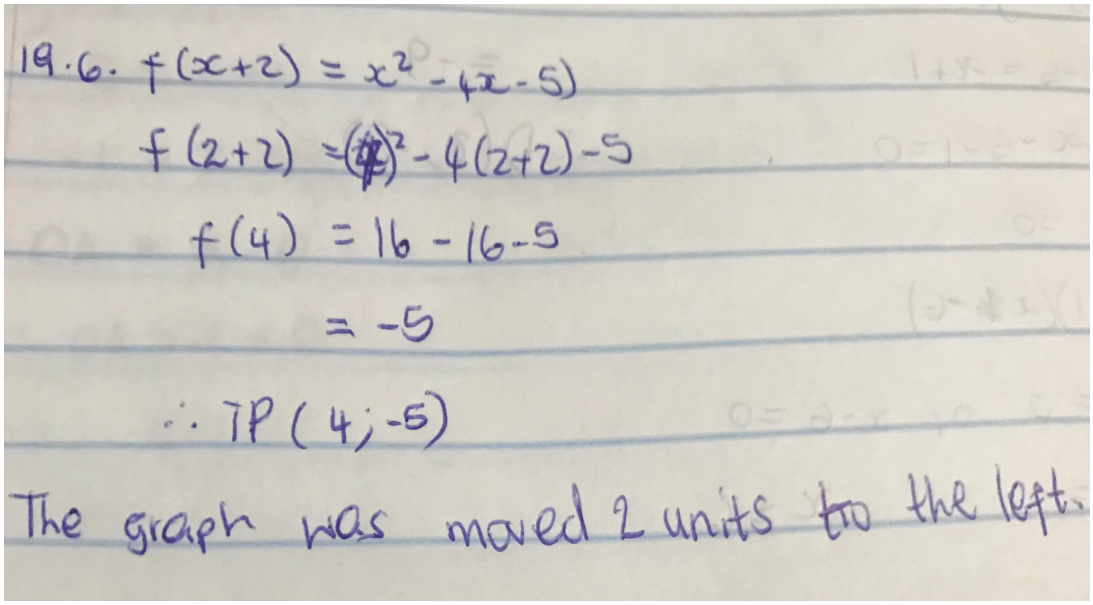


Figure 4.16: L19 response to Question 19.6

The response by L19 for Question 19.6 showed a complete breakdown. L19 did not demonstrate that she understands what it means to say $f(x+2)$. The researcher expected her to replace x with $x+2$ everywhere on the right hand side, first step in Figure 4.17 above. She should have realised that $x+2$ is the value of the function's argument at which the function is being evaluated. L19 was able to describe the translation caused by $f(x+2)$ correctly by mentioning that the graph was moved *2* units to the left. L19 can be categorised as operating at pre- action level as she could not do the substitution and simplify step by step.

The researcher demonstrated to the class how the question could have been answered in two ways. The two ways will determine whether a learner is operating at object or schema levels. A learner that will immediately understand that $f(x+2)$ is a horizontal shift of *2* units to the left and easily figure out the coordinates of the turning point of the new graph will be operating at the schema level of the APOS theory. A learner that will replace x with $x+2$ in $f(x)$ and describe the translation taking place can be said to be operating at the object level.

L18 had the following attempt to Question 19.6, Figure 4.17 below:

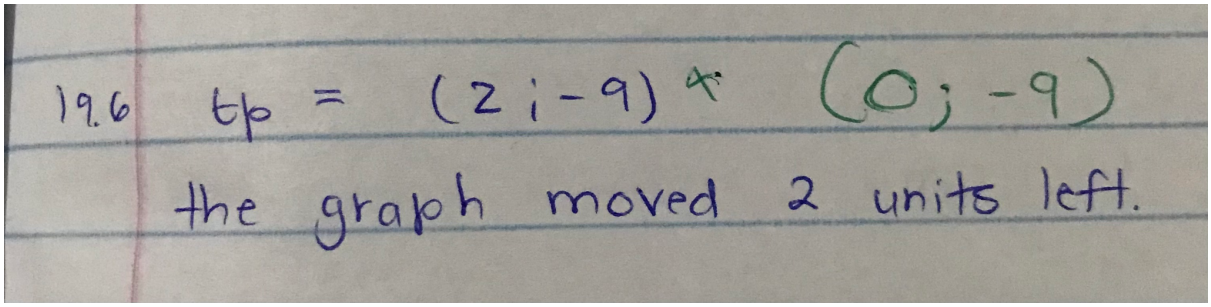


Figure 4.17: L18 response to Question 19.6

The following exchange ensued with L18, as in Box 4.9 below:

Researcher: *Do you understand this question?*
Researcher: *For the old graph, what was the turning point?*
L18: $D(2;-9)$
Researcher: *What does it mean to say* $f(x+2)$
L18*: Graph moves two units to the left. Aaaah I went right. My bad.*
Researcher*: Be careful next time. We saw on GeoGebra that if we add two units to* x*, the graph will move 2 units to the left. If* D *was our turning point, what is the new turning point*
L18*: (0;-9)*

Researcher: *So you see how quick you can get to answer. Now suppose you didn't understand this question in this way. We are saying,* $f(x)$ *is given as* $f(x) = x^2 - 4x - 5$*. Now* $f(x+2) =$ *what?*

L18: *in place of* x *we must put* $x + 2$

Researcher*: (writes on paper and asks L18 to complete)* $f(x+2) = (x+2)^2 - 4(x+2) - 5$

Box 4.9: Interview excerpt for L18 Question 16.9

L18 gave the coordinates of the turning point of $f(x)$ and not $f(x+2)$ as demanded by the question. Surprisingly, she managed to describe the translation.

Sub-theme 2

The deduced sub-theme from these learners was that they *failed to apply learnt concepts in the solution of new problems.*

The first 3 lessons showed that most learners were on the action stage of the APOS theory as they could only perform transformations of functions on GeoGebra but were not always able to apply these actions without the need to apply each step when it came to written work. They could draw the graphs on GeoGebra and proceed to move sliders but at times fail to write down their observations in a coherent way. The researcher observed that learners were generally active in the classroom activities. The participants in the study were feeling excited when using GeoGebra, initially as they were moving sliders and eventually as they were confirming their answers. The three lessons can potentially help to answer the second research question of this study on how the use of GeoGebra enhances learners' conceptions of transformations of functions.

4.3 Analysis of homework activity

The homework and genetic decomposition applied are on Appendix 1. Written responses for each question in the homework were given a predetermined code and categorised.

4.3.1 Analysis of Question 1

The predetermined code for Question 1 was the process component from APOS theory. The genetic decomposition expected learners to draw the graphs of the given functions without the need to perform each and every step of the transformation.

The written response for L22 is shown in Figure 4.18 below:

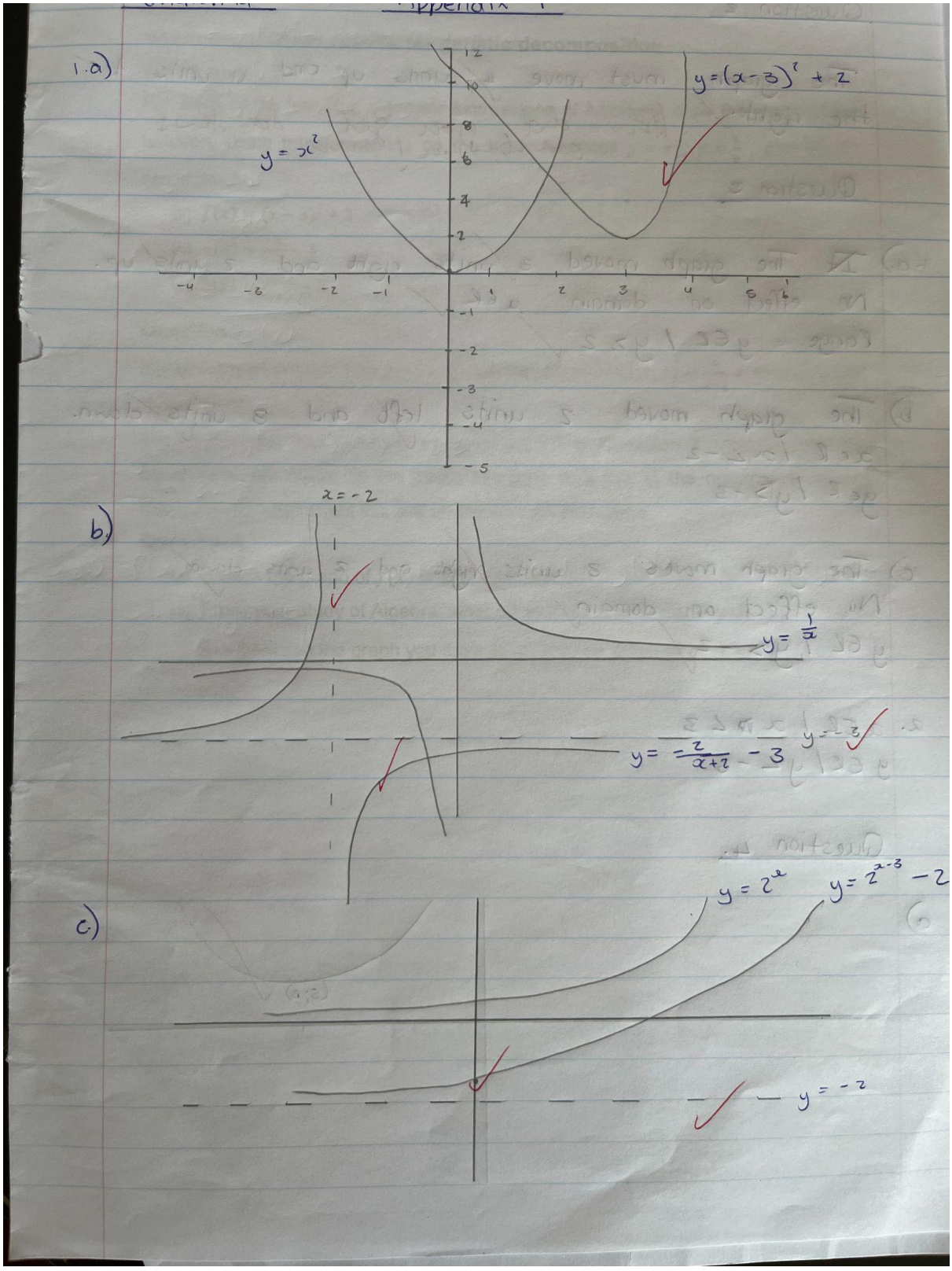


Figure 4.18: L22 response to Question 1

L22 was able to draw the graphs as expected by the genetic decomposition which showed that she could figure out transformations done on the basic functions by

merely inspecting each of the given algebraic representations of functions. L22 was able to use her understanding of the effects of changing parameters a, p and q. At this stage L22 was working at the process level of the APOS theory. 7 learners answered this first question by using transformations on the basic graphs like what L22 did.

Sub-theme 1

The sub- theme as deduced by the researcher was that the learner *was able to work mentally, having internalised actions.*

L16, as shown in Figure 4.19 below, showed all workings for intercepts and turning points.

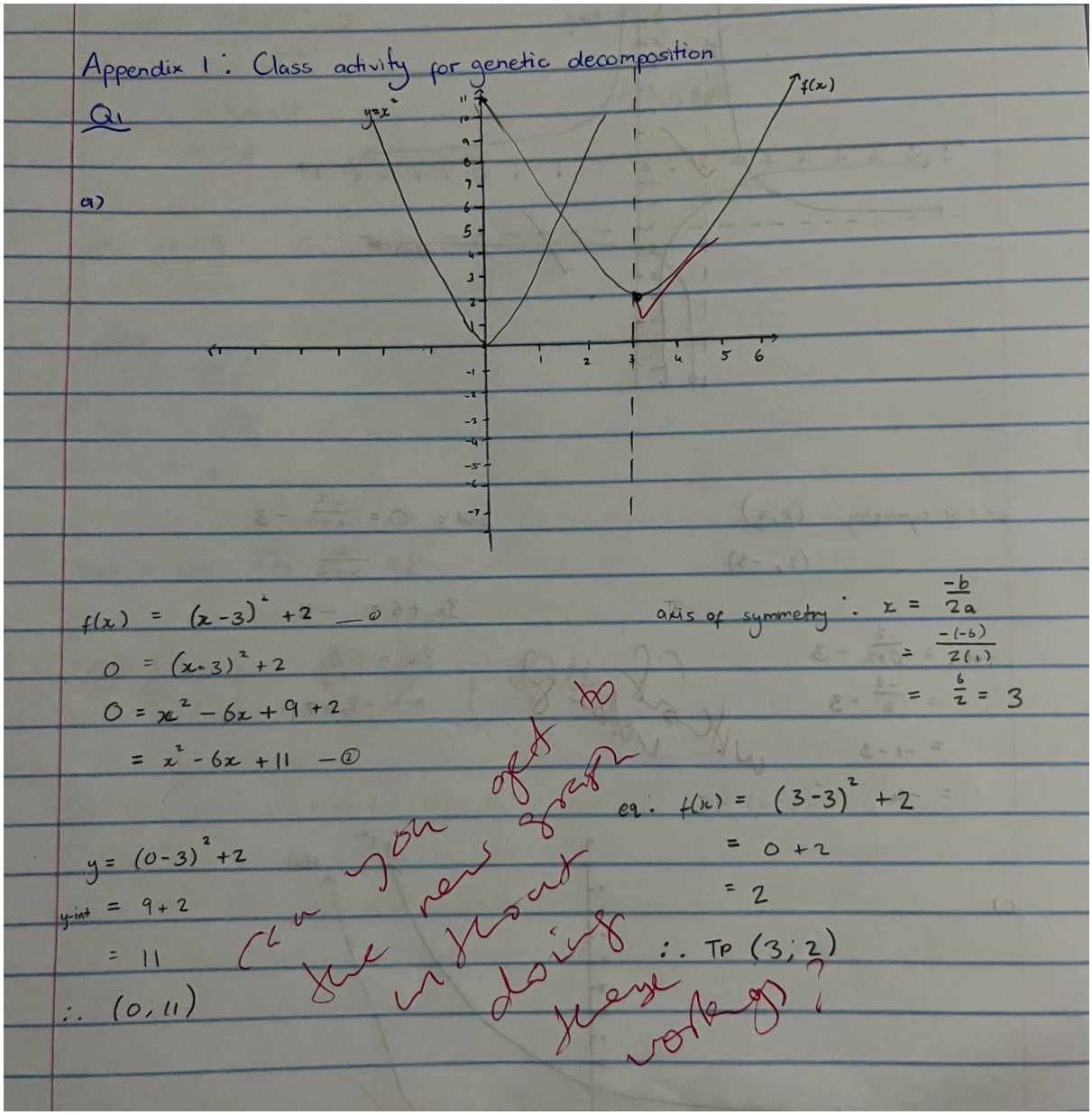


Figure 4.19: L16 response to Question 1

According to the genetic decomposition, L16 was said to be operating at the action level of the APOS theory. She demonstrated that even though she was able to draw the transformed graph, she was unable to place the transformed graph without showing each step because she lacked the necessary mental structures. When asked about why she had to show all the steps, her responses are captured in the Box 4.10 below:

Researcher: *I see for the first question, you have a lot of workings there. Does it mean that when you are looking at the graph, like at (a), you see f(x) is x minus three squared plus two? You can't draw the graph straight away? You need to do these workings first?*
L16: *Yes, to a certain extent. I need to find the asymptotes.*
Researcher: *Parabola, no asymptotes hey. Okay.*
L16: *Alright, x intercepts.*
Researcher: *Okay. And here you have the two graphs, this is the graph of x squared. This is the original graph. Okay, so what you must do is, you see where there is this x, we put x minus three? What is the effect of changing x to x minus three?*
L16: *the graph moves three units to the right*
Researcher: *Then the plus two?*
L16: *it moves two units up.*
Researcher: *Do you see we don't need these workings?*
L16*: Yes sir*

Box 4.10: L16 interview responses

The answers demonstrate that, although knowing how to obtain transformed graphs, she did not possess the necessary mental structures for the process level.

Sub-theme 2
The deduced sub-theme from L16 was that the learner was *unaware of the process in its totality.*

L14 totally failed to make a good attempt at the first question. The researcher coded L14 as pre-action level. He could not get even the shape of a parabola correct. For the parabola, his only correct calculation was for the y-intercept. His response to the written homework is shown in Figure 4.20 below:

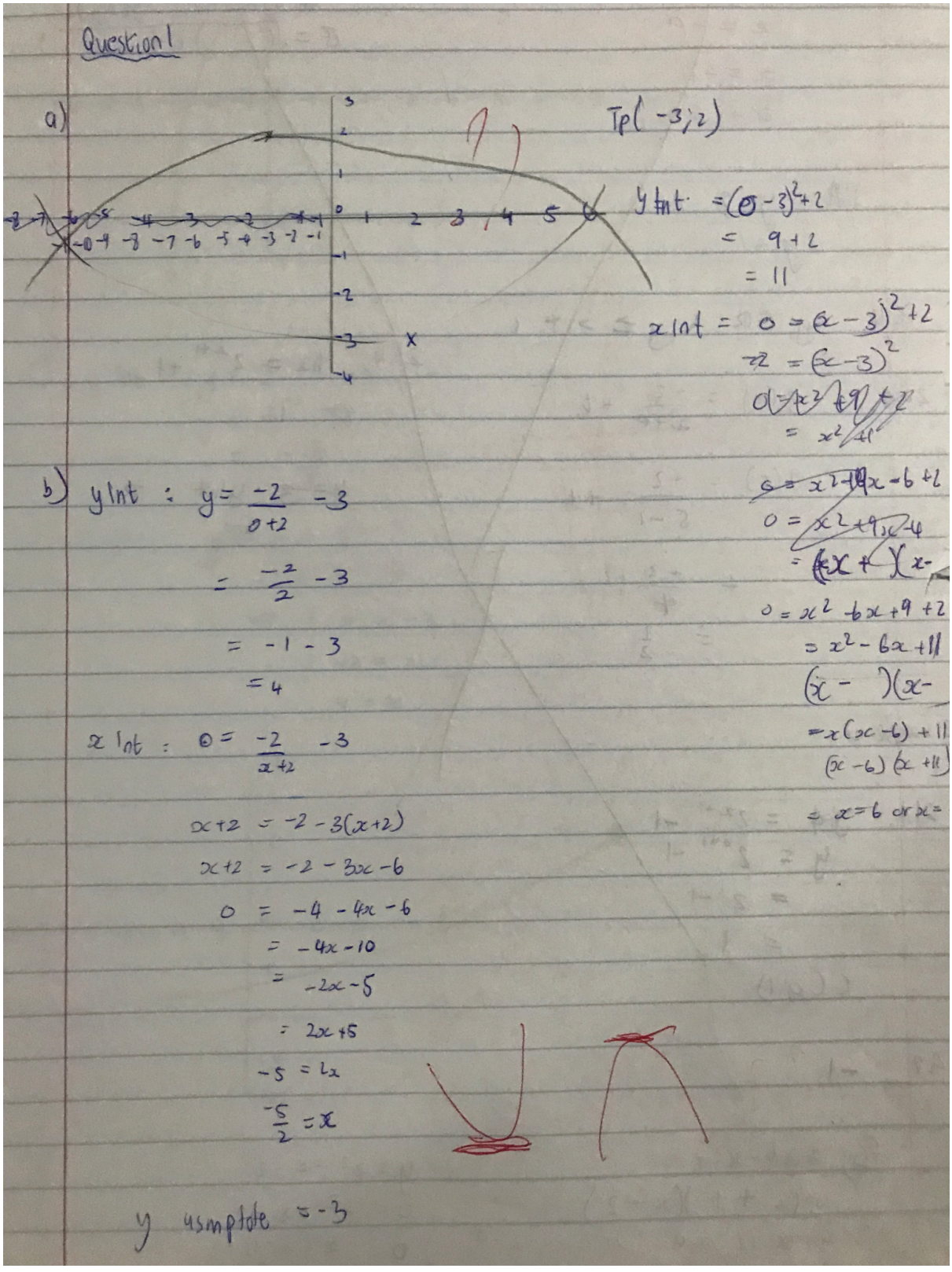


Figure 4.20: L14 response to Question 1

The interview for L14 was also quite informative as can be seen in Box 4.11 below:

Researcher: *what would you say are your challenges when it comes to this section on transforming functions?*

L14: *it would be mostly drawing the graphs. I forget which type of graph I'm drawing and also like everything each graph needs.*

Researcher: *like if it is a parabola, like question 1a, you are saying when you look at this equation, you don't know what type of function it is?*

L14: *for this one I know it's a parabola, I will get the x and y intercepts, but sometimes I will forget to get the turning point.*

Researcher: *okay. So without making the calculations, like I see you made calculations like this, but the question was saying using this basic graph. Are you able to tell how the graph of x squared looks like? Do you know how the graph of x squared looks like, because we are saying for all parabolas y=x squared is the mother graph. Every other parabola that you get, is coming from this one.*

L14: *okay*

Researcher: *now, do you see where there is x, we have put x-3, what does that mean?*

L14: *that the graph would be on the lower side … or that the horizontal asymptote is 3*

Researcher: *asymptote for parabola?*

L14: *the turning point would be 3*

Researcher: *3 what? Remember you must give x and y coordinate*

L14: *(3;2)*

Researcher: *turning point must be (3:2) isn't it? And also this graph, when you are just looking at the equation, will the graph have a minimum point or will it have a maximum point. The turning point, is it a maximum or is it a minimum? Do you understand that question?*

L14: *no*

Researcher: *okay. Remember when we were saying the graph is u-shaped or the graph is n-shaped? So if it is u shaped, we will call this turning point here, a minimum. If it is n shaped , this turning point is a maximum. Whether the graph has a minimum point or maximum point, we see it by the number that is there before the bracket, in this case it is 1.*

L14: *okay*

Researcher: *if it is positive, the graph will have a minimum, if it is negative, the graph will have a maximum. Do you remember we were doing this on GeoGebra. when we were pushing sliders for 'a'. when 'a' is negative, when 'a' is positive what happens? Do you remember?*
L14: *yes. So its called the minimum because this is the lowest point?*
Researcher: *yes!!*
L14: *oooh*

Box 4.11: L14 responses in interview

Responses in the interview by L14 were quite revealing. Besides not being able to relate algebraic representations to the general shape of the graph, L14 only discovered in the interview why a parabola is said to have a minimum or maximum. He showed a lack of understanding of the basics of functions.

Sub-theme 3
The sub-theme deduced from L14 was that he *failed to apply learnt concepts in the solution of new problems.*

4.3.2 Analysis of Question 2

The predetermined code for Question 2 was the process level of the APOS theory. The genetic decomposition expected learners to be able to describe changes in the function $p(x)$ to $k(x)$, without the need to perform every step of the transformation.

L19 drew the two graphs but did not mention the transformations that had taken place as shown in Figure 4.21 below:

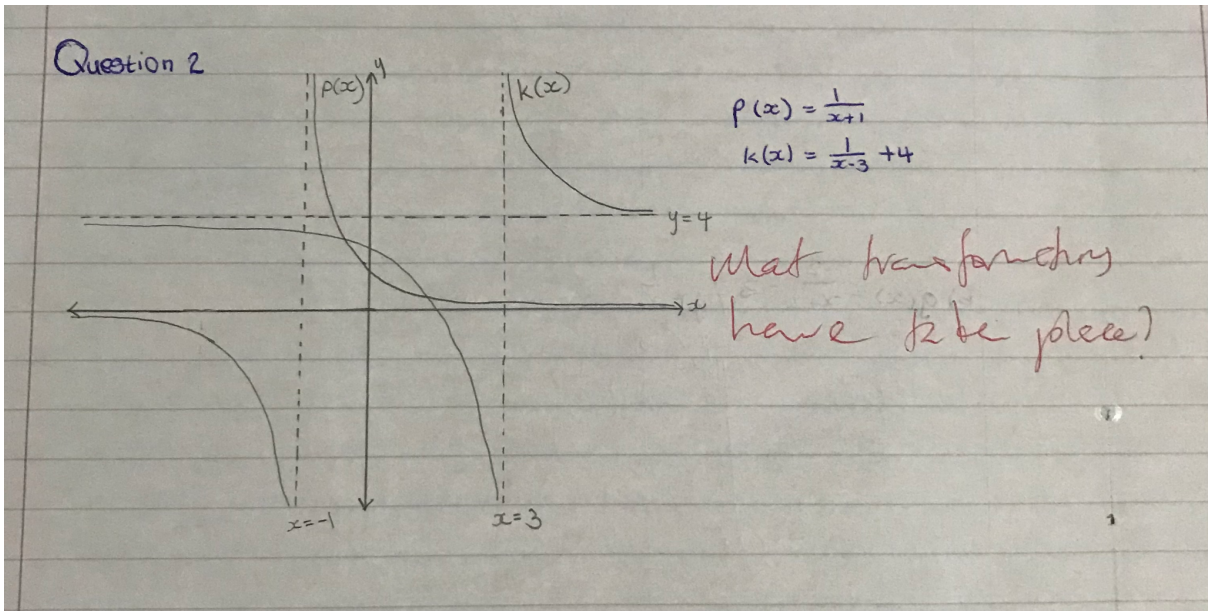


Figure 4.21: L19 response to Question 2

The interview excerpt with L19 in Table 4.12 below, revealed that even though she could describe the transformations that had taken place, the fact that she did not mention these when she was writing the work, means she probably did not understand the demands of Question 2.

Researcher: *then at question 2, you drew the graphs, are you also able to explain verbally what is happening as you move from p(x) to k(x)*

L19: *so first p(x) has x+1 as a denominator, then k(x) has x-3, so now that means that in order for us to get from 1 to -3 then we had to minus 4. That means that p is now -3 so it was moved 4 units to the right. So the asymptote, the vertical asymptote went from being -1 to being 3*

Researcher: *and also the horizontal?*

L19: *the horizontal asymptote went from being y=0 to y=4.*

Researcher: *so the asymptote shifted 4 units up?*

L19: *yes*

Box 4.12: L19 interview responses

A similar situation occurred with L13 as shown in Figure 4.22 below:

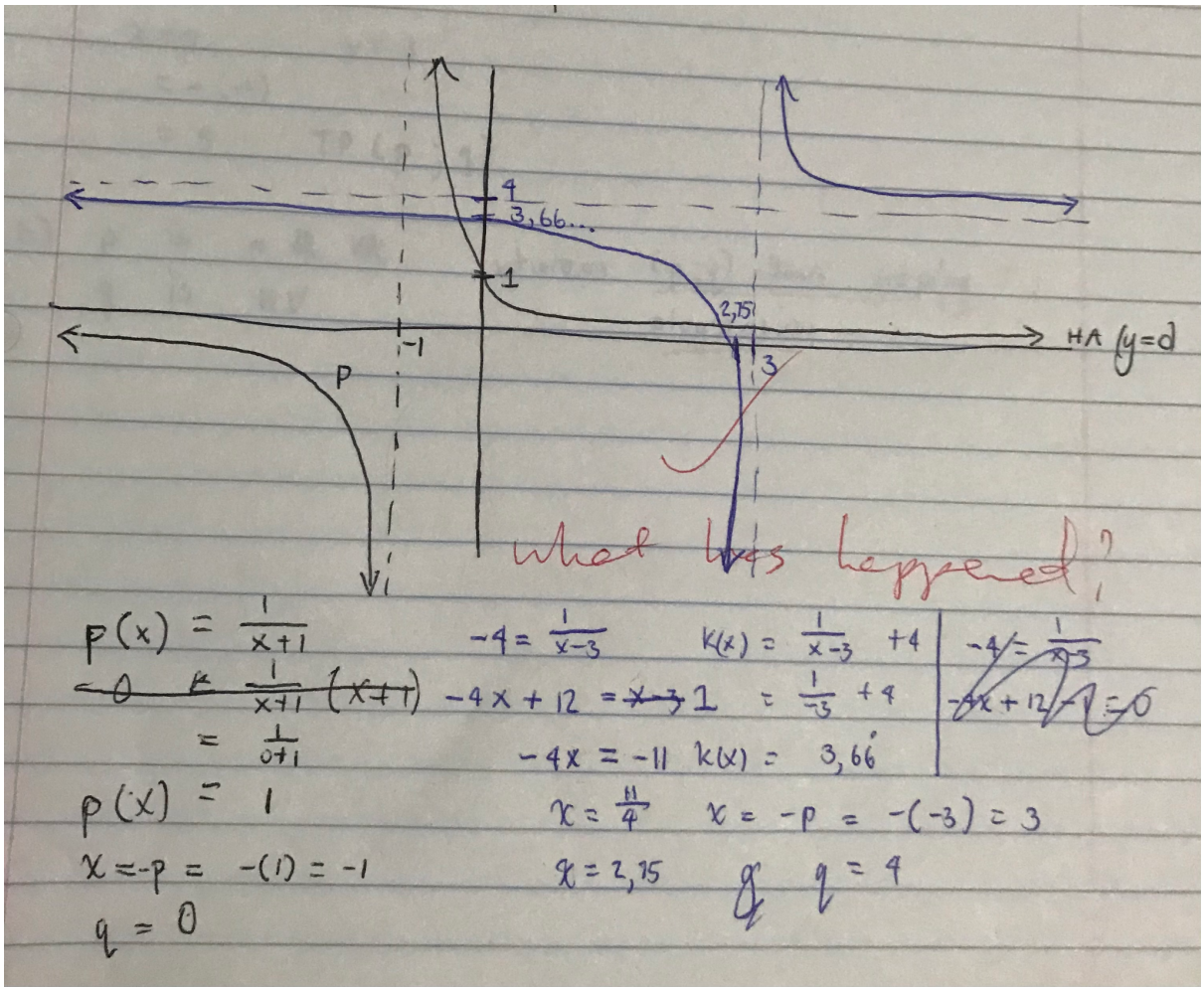


Figure 4.22: L13 response to Question 2

The interview with L13 in Box 4.13 also revealed that he could describe the transformations that had taken place as was observed with L19. Again the researcher attributed this to not understanding the demands of Question 2.

R: *How does p(x) change to k(x), what transformations take place?*
L13: *sir, so the first transformation is that p(x) moves four units to the right, so now, the vertical asymptote, it's no longer at x=-1. It's now at x=3.*
R: *meaning it has gone 1, 2, 3, 4 units to the right*
L13: *yes.*
R: *So if you subtract 4 on the equation, on the graph, it's shifting to the right by 4 units. We remember that, we saw that on GeoGebra. Then also, how about the*

vertical shift?

L13: *sir, the vertical shift, now, that, so now, p(x) has moved, it's no more p(x), but that graph that moved four units to the right is also moving four units up. So now the horizontal asymptote is no more at y=0, now it's at y=4.*

R: *okay*

L13: *Yes.*

Box 4.13: L13 responses to interview

Sub-theme 1

The deduced sub- theme from L19 and L13 was *not understanding the demands of a question.*

L22's answer to Question 2 was as expected in the genetic decomposition. Her response is shown in Figure 4.23 below:

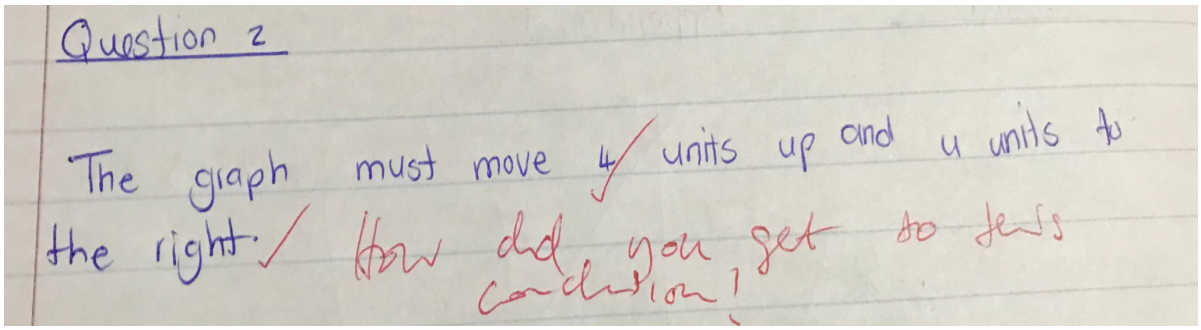


Figure 4.23: L22 response to Question 2

In the interview, L22 was able to explain how she got to her conclusion about the transformations that had taken place. Her interview response is shown in Box 4.14, below:

Researcher: *I see in the first question you did the graphs quite well, on question two you said the graph must move 4 units up and 4 units to the right, so my question to you is how did you get to this conclusion?*

L22*: For p(x) I can see that there is no q value which means that q would have been plus zero and then if it says +4 it means from zero to 4 that's 4 units up and then for x+1, the denominator, we know that the actual formula is x=p which means*

that x= -1 and then this x=3. We know that -1 lies on the left side of the graph, the axes, no the x-axis, oooh the y-axis and then x=3 will be on the right side so then if you go from -1 to 3 that's 4 units to the right.

Box 4.14: L22 interview response to Question 2

Sub-theme 2

The deduced sub-theme from L22 was that she could *describe the change in a given function as a result of the application of transformation without the need to perform each step.*

There were two learners that really struggled with Question 2. These were L14 and L16 respectively. See Figure 4.24 and 4.25

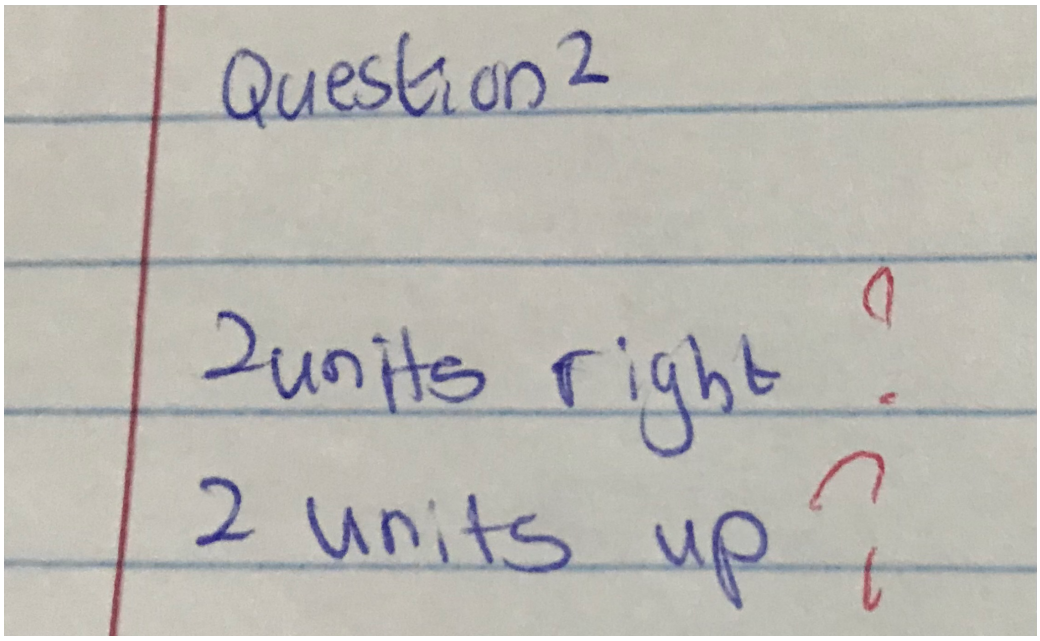


Figure 4.24: L14 response to Question 2

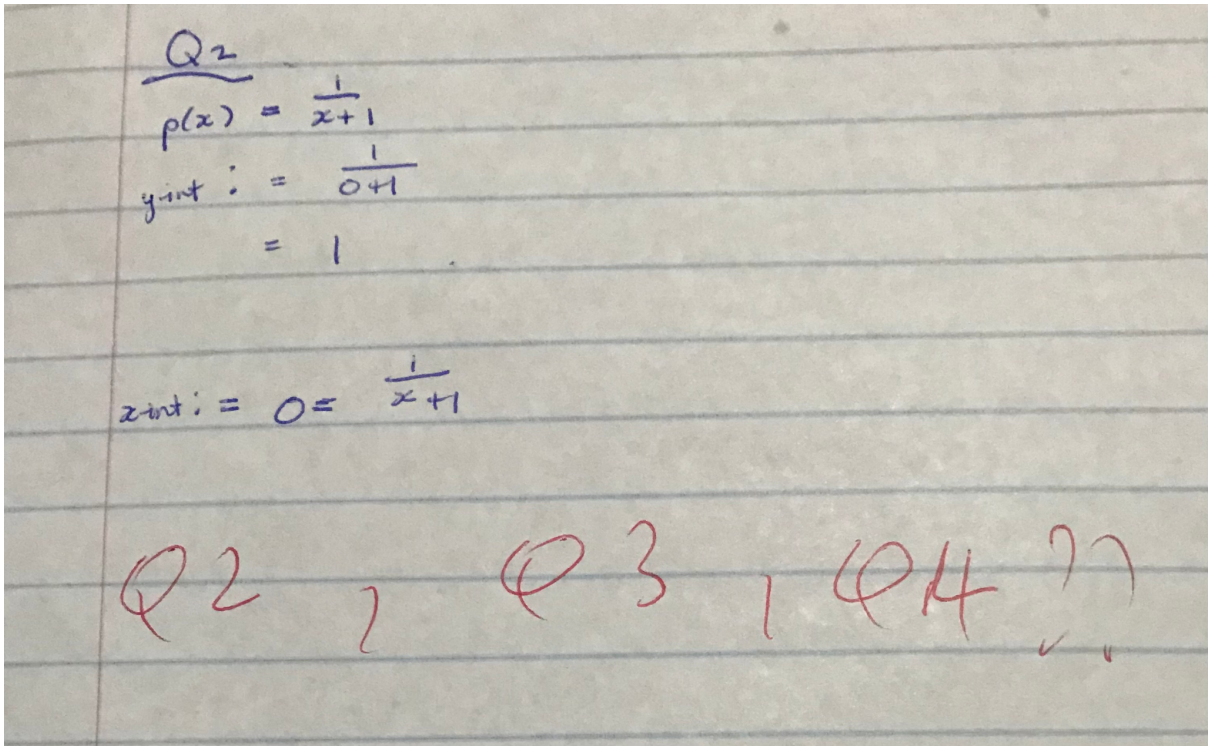


Figure 4.25: L16 response to Question 2

L16 had this to say in the interview Box 4.15:

Researcher: *I see your work is incomplete for questions two, three and four. Why?*
L16*: Oh, it was difficult.*
Researcher: *Can you answer the questions now?*
L16: *No, I'm really bad when it comes to it.*
Researcher: *So can I ask you to do this question two, three and four over the weekend, then give me your work on Monday.*
L16*: Okay sir.*
Researcher: *Maybe my last question will be how challenging are these transformations of functions?*
L16: *Transformations themselves are not difficult. But when you're given a function and you have to find specific points for your given point, and you have to substitute them into an equation, then it becomes a bit trickier.*
Researcher: *Okay. GeoGebra, do you want to say anything?*
L16: *Really helped. Instead of like doing it yourself in the book, and that plotting*

doesn't make sense because you didn't …(inaudible)
Researcher: *but remember, you must also know how to do the sketches on your own, in the exam or in the test, you won't be having GeoGebra, it is just for learning purposes.*

Box 4.15: L16 interview responses to Question 2

L14 and L16 struggled to accommodate the newly learnt transformation of functions concepts with previous content on functions. They were exposed to opportunities for learning transformations of functions on GeoGebra in class but they still could not relate or apply that experience in the homework. The researcher's analysis of these two learners was that they lacked the mental constructions needed for transforming functions.

Sub-theme 3
The deduced sub- theme from L14 and L16 was that they *failed to apply learnt concepts in the solution of new problems.*

4.3.3 Analysis of Question 3

The predetermined code for Question 3 was the object level of the APOS theory. This was because the question required that learners were able to reflect on all processes, and think of them as a whole, in any representation or context. Learners had to think of how transforming functions affected domain and range. In this question, the majority of learners were comfortable with writing verbally, the transformations made on the basic functions to arrive at the resulting functions. They then struggled with commenting on what effect these transformations had on the domain and range. This can be seen in the written responses by L24 and L4, Figure 4.26 and Figure 4.27 respectively.

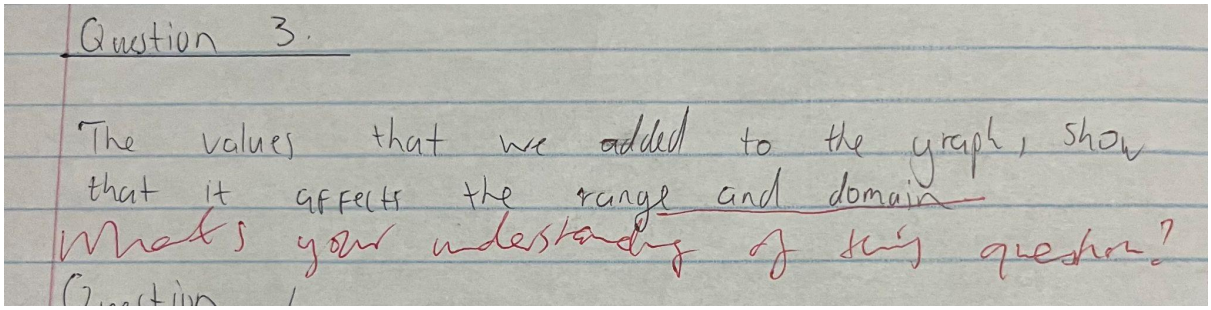


Figure 4.26: L24 response to Question 3

In an attempt to check the understanding that L24 had on domain and range, the interview excerpt with the researcher is shown in Box 4.16 below:

Researcher: *Okay, question three. You did not comment about domain and range. Do you know these things? For all the graphs?*
L24: *We were taught about them. But not for all the graphs*
Researcher: *which graph was excluded.*
L24: *I think we were taught about them. But then I got stuck about the domain and range*
Researcher: *domain and range. So you must read more, or next week, you can come to me, I can help you with that.*
L24*: thank you sir*
Researcher: *domain and range. Domain affects the input, that is the x values. Range affects the output, that is the y values.*
L24: *Yes.*
Researcher: *So when you're checking domain and range, those are things that we look for. But I'll assist you, maybe next week if you come.*
L24*: Okay, sir*

Box 4.16: L24 interview responses to Question 3

Sub-theme 1
From L4 and L24 the sub- theme deduced was that learners were *unaware of the process in its totality.*

L4's written response was as shown in Figure 4.27 below:

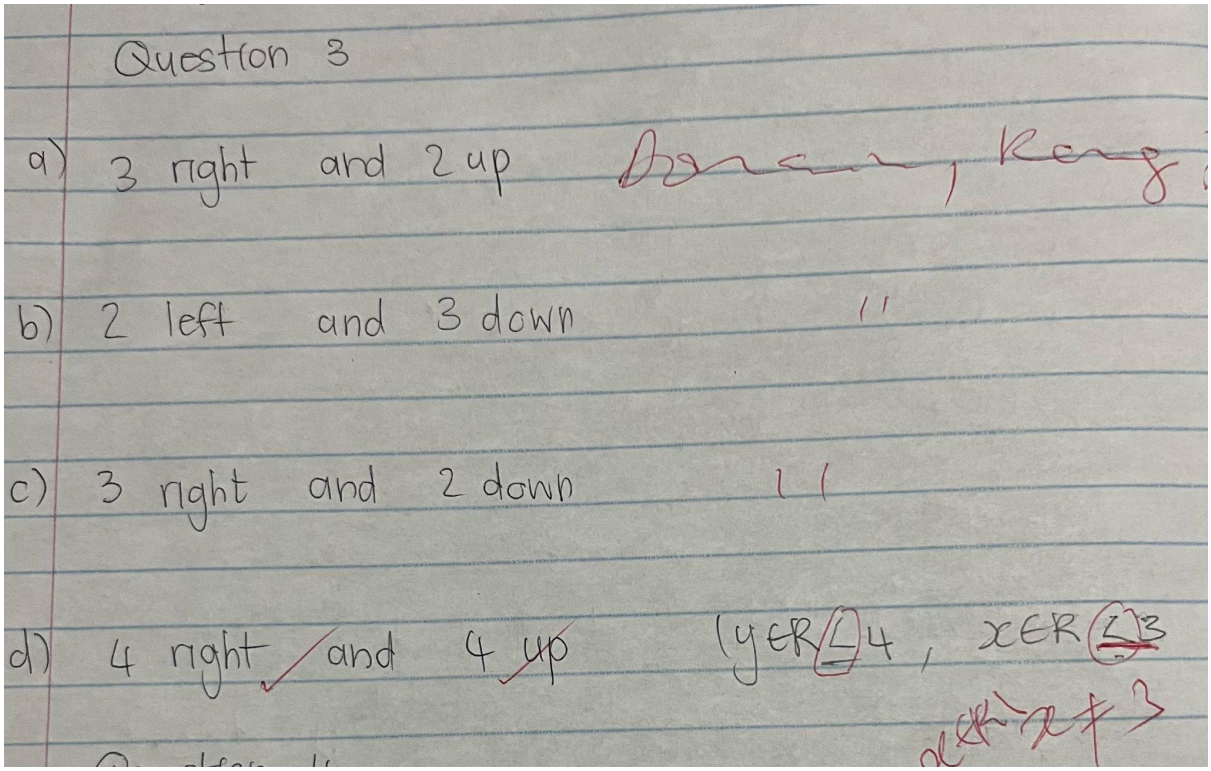


Figure 4.27: L4 response to Question 3

The researcher was interested in finding out the challenges that had been encountered by L4 in Question 3. Box 4.17 below shows the interview transcript.

Researcher: *Question 3, you were supposed to verbally give the transformations made. I have seen you have verbal transformations there but you left out the domain and range. Do you understand domain and range?*
L4: *not exactly sir. I struggled, that's why I left it out*
Researcher: *then also for the 4th graph, you are saying the domain is y less or equal to 4. Now in the 4th graph, it's a hyperbola, when you are giving the domain or the range, do we say y less than or y greater than, x less than or x greater than? If it is a hyperbola? (draws hyperbola on paper). When you want to give domain and range, what do you do?*
L4: *you look at the asymptotes and then that is your maximum value.*
Researcher: *does a hyperbola have a maximum?*
L4: *I don't know how to put it sir, but its where the …*
Researcher: *do you know how to explain the domain?*

L4: *domain is the x value*

Researcher: *you look at the input values, right? You are saying on the x-axis, does the graph exist for every value of x?*

L4: *it exists*

Researcher: *do you see the graph will not exist at the asymptote? (demonstrating on drawn hyperbola)*

L4: *yes*

Researcher: *but immediately after the asymptote, the graph is existing again. So how must you give your domain?*

L4: *that the graph does not exist at the value of 3*

Researcher: *which is the asymptote, so you were not supposed to write it as an inequality like you did there*

L4: *so how was I supposed to write it?*

Researcher: *you were just supposed to write x cannot be 3 where x is a real number*

L4: *oooh okay. And also y is a real number but it can't be 4*

Researcher: *everywhere on the y values, the graph exists, but when y is equal to 4, you can't touch the graph.*

L4: *thank you sir, I have learnt something*

Box 4.17: L4 responses to interview

L4's response to Question 3 was the same as for many other learners in the class. They were confident in writing the transformations that had taken place verbally for all the graphs. However, they could not explain the effects that these transformations had on the domain and range. This implied that they did not have an object level conception of reflecting on all processes and being able to think of them as a whole, in any representation or context as suggested by the genetic decomposition.

Sub-theme 2

The deduced sub- theme was that learners *failed to apply learnt concepts in the solution of new problems.*

For a learner operating at the object level, L19 was the only one amongst the learners that was able to display the expectations that are mentioned in the genetic decomposition. Her written response is shown in Figure 4.28 below:

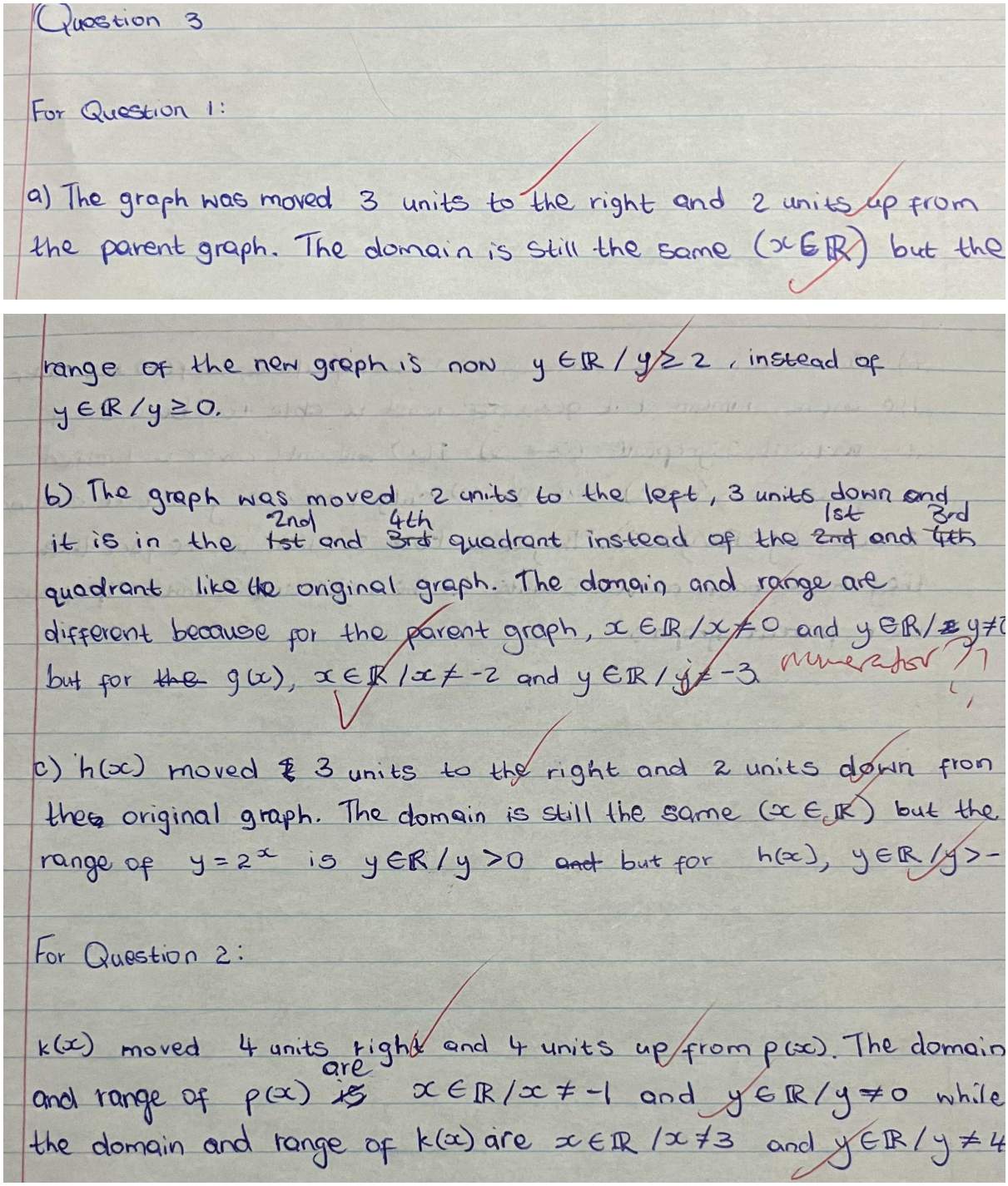


Figure 4.28: L19 response to Question 3

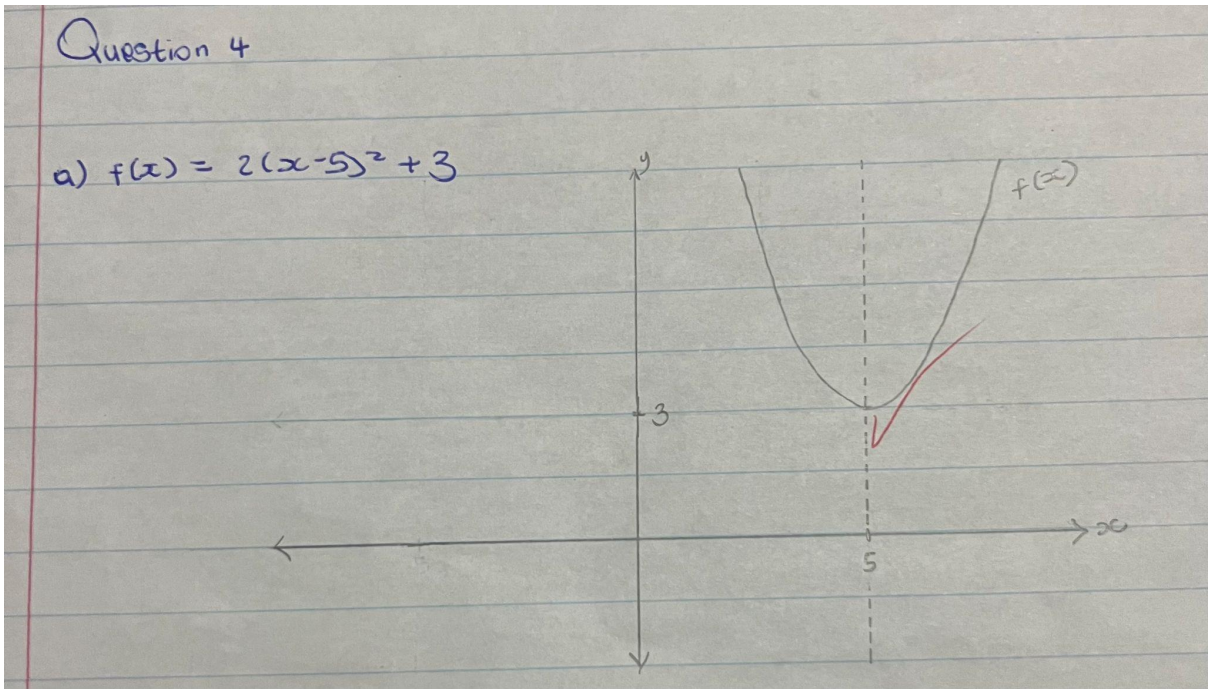
Sub-theme 3

The deduced sub- theme from L19 was that she could *describe the change in a given function as a result of the application of transformations without the need to perform each step.*

4.3.4 Analysis of Question 4

The predetermined code for Question 4 was the schema level of the APOS theory. According to the genetic decomposition, learners operating at this level were expected to be able to form perceptions by considering the interconnection of several actions, processes, objects and other previously constructed schema.

Many learners answered Question 4(a) confidently. At Question 4(b), only two learners were able to respond to the demands of the question in a convincing way. These were L19 and L13 as shown in Figure 4.29 and Figure 4.30 below:



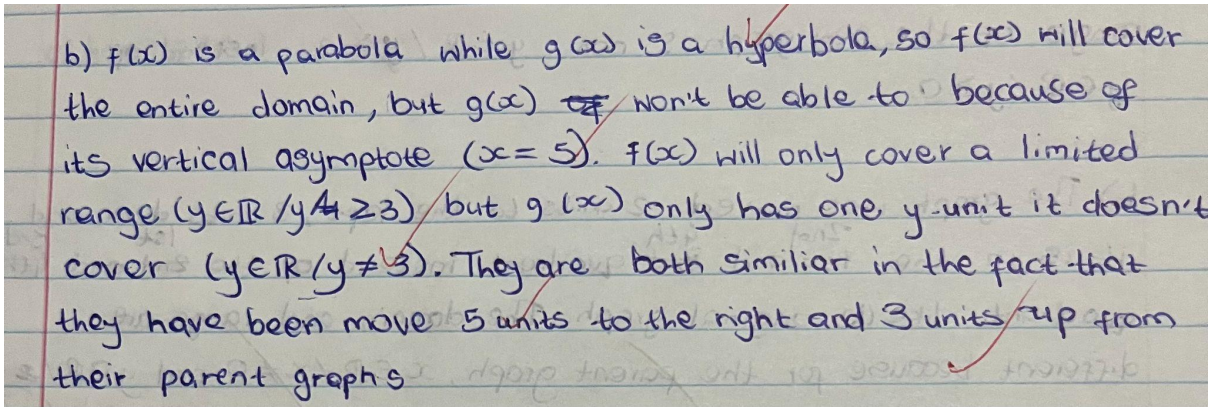


Figure 4.29: L19 response to Question 4

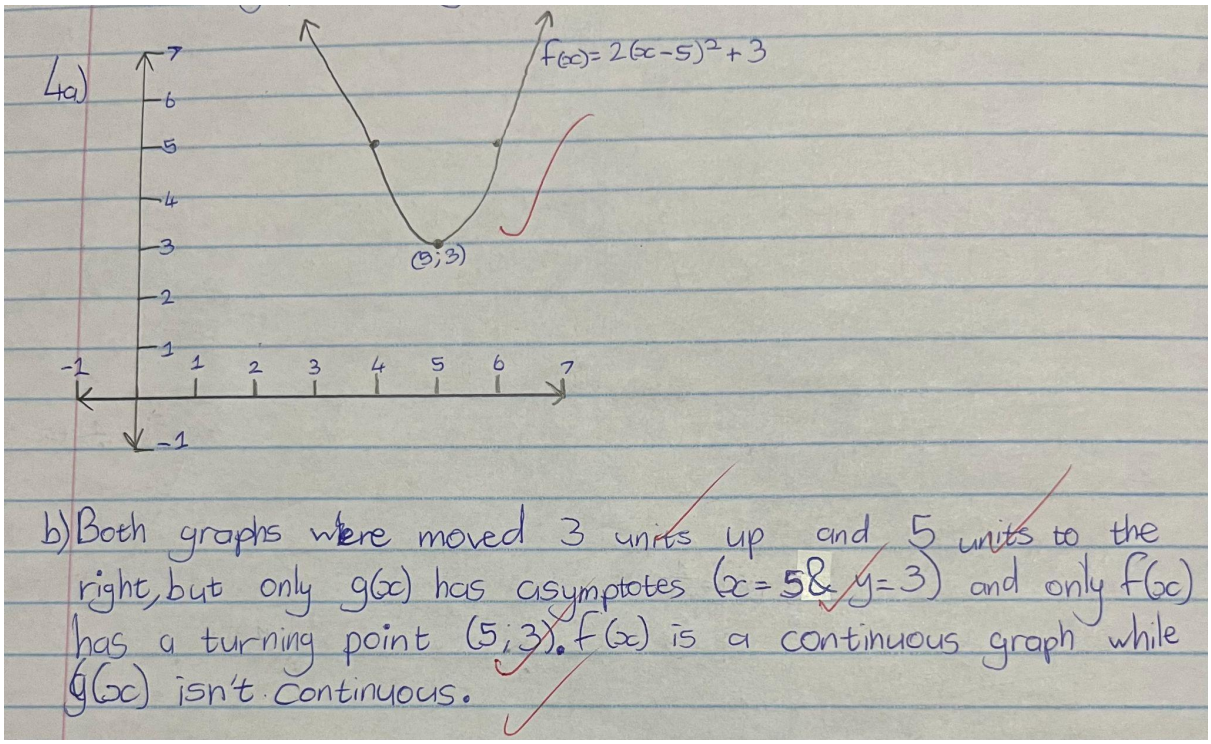


Figure 4.30: L13 response to Question 4

The transformations of functions schema had developed in L19 and L13 as they easily drew the required sketch in 4(a) and also managed to compare and contrast the two functions in 4(b). Since the schema had grown, they were able to incorporate other previously created schema into their mental frameworks. They were able to do this because they could articulate their reasoning, build the necessary knowledge of transforming functions, and demonstrate understanding of procedures through reasoning rather than memorization.

Sub-theme 1

The deduced sub- theme from L19 and L13 was that they were able to *form conceptions by considering the interconnection of several actions, processes, objects and other previously constructed schema.*

For Question 4(b), the researcher identified L15 and L26 to be in the same category as most of the learners as they could not compare and contrast the two graphs as shown in Figure 4.31 and Figure 4.32 respectively.

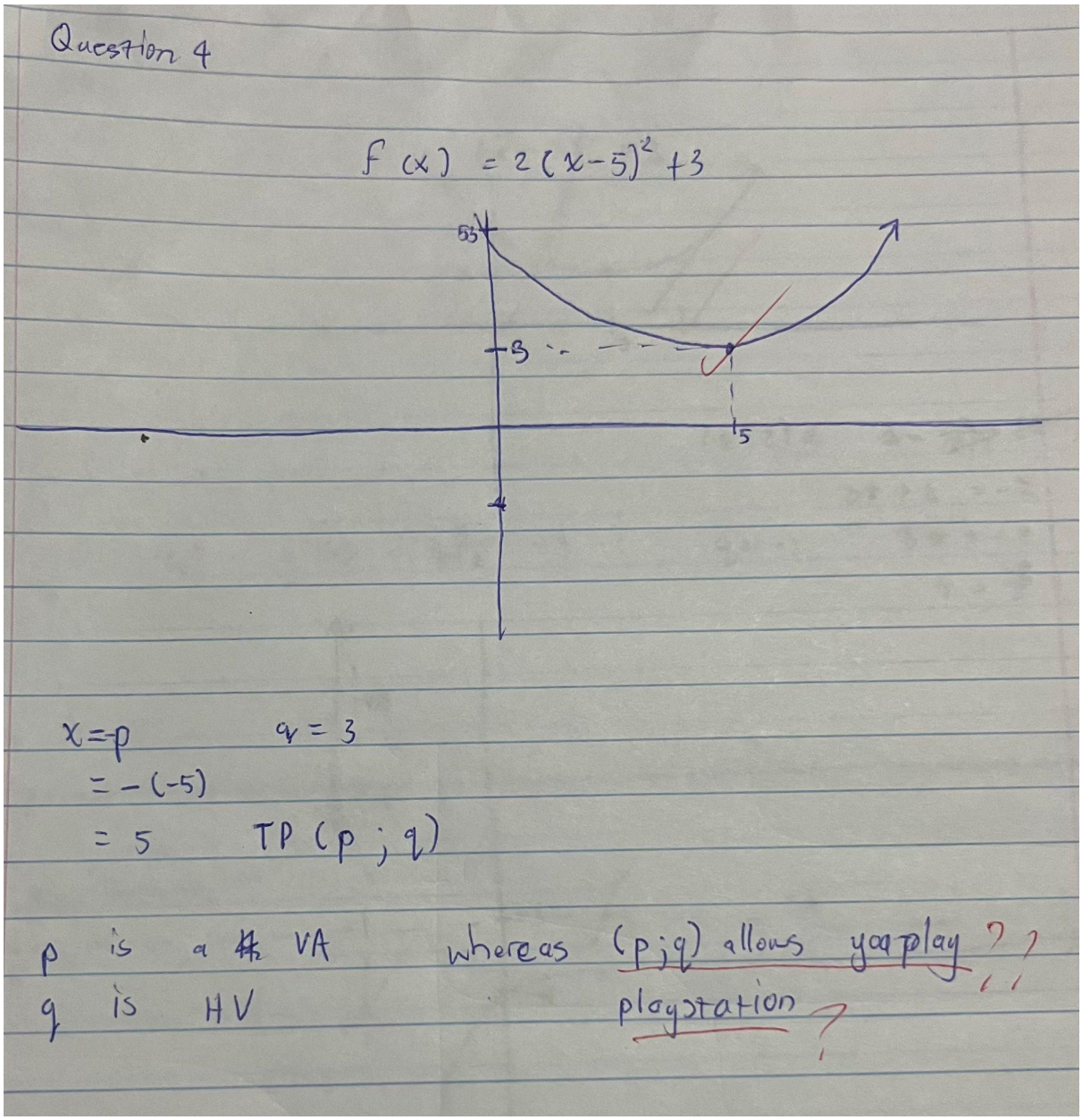


Figure 4.31: L15 response to Question 4

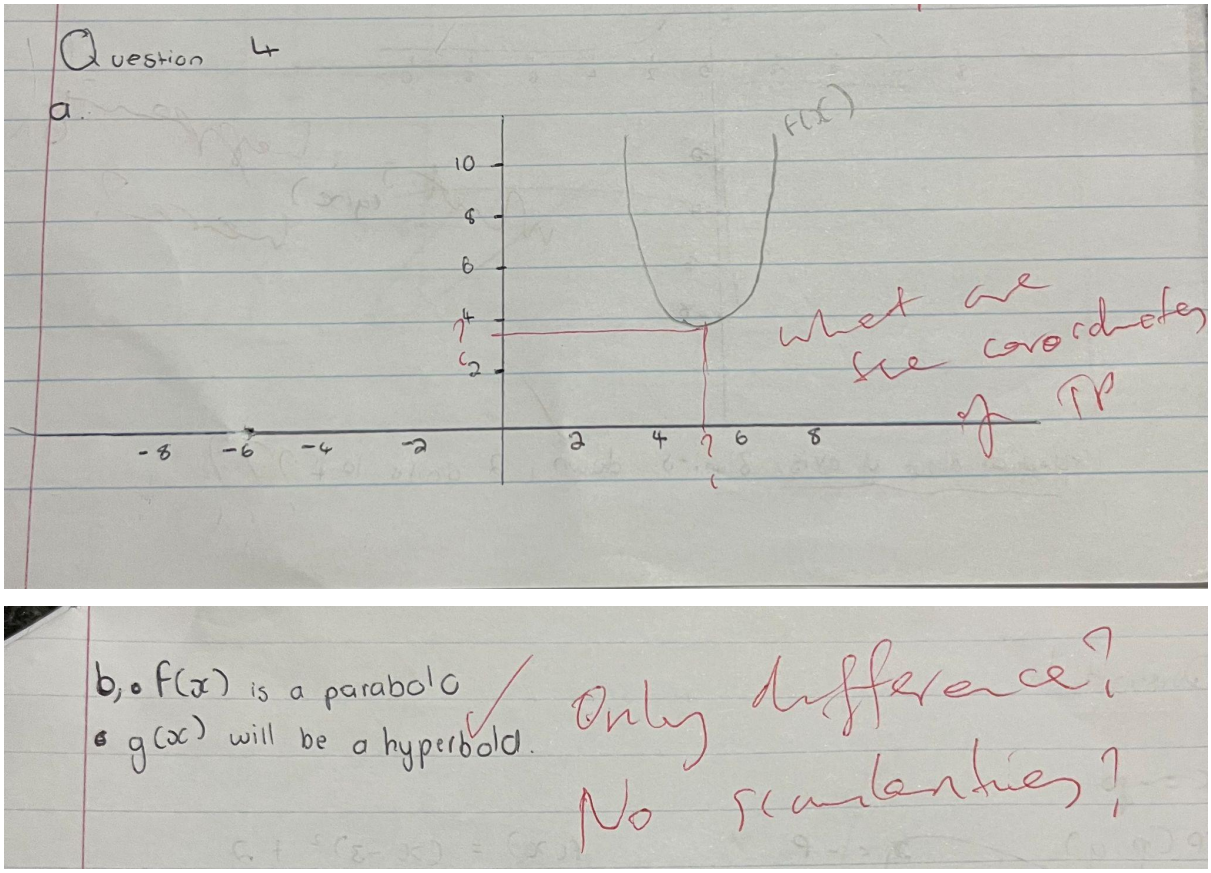


Figure 4.32: L26 response to Question 4

L15 answered 4(a) well, including showing the coordinates of the turning point. He completely struggled with the comparison of the two functions. L26 got the shape of the graph correct but with incorrect coordinates of the turning point. These two learners, and many others in the class, had difficulty forming conceptions by not realising the interconnection of several actions, processes, objects and other previously connected schema.

Sub-theme 2

The deduced sub-theme was that learners were *unaware of the process in its totality.*

4.4 Triangulation of results

The sub themes from the data analysis conducted for the classroom observations, task-based interviews and homework were integrated as shown below so as to establish the final themes. The final themes were then used to address the research questions.

The first research question was:

- What are the challenges faced by Grade 11 learners when transforming functions?

The sub-themes that responded to this first question were firstly, failure to apply learnt concepts in the solution of new problems. Secondly, learners were unaware of the processes in their totality. The third one was that learners' algebraic skills were poor. The fourth one was that of giving partial descriptions of transformations. The fifth sub-theme that came up was that of learners not understanding the demands of a question. The main theme that was deduced from the first and fourth sub- themes is that ***learners do not easily recognise transformations on functions***. From the second, the third and the fifth sub-themes, the other main theme deduced was that ***learners' mental constructions were based on isolated facts or procedures.***

The second research question was:

- How does the use of GeoGebra enhance learners' conceptions of transformations of functions?

The sub- themes that responded to the second question were firstly that learners could describe the change in shape of the graph of the basic function after moving sliders. Secondly, learners were able to visualise properties of basic graphs and their transformations. Lastly, repetition of (ACE Cycle) of actions on GeoGebra leads learners to internalising actions. The main theme that was deduced from these sub-themes for the second research question was that GeoGebra is an ***effective tool in supporting conceptual understanding in the learning of transformations of functions.***

4.5 Discussion of the findings

The APOS theory drove the design and implementation of instruction in this study. In agreement with Daher and Anabousy (2015), the implementation of instruction created opportunities for data collection and analysis. Collection and analysis of data involved a comparison of constructions envisaged in the genetic decomposition with what learners were doing, saying or writing. A genetic decomposition of transformations of functions was used as the analysis tool while the ACE cycle

guided instruction. Deductive thematic analysis was used to generate sub- themes and eventually the main themes of the findings.

The first step of the cycle were the activities on GeoGebra and in the worksheets that were designed to foster learners' development of mental structures necessary for transforming functions. Learners then discussed their solutions or challenges that they encountered and listened to explanations by fellow learners or the teacher. There was homework given to reinforce the knowledge obtained in the worksheets and classroom discussions.

The first research question focused on the challenges faced by Grade 11 learners when transforming functions. In unpacking the question through the task-based interviews and the homework activity, the researcher discovered that learners mainly failed to apply learnt concepts in the solution of new problems. For example, L19 in Question 8 of the first worksheet, she failed to notice that $y = ax^2 + bx + c$ and $y = a(x + p)^2 + q$ represent the same function. This is despite the fact that she has been dealing with the quadratic trinomial from Grade 9 and has also completed the square in Grade 11.

It was also evident that most learners were unaware of processes in their totality. L1 on Question 9 of the worksheet on translation of graphs, Figure 4.13, got the vertical translation correct but the horizontal translation was incorrect. Similarly, L24 and L4 in Question 3 of the homework activity, managed to describe transformations eloquently but failed to explain the effect of these transformations on the domain and range. L19, L1, L24 and L4's experiences corroborate Bhagat and Chang (2015)'s observation that learners with a low understanding of mathematics were not in a position to take advantage of tools such as GeoGebra. Venter (2020) also reported that learners could not apply previously owned knowledge as there was a barrier in learning that arose because of the limitations of learners' understanding of a concept.

In many cases, learners gave partial descriptions of transformations, a case in point being L18. On Question 5 of the worksheet on translations, L18 did not consider all parameters when asked about the effects of each of the parameters in each of the

different functions dealt with in Question 1, as was seen in Figure 4.6. The same happened in Question 2 of the homework activity. When required to perform transformations, L19 and L13, Figure 4.22 and Figure 4.23 respectively, instead of writing about the transformations that would move $p(x)$ to $k(x)$, they just drew the two functions on the same cartesian plane. This confirms Baker et al (2001)'s reference to having carried out studies on learners' understanding of transformations of functions and found out that transformations of functions is a difficult concept for learners. There were also cases where it appeared that learners struggled to accommodate new learnt concepts with previous ones. This was evident in both the worksheets and the homework exercise. As an example, L16 in Figure 4.16, when she was answering Question 19.6 on the worksheet on deductions from sketch graphs, she could not determine the turning point of $f(x) = x^2 - 1$. Parabolas of this form are dealt with extensively in Grade 10. Final themes answering the first research question were that learners lacked important competencies and that learners' mental constructions were based on isolated facts or procedures.

The analysis of lesson observations and task-based interviews that were being done during the lessons managed to respond to the second research question on how the use of GeoGebra enhances learners' conception of transformations of functions. The researcher found out that GeoGebra is an effective tool in supporting conceptual understanding. GeoGebra also appeared to be a good motivator for learners in the learning of transformations of functions. There was an increased level of interest in learners about transforming functions. Similarly, Arbain and Shukor (2015), found out that learners will have positive perceptions of GeoGebra software in terms of enthusiasm, confidence and motivation.

The study investigated how learners construct their knowledge when transforming functions using the four stages of the APOS theory. Each level outlined the phases of knowledge development. The researcher came to the conclusion that two students fully comprehended the subject even when they reached the schema level of the homework assignment.In this study, learners' responses revealed that most of them functioned at an action level, as in all four questions in the written homework and in the interviews. In the lessons, the researcher noted that when translating functions,

learners would come up with rules, for example, $f(x+2)$, the learners would say this means the function is translated *2* units to the left. Surprisingly, one would think that they could complete all questions where these rules were required. In the worksheet on deductions from sketch graphs and in the homework activity, there were learners who could not apply this 'rule' that they repeated a lot of times.

There were a few learners that could interiorise some of the transformation concepts into a process. In Question 1 and 2 of the homework exercise, they interiorised transformations from one function to another into a process whereby internally, they visualised the transformation and encapsulated the resulting function as an object which they could even reverse. Looking at the questions in the homework exercise, it was apparent that transforming functions was partially assimilated by most learners into their cognitive structures. At most times, they could not successfully coordinate the newly learnt concepts with their existing actions, processes and schemas. It could be said that mostly, their knowledge construction was limited to action conception of transformations of functions as they could only complete questions that required external cues. This might also have been due to the stimuli created within the structured nature of the questions that they were dealing with.

4.6 Chapter summary

The findings obtained from the observations, the task-based interviews and the homework exercise revealed that learners were able to visualise properties of basic graphs and their transformations on GeoGebra, many would still struggle to answer questions when working on the worksheets or homework exercise. The GeoGebra intervention prioritised visual representation, which has been shown in earlier studies to extend and deepen learners' conceptual understanding. The researcher was able to successfully implant various characteristics of transforming functions, such as horizontal and vertical translations, in the trainees' thoughts through the intervention. The findings demonstrate that the GeoGebra exercises created for the students seemed to inspire and capture their attention while aiding in the development of the mental structures as outlined in the genetic breakdown of transformations of functions. This can be also interpreted by the classroom observation analysis and

task-based interviews that were carried out during lessons. The next chapter has the summary, recommendations and conclusion of the study.

CHAPTER 5

SUMMARY, RECOMMENDATIONS AND CONCLUSION

5.1 Introduction

The previous chapter dealt with the findings and interpretation of this study. Data from classroom observations, task-based interviews and written work from the daily lessons and homework was analysed using predetermined codes from the APOS theory and indicators from the genetic decomposition. The generated themes were then used to respond to the research questions in the study. This chapter will summarise the results, discuss the implications of these results and make suggestions on the way forward. Recommendations based on this study will be provided and to conclude the study, a summary of this chapter will be presented.

5.2 Summary of the results

The study explored learners' conception of transformations of functions using a technological tool, Geogebra. The research questions were:

- What are the challenges faced by Grade 11 learners when transforming functions?
- How does the use of GeoGebra enhance learners' conceptions of transformations of functions?

A qualitative approach, with an interpretive case study design were adopted in an endeavour to respond to these questions. The APOS theory guided the study, both as a theoretical framework and as a teaching method. Data was collected from lesson observations, task-based interviews and written scripts for daily work and a homework activity. The data generated was analysed based on predetermined codes, namely action, process, object and schema, from the APOS theory, with a genetic decomposition of transformations of functions providing indicators for the codes. Deductive thematic analysis was used to find sub-themes and main themes that eventually gave answers to the research questions. The written scripts for the daily work and homework gave answers to the first research question. The challenges faced by Grade 11 learners when transforming functions were that a majority of the learners

did not easily recognise transformations on functions. Learners also showed that their mental constructions were based on isolated facts or procedures.

The failure to recognise transformations of functions stemmed from the failure by learners to apply learnt concepts in the solution of new problems and also when they gave partial descriptions of transformations of functions. Examples can be drawn from L19 in Question 8, Figure 4.12, on the worksheet on translating graphs using GeoGebra, failed to notice that $y = ax^2 + bx + c$ and $y = a(x + p)^2 + q$ represent the same function. This after dealing with quadratic trinomials from Grade 9 and also completing the square earlier in Grade 11. In the same question L19 failed to describe the effects of changing coefficients a and b despite the fact that he was working on GeoGebra. L16 in Question 19.6, Figure 4.16, substituted for $f(x + 2)$ correctly but made mistakes when simplifying. L19 in Figure 4.17 for the same question, did not demonstrate that she understands what it means to say $f(x + 2)$. It was a complete breakdown.

The other main theme of learners' mental constructions being based on isolated facts or procedures came up when learners showed that they were unaware of the process in its totality and not understanding demands of questions was exhibited in the following examples. L19 in Question 2 of the homework, Figure 4.22 was supposed to mention the transformations that had taken place on $p(x)$ so that it becomes $k(x)$. She drew the graphs instead. This showed that she did not understand the demands of this question. In Question 9 of the worksheet on translating graphs, L1 in Figure 4.13, correctly did the vertical translation but failed on the horizontal translation. This showed that she was unaware of the process in its totality. Poor algebraic skills were also shown in the same question by L1 when she said $-2x - 2 = -4x$.

From the lesson observations, homework and task-based interviews, where learners were mostly working on GeoGebra and responding to questions by the researcher, answers to the second question were found. The results showed that GeoGebra is an effective tool in supporting conceptual understanding in the learning of transformations of functions.

One sub-theme that bothered the researcher indicated that learners were unaware of processes in their totality, mainly on how learners found the horizontal translation challenging. Examples from L1 when answering Question 1 from the Translations worksheet correctly applied the vertical translation but the horizontal translation was completely misunderstood. L13 also had an almost similar experience when answering the same question. He first completed the square and when it was time to effect the translations, he did the vertical translation well but the horizontal translation was not done well. The finding is consistent with Zazkis et al (2003) who wrote extensively about learners grappling with horizontal translations, referring to it as perceived inconsistency. The horizontal translation caused these learners to be unaware of the process as a whole. The researcher concurs with Zazkis et al (2003) that more exploration on the intricacies of the inconsistency between the intuitive presupposition and the actual result must be studied further so that learners can have a complete awareness of both the horizontal and vertical translations.

The researcher also noticed that visualisation of transformations of functions on GeoGebra helps learners with better understanding of the concept. As an example, L19 was giving partial descriptions in the worksheet on translations in Question 8 Figure 4.12. She also had a complete breakdown on Question 19.6 of the worksheet on deductions from graphs. However, tremendous progress was noticed in her homework as indications from her work were classified to be at schema level by the researcher, refer to Figure 4.30. Learners were generally active in the classroom activities. They looked excited when using GeoGebra. GeoGebra came out as a motivator in the learning of transformations of functions. This was evident in learners that were normally quiet and reserved in class being involved in classroom discussions and were solving problems with peers as well as with the researcher. There was an increased level of interest about the topic. Interest in doing work, classroom participation and learners being generally good towards each other. Learners were happy learning with GeoGebra.

The researcher discovered that including GeoGebra into the classroom while he was teaching learners on function transformations boosted learner satisfaction at all levels of instruction while promoting the development of abilities that are crucial for facilitating active learning. Action, Process, Object, and Schema were the four stages of

mathematical comprehension that were covered in the lessons, which were taught utilising the ACE theory, a pillar of APOS theory. These four APOS levels were utilised in the previous chapter to examine the stages at which students built their knowledge in order to comprehend the idea of changing functions. The APOS theory was applied in this study as a teaching tool and as a tool for data analysis. This application of the APOS theory supported the claim made by Dubinsky and McDonald (2001) that research in mathematics education is supported in a number of ways when it is grounded in a theoretical perspective. When a learner reached the schema stage, the researcher determined that he or she had fully grasped the concept. Each level of the APOS theory highlighted the step of knowledge construction. The ramifications, ideas, and advice resulting from the study's findings are presented in this chapter.

5.3 Implications and suggestions

The findings imply that GeoGebra could enhance conceptual understanding of transformations of functions in Grade 11 classes. If mathematics teachers could adopt the use of GeoGebra in the teaching of transformations of functions, learners could be motivated to learn. It could also be proposed that GeoGebra be used in other mathematical topics, particularly those that pose some challenges to learners such as Euclidean Geometry, Probability, Trigonometry, to mention a few.

Besides the positives from the integration of GeoGebra in teaching and learning transformation of functions, the researcher found out that learners' conception of transformations of functions was generally not satisfactory when it came to answering questions that were on the worksheets and homework activity. Particularly, learners' process, object and schema conceptions were poor. The reason for the poor conception might have been influenced by learners' poor acquisition of these conceptions. Poor conception could also have been because learners with low understandings of mathematics were not in a position to take advantage of tools such as GeoGebra as they felt obliged to use them.

The APOS theory and the formulation of a genetic decomposition of transforming functions provided a valuable way of exploring the learning of transforming functions

with the Grade 11 class. The findings of the study revealed that some aspects of the APOS theory, specifically process, object and schema, were not fully operational because most of the learners were successful with drawing graphs on GeoGebra, move sliders and attempt questions where they could use formulas, substitute and follow steps. Mathematics teachers need to focus on activities such as investigations that improve learners' object and schema conceptions of function transformations. The investigations should have instructions and tasks that frame the activity and clarify goals without revealing solution methods.

The researcher also suggests that there is a need for mathematics teachers to develop skills in using technology, GeoGebra in this study, so as to support its continued integration into mathematics teaching and learning. Learning using GeoGebra has many benefits, as was deduced in the sub-themes in the previous chapter. These benefits can also include providing greater learning opportunities for learners, enhancing learner engagement and promoting inquiry based learning. This was confirmed in this study. In planning for instruction, mathematics teachers must devise genetic decompositions for various tasks as these inform teachers of the mental constructions of their learners. The instructional design informed by the genetic decomposition encourages learners to construct solutions through creative thinking, and if supported by GeoGebra, can improve mathematics learning. The feedback from this instruction will help teachers gauge whether effective teaching is taking place.

The DBE (2018, 2019) diagnostic reports indicate that most South African learners were unable to transform functions at matric level. This inability of learners calls for more studies to be undertaken to find out the challenges which learners encounter as they are transforming functions. Furthermore, studies on conceptions and reasoning patterns are needed for a strong and flexible understanding of why learners continue to struggle with transformations of functions. This study used a case study design. Future research could be conducted that follow mixed methods designs, or use other qualitative or quantitative designs, in order to obtain a deeper understanding of the phenomenon discussed.

5.4 Conclusion

The purpose of this study was to investigate whether the use of GeoGebra enhances Grade 11 learners' conception of transformations of functions.
From the findings, the following conclusions are drawn:

a) Repetition of ACE cycle actions on GeoGebra leads learners to internalising actions.
   GeoGebra allows for a lively classroom where principles of cooperative and collaborative learning are evident.
b) Visualising transformations of functions on GeoGebra helps learners with better understanding of the concept.
   GeoGebra is an effective tool to support learners achieve principles of constructivist learning.
c) This study also established challenges such as learners showing lack of understanding of basic functions, struggling to accommodate newly learnt transformations of functions concepts with previously owned content and not understanding the demands of questions. This study recommends that more opportunities be provided for learners to experience diverse function types emphasizing multiple representations of the same functions.

5.5 Recommendations

The action level of the APOS theory needs to be improved for learners. It should be noted that the worksheets and homework activities had questions that were intended to guide students through the process, object and schema stages. Future research on the poor object and schema conception of function transformations is promising. Future studies on how GeoGebra improves conceptual understanding in learning transformations of functions must also be thorough and conducted over longer time periods with far larger sample sizes at various schools with various racial and socioeconomic compositions that accurately represent the entire South African economy. This study suggests that mathematics teachers should be encouraged to use this software in their lessons. The software should be introduced to teachers so they can experience its impact on both themselves and their learners.

The researcher concurs with Baye, Ayele, and Wondimuneh, (2021), who cautioned on the use of GeoGebra alone. They put forward that Geogebra, on its own, is unlikely to bring a significant change in learners' conceptual understanding. They recommended that mathematics teachers must integrate GeoGebra with other worthwhile teaching approaches. Mudaly and Uddin (2016) promoted the idea of investigations as a chance for learners to develop their reasoning, abstraction, and conjecture-making and justification skills. Future research should also expand this study to include additional mathematics topics and grade levels to determine if it can provide outcomes that are comparable. The results of these research may contribute to raising the standard of teaching mathematics in South Africa.

This study found out that visualisation of transformations of functions on GeoGebra helps learners with better understanding of the concept. The researcher suggests using GeoGebra-assisted teaching to teach and learn about function transformations in light of this study's finding. Any teaching and learning strategy that inspires learners to learn will go a long way toward addressing South Africa's issue of low accomplishment in mathematics in general and in transformations of functions in particular. Motivation is a vital factor in determining learners' conceptual understanding. Olsson (2017), highlighted that GeoGebra supports collaborative and creative mathematics thinking and problem solving by providing learners with a shared workspace and feedback on their actions. This was echoed by Shadaan and Leong (2013), who observed that the use of GeoGebra allowed for a lively classroom where the principles of cooperative and collaborative learning were evident.

Since it gives learners the opportunity to actively participate in their learning, explore, form hypotheses, communicate with and learn from their classmates, and ask questions, cooperative learning is very significant (Shadaan and Leong, 2013). Additionally, several participants in this study attested to the fact that GeoGebra aided them in connecting various forms of representation, such as graphical representation and algebraic formulations of functions, and thereby improved their knowledge. In today's environment, young children can be found spending hours using their iPads, cell phones, digital games, and other modern devices. Therefore, it is essential to use technology, particularly the GeoGebra software. It is advised that all mathematics teachers use visual examples while instructing their learners. It is not enough for

teachers to just introduce technology in the form of computers, calculators, simulation games, and other tools into the classroom; they also need to know when and how to use it (Bray and Tangney, 2017). Additionally, when one sort of representation dominates learners' thinking, this creates significant learning challenges.

Overall, this study recommends GeoGebra as an effective tool to support mathematics teachers and their learners in mathematics classrooms to achieve the principles of constructivist learning. The researcher also recommends that further research on GeoGebra use be carried out so as to explore its capabilities to support retention of knowledge of transformations of functions.

The researcher suggests that in order to improve knowledge retention, future researchers should diagnose learners' action and process conceptions of function transformations and analyse the connections between the mental constructions from one level to the next. The studies reviewed thus far have not focused on the problems of conceptual understanding in the transformation of functions. The current study is therefore intended to address this gap and explore how GeoGebra may be used to enhance conceptual understanding in the transformation of functions. Future studies could also examine how learners in secondary school mathematics employ algebraic techniques to respond to questions about functions and function transformations. These topics make for compelling research topics in the future.

The DBE (2011) emphasises the constructivist approach as an intervention for understanding mathematical concepts.

5.6 Chapter summary

This chapter summarises the results of the study and highlights the implications thereof. Suggestions on dealing with challenges found in the study are provided. Current research in mathematics education is providing evidence of the considerable difficulties learners face when learning the concept of transforming functions in particular, and the concept of functions in general. This study recommended that

future research could be conducted that follows mixed methods designs, or use other qualitative or quantitative designs, in order to obtain a deeper understanding of the phenomenon discussed. Future studies on how GeoGebra enhances conceptual understanding in learning transformations of functions must also be comprehensive and undertaken over longer periods, using far larger sample sizes, at different schools of different ethnic composition and socio-economic status that reflect the entire South African economy.

As a researcher, I have greatly benefited from the classroom-based nature of this project in terms of honing my teaching techniques. The expertise of how to handle function transformations at all levels of the APOS theory is something I hope these findings will help with.

REFERENCES

Akcay, A. O. (2017). Instructional Technologies and Pre-Service Mathematics Teachers' Selection of Technology. *Journal of Education and Practice*, *8*(7), 163-173.

Adler, P. A., & Adler, P. (1994). Observational techniques. *Handbook of qualitative research* (pp. 377–392). Sage Publications, Inc.

Arbain, N., & Shukor, N. A. (2015). The effects of GeoGebra on students achievement. *Procedia-Social and Behavioral Sciences*, *172*, 208-214.

Amawa, I., Yerizon, Y., Sri, N., & Putra, R. T. (2019). Development of students' worksheet based on APOS theory approach to improve student achievement in learning system of linear equations. *International Journal of Scientific dan technology Research*, *8*(4), 287-292.

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). APOS theory: A framework for research and curriculum development in mathematics education.

Asiala, M., Brown, A., De Vries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and development in undergraduate mathematics education. *Research in Collegiate Mathematics Education*,

Assad, D. A. (2015). Task-based interviews in mathematics: Understanding student strategies and representations through problem solving. *International Journal of Education and Social Science*, *2*(1), 17-26.2, 1-32.

Baker, B., Trigueros, M., & Hemenway, C. (2001). On transformations of functions. In *Proceedings of the Twenty-Third Annual Meeting, North American Chapter of the International Group for PME* (Vol. 1, pp. 91-98).

Bansilal, S., Mkhwanazi, T., & Brijlall, D. (2014). An exploration of the common content knowledge of high school mathematics teachers. *Perspectives in Education*, *32*(1), 34-50.

Baxter, J., & Eyles, J. (1997). Evaluating qualitative research in social geography: establishing 'rigour'in interview analysis. *Transactions of the Institute of British geographers*, *22*(4), 505-525.

Bayazit, I., Aksoy, Y., & Ilhan, O. (2010). GeoGebra as an instructional tool to promote students' operational and structural conception of function. In *First North American GeoGebra Conference* (p. 117).

Baye, M. G., Ayele, M. A., & Wondimuneh, T. E. (2021). Implementing GeoGebra integrated with multi-teaching approaches guided by the APOS theory to enhance students' conceptual understanding of limit in Ethiopian Universities. *Heliyon,* 7(5), e07012.

Belgheis, S., & Kamalludeen, R. (2018). The Intention to Use GeoGebra in the Teaching of Mathematics among Malaysian Teachers. *Malaysian Online Journal of Educational Technology*, *6*(1), 109-115.

Bhagat, K. K., & Chang, C. Y. (2015). Incorporating GeoGebra into Geometry learning-A lesson from India. *Eurasia Journal of Mathematics, Science and Technology Education*, 11(1), 77-86.

Borba, M. C., & Confrey, J. (1996). A student's construction of transformations of functions in a multiple representational environment. *Educational Studies in Mathematics*, *31*(3), 319-337.

Borji, V., Alamolhodaei, H., & Radmehr, F. (2018). Application of the APOS-ACE theory to improve students' graphical understanding of derivative. *EURASIA Journal of Mathematics, Science and Technology Education*, *14*(7), 2947-2967.

Boz-Yaman, B., & Yigit-Koyunkaya, M. (2019). Examination of Pre-Service Mathematics Teachers' Mental Constructions of Function Transformation. *Acta Didactica Napocensia*, *12*(1), 33-56.

Bray, A., & Tangney, B. (2017). Technology usage in mathematics education research–A systematic review of recent trends. *Computers & Education*, *114*, 255-273.

Brown, P. A. (2008). A review of the literature on case study research. *Canadian Journal for New Scholars in Education/Revue canadienne des jeunes chercheures et chercheurs en education*,*1*(1).

Bu, L., & Robert, S. (2011). GeoGebra for model-centered learning in mathematics education: An introduction. In *Model-Centered Learning* (pp. 1-6). Brill Sense. Netherlands.

Burnett, C. S. (2020). Students' understandings of the Transformations of Functions. Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020.

Carlson, M. P., Madison, B., & West, R. D. (2015). A study of students' readiness to learn calculus. *International Journal of Research in Undergraduate Mathematics Education*, *1*(2), 209-233.

Charles-Ogan, G., & Ibibo, G. (2018). GeoGebra: A Technological Soft Ware for Teaching and Learning of Calculus in Nigerian Schools. *American Journal of Applied Mathematics and Statistics,* 6(3), 115-120.

Chamberlain Jr, D., & Vidakovic, D. (2015). APOS Theory in the Mathematical Classroom.

Chen, C. L. (2020). Predicting the Determinants of Dynamic Geometry Software Acceptance: A Two-Staged Structural Equation Modeling—Neural Network Approach. *International Journal of Information and Education Technology*, *10*(6).

Chimhande, T., Naidoo, A., & Stols, G. (2017). An analysis of Grade 11 learners' levels of understanding of functions in terms of APOS theory. *Africa Education Review*, *14*(3-4), 1-19.

Clarke, V., Braun, V., & Hayfield, N. (2015). Thematic analysis. *Qualitative psychology: A practical guide to research methods*, *222*(2015), 248.

Conway, C., Pellegrino, K., & West, C. (1998). Case study research in music education. *teaching and learning*, 37.

Creswell, J. W., & Miller, D. L. (2000). Determining validity in qualitative inquiry. *Theory into practice*, *39*(3), 124-130.

Creswell, J. W. (2002). *Educational research: Planning, conducting, and evaluating quantitative*. Upper Saddle River, NJ: Prentice Hall.

Curtin, M., & Fossey, E. (2007). Appraising the trustworthiness of qualitative studies: Guidelines for occupational therapists. *Australian occupational therapy journal*, *54*(2), 88-94.

Daher, W., & Anabousy, A. (2015). Students' Conceptions of Function Transformation in a Dynamic Mathematical Environment. *International*

Denbel, D. G. (2015). Functions in the Secondary School Mathematics Curriculum. *Journal of Education and Practice*, 6(1), 77-81.

Denzin, N. K., & Lincoln, Y. S. (Eds.). (2011). *The Sage handbook of qualitative research*. sage. *Journal for Mathematics Teaching & Learning*.

Department of Basic Education.(2011). National Curriculum Statements Grade 10-12 (Mathematics). Pretoria: Department of Basic Education.

Department of Basic Education. “Schools Subject and Diagnostic Report”. 2018. [Online]. Available From: https://www.education.gov.za/Resources/Reports.aspx

Department of Basic Education (2019). Schools Subject and Diagnostic Report. Pretoria: Department of Basic Education.

Department of Basic Education (2020). Schools Subject and Diagnostic Report. Pretoria: Department of Basic Education.

Department of Health (2015). *Ethics in health research: principles, structures and processes* (2nd ed.)

Digel, S., & Roth, J. (2022). Fostering a concept of function with combined experiments in distance and in-class learning.

Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In *The teaching and learning of mathematics at university level* (pp. 275-282). Springer, Dordrecht.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational studies in Mathematics*, *27*(3), 267-305.

Ellinger, A. D., Watkins, K. E., & Marsick, V. J. (2005). Case study research methods. *Research in organizations: Foundations and methods of inquiry*, 327-350.

Gcasamba, L. C. (2014). *A Discursive Analysis of Learners' Mathematical Thinking: The Case of Functions* (Doctoral dissertation, University of the Witwatersrand, Faculty of Science, School of Science Education).

Guncaga, J., & Majherová, J. (2012). GeoGebra as a motivational tool for teaching and learning in Slovakia. *North American GeoGebra Journal*, *1*(1).

Harel, G., & Soto, O. (2017). Structural reasoning. International Journal of Research in Undergraduate Mathematics Education, 3(1), 225-242.

Harper, M., & Cole, P. (2012). Member checking: Can benefits be gained similar to group therapy. *The Qualitative Report*, 17(2), 510-517.

Hohenwarter, M., Preiner, J., & Yi, T. (2007). Incorporating GeoGebra into teaching mathematics at the college level. In *Proceedings of the International Conference for Technology in Collegiate Mathematics 2007*.

Joshi, D. R., & Singh, K. B. (2020). Effect of Using GeoGebra on Eight Grade Students' Understanding in Learning Linear Equations. *Editorial from Bronisław Czarnocha*, 76.

Khokhar, S., Pathan, H., Raheem, A., & Abbasi, A. M. (2020). Theory Development in Thematic Analysis: Procedure and Practice. *Review of Applied Management and Social Sciences*, *3*(3), 423-433.

Kiger, M. E., & Varpio, L. (2020). Thematic analysis of qualitative data: AMEE Guide No. 131. *Medical teacher*, *42*(8), 846-854.

Korstjens, I., & Moser, A. (2018). Series: Practical guidance to qualitative research. Part 4: Trustworthiness and publishing. *European Journal of General Practice*, *24*(1), 120-124.

Kumatongo, B., & Muzata, K. K. (2021). Research paradigms and designs with their application in education. *Journal of Lexicography and Terminology (Online ISSN 2664-0899. Print ISSN 2517-9306).*, *5*(1), 16-32.

Lage, A. E., & Gaisman, M. T. (2006). An analysis of students' ideas about transformations of functions. In *Proceedings of the Twenty Eighth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 23-30).

Larson, R. E., Edwards, B. H., & Hostetler, R. P. (2001). Calculus, McDougal Littell

Lewis, A. (2002). The mixing approach to stochastic volatility and jump models. *Wilmott*.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. sage.

Lukáč, S., & Sekerák, J. (2017). The use of the system GeoGebra for graphical interpretation of functional dependencies in mathematics teaching. In *2017 15th International Conference on Emerging eLearning Technologies and Applications (ICETA)* (pp. 1-7). IEEE.

Maguire, M., & Delahunt, B. (2017). Doing a thematic analysis: A practical, step-by-step guide for learning and teaching scholars. *All Ireland Journal of Higher Education*, *9*(3).

Maharaj, A. (2010). An APOS analysis of students' understanding of the concept of a limit of a function. *Pythagoras*, 2010(71), 41-52.

Maharaj, A. (2013). An APOS analysis of natural science students' understanding of derivatives. *South African Journal of Education*, *33*(1).

Masri, R. M., Ting, S. H., Zamzamir, Z., & Ma'amor, R. L. Z. R. (2017). The effects of using GeoGebra teaching strategy in Malaysian secondary schools: A case study from Sibu, Sarawak. *Geografia-Malaysian Journal of Society and Space*, *12*(7), 13-25.

McCulloch, A. W., Hollebrands, K., Lee, H., Harrison, T., & Mutlu, A. (2018). Factors that influence secondary mathematics teachers' integration of technology in mathematics lessons. *Computers & Education*, *123*, 26-40.

McMillan, J. H., & Schumacher, S. (2010). Research in Education: Evidence-Based Inquiry, MyEducationLab Series. *Pearson*.

Merriam, S. B. (2002). Introduction to qualitative research. *Qualitative research in practice: Examples for discussion and analysis*, *1*(1), 1-17.

Mthethwa, M., Bayaga, A., Bossé, M. J., & Williams, D. (2020). GeoGebra for learning and teaching: A parallel investigation. *South African Journal of Education*, *40*(2).

Mudaly, V., & Uddin, R. S. (2016). Technology in Mathematics: Use of GeoGebra applets. International Scientific Researches Journal, 72(9).

Mushipe, M., & Ogbonnaya, U. I. (2019). Geogebra and Grade 9 learners' achievement in linear functions. International Journal of Emerging Technologies in Learning.

Nagle, C., Martínez-Planell, R., & Moore-Russo, D. (2019). Using APOS theory as a framework for considering slope understanding. *The Journal of Mathematical Behavior*, *54*, 100684.

National Research Council, & Mathematics Learning Study Committee. (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.

Njie, B., & Asimiran, S. (2014). Case study as a choice in qualitative methodology. *Journal of Research & Method in Education*, *4*(3), 35-40.

Nocar, D., & Zdráhal, T. (2016). ICT Tools Used in Teaching and Learning Concept of Function in School Mathematics, Olomouc: Palacký University.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. Making the connection: Research and teaching in undergraduate mathematics education, 27, 42.

Ogbonnaya, U. I., & Mushipe, M. (2020). The Efficacy of GeoGebra-Assisted Instruction on Students' Drawing and Interpretations of Linear Functions. *International Journal of Learning, Teaching and Educational Research*, 19(9), 1-14.

Oktaç, A. (2022). What's new with APOS theory? A look into levels and Totality. *Avances de investigación en educación matemática*.

Oktaç, A., Trigueros, M., & Romo, A. (2019). APOS theory: Connecting research and teaching. *For the learning of mathematics*, *39*(1), 33-37.

Olsson, J. (2017). *GeoGebra, enhancing creative mathematical reasoning* (Doctoral dissertation, Umeå universitet).

Patton, M. Q. (1990). *Qualitative evaluation and research methods*. SAGE Publications, inc.

Ponte, J. P. D. (1992). The history of the concept of function and some educational implications. *The Mathematics Educator*, 3-8.

Rocha, H. (2020). Graphical representation of functions using technology: a window to teacher knowledge. Teaching Mathematics and its Applications. *An International Journal of the IMA*, 39(2), 105-126.

Salinas-Hernández, U., Sacristán, A. I., & Trouche, L. (2018). Technology integration into mathematics classrooms: case study of a high school teacher's use of GeoGebra. In *Proceedings of the Re (s) sources International Conference* (pp. 88-91).

ŞEFİK, Ö., UZUN, Ö. E., & Şenol, D. O. S. T. (2021). Content Analysis of the APOS Theory Studies on Mathematics Education Conducted in Turkey and Internationally: A Meta-Synthesis Study. *Necatibey Eğitim Fakültesi Elektronik Fen ve Matematik Eğitimi Dergisi*, *15*(2), 404-428.

Seloraji, P., & Eu, L. K. (2017). Students' Performance in Geometrical Reflection Using GeoGebra. *Malaysian Online Journal of Educational Technology*, *5*(1), 65-77.

Shadaan, P., & Leong, K. E. (2013). Effectiveness of Using GeoGebra on Students' Understanding in Learning Circles. *Malaysian Online Journal of Educational Technology*, *1*(4), 1-11.

Shenton, A. K. (2004). Strategies for ensuring trustworthiness in qualitative research projects. *Education for information*, *22*(2), 63-75.

Siyepu, S. W. (2013). Students' interpretations in learning derivatives in a university mathematics classroom. In *Proceedings of the 19th Annual congress of the association for mathematics education of South Africa* (Vol. 1, pp. 183-193).

Stols, G., & Kriek, J. (2011). The understanding of teachers' use of the Geometer's Sketchpad in their classrooms.

Tatira, B. (2021). Mathematics Education Students' Understanding of Binomial Series Expansion Based on the APOS Theory. *EURASIA Journal of Mathematics, Science and Technology Education*, *17*(12), em2035.

Thomas, R. V. (2016). The Effects of Dynamic Graphing Utilities on Student Attitudes and Conceptual Understanding in College Algebra, Fayetteville: University of Arkansas.

Thurmond, V. A. (2001). The point of triangulation. *Journal of nursing scholarship*, *33*(3), 253-258.

Triantafyllou, E., & Timcenko, O. (2014). Opportunities and challenges of using technology in mathematics education of creative engineering studies. In *International Conference on Human-Computer Interaction* (pp. 171-176). Springer, Cham.

Tumenbayar, D., Amarzaya, A., & Navchaa, T. (2021). Student's Conceptions of Function Transformation. *Open Education Studies*, *3*(1), 26-32.

Ubah, I. J. A., & Bansilal, S. (2018). Pre-service mathematics teachers' knowledge of mathematics for teaching: quadratic functions. *Problems of Education in the 21st Century*, 76(6), 847.

Uddin, R. S. (2011). *Geogebra, tool for mediating knowledge in the teaching and learning of transformation of functions in mathematics* (Doctoral dissertation).

Uwurukundo, M. S., Maniraho, J. F., & Tusiime, M. (2020). GeoGebra integration and effectiveness in the teaching and learning of mathematics in secondary schools: A review of literature. *African Journal of Educational Studies in Mathematics and Sciences*, *16*(1), 1-13.

Van Wynsberghe, R., & Khan, S. (2007). Redefining case study. *International Journal of Qualitative Methods*, *6*(2), 80-94.

Venter, C. (2020). Learning the function concept by exploring digital images as functions. *Pythagoras*, *41*(1),a524. http://doi.org/10.4102/pythagoras.v41i1.524.

Wassie, Y. A., & Zergaw, G. A. (2018). Capabilities and Contributions of the Dynamic Math Software, GeoGebra---A Review. *North American GeoGebra Journal*, 7(1), 68-86.

Yazan, B. (2015). Three approaches to case study methods in education: Yin, Merriam, and Stake. *The qualitative report*, *20*(2), 134-152.

Zazkis, R., Liljedahl, P., & Gadowsky, K. (2003). Conceptions of function translation: Obstacles, intuitions, and rerouting. *The Journal of Mathematical Behavior*, *22*(4), 435-448.

Zulnaidi, H., & Zakaria, E. (2012). The effect of using GeoGebra on conceptual and procedural knowledge of high school mathematics students. *Asian Social Science*, *8*(11), 102.

Zulnaidi, H., & Zamri, S. N. A. S. (2017). The effectiveness of the GeoGebra software: The intermediary role of procedural knowledge on students' conceptual knowledge and their achievement in mathematics. *Eurasia Journal of Mathematics, Science and Technology Education*, 13(6), 2155-2180.

Appendix 1: Homework activity and genetic decomposition

Question 1

For each of the following algebraic expressions of functions, draw the graph of the function, using transformations on the basic functions $y = x^2$, $y = \frac{1}{x}$, and $y = 2^x$ respectively:

(a) $f(x) = (x-3)^2 + 2$

(b) $g(x) = \frac{-2}{x+2} - 3$

(c) $h(x) = 2^{x-3} - 2$

Question 2

Perform transformations on $p(x) = \frac{1}{x+1}$ so that it becomes $k(x) = \frac{1}{x-3} + 4$.

Question 3

For each of the four functions in **Question 1** and **Question 2**, write verbally, the transformations made on the basic functions to arrive at the resulting functions. Comment on what effect this has on the domain and range.

Question 4

(a) Sketch the graph of $f(x) = 2(x-5)^2 + 3$

(b) From your study of Algebra, what differences and similarities do you expect to see between the graph you drew in (a) and the graph of $g(x) = \frac{2}{x-5} + 3$?

Genetic decomposition (to be used by the researcher)

Level	**Indicators**
Action	Learner can perform operations on GeoGebra and on functions and variables step by step, and these operations can be applied either in the analytical or graphical representation context, relying on memorised facts or external signs.
Process	Learner is able to describe changes in the basic functions as a consequence of the application of the transformation without the need to perform each step of the transformation. Also able to reverse the

	process to identify the function on which a set of transformations was applied.
Object	Learner can reflect on all processes and is able to think of them as a whole, in any representation or context.
Schema	Learner can form conceptions by considering the interconnection of several actions, processes, objects and other previously constructed schema.

Appendix 2: Capricorn District Department of Basic Education Letter

11 Feathers Hill
Vharanani Street
Bendor
Polokwane
0699
11 February 2021

The Circuit Manager
Department of Basic Education
Capricorn District
Polokwane
0699

Dear sir/ Madam

RE: **Requesting permission to conduct research**

I am an MEd (Mathematics Education) student at University of Limpopo. As part of my study, I am carrying out research on **Using GeoGebra to enhance Grade 11 learners' conception of transformations of functions**.

I kindly ask for permission to conduct research at a school in your circuit. The research will involve teaching grade 11 learners transformation of functions using GeoGebra, administering class activities and conducting task based interviews. I assure that the research process will not tamper with the normal teaching and learning activities of the school. In addition, anonymity of the school and the participants will be maintained throughout the research process.

If you have any comments or questions about this research, please contact my supervisor Dr P Mutodi at paul.mutodi2@ul.ac.za

Yours Faithfully
Tendai David Matiyenga

Appendix 3: Headmaster consent and permission letters

11 Feathers Hill
Vharanani Street
Bendor
Polokwane
0699
11 February 2021

The Principal
Mitchell House Prep School
Polokwane
0699

Dear Sir/ Madam

RE: **Requesting permission to conduct research**

I am an MEd (Mathematics Education) student at University of Limpopo. As part of my study, I am carrying out research on **Using GeoGebra to enhance Grade 11 learners' conception of transformations of functions**.

I kindly ask for permission to conduct research at your school. The research will involve teaching Grade 11 learners Transformations of Functions using GeoGebra, administering class activities and conducting task based interviews. I assure that the research process will not tamper with the normal teaching and learning activities of the school. In addition, anonymity of the school and the participants will be maintained throughout the research process.
If you have any comments or questions about this research, please contact my supervisor Dr P Mutodi at paul.mutodi2@ul.ac.za

Yours Faithfully
Tendai David Matiyenga
(Researcher)

Gmail Tendai Matiyenga

Research Consent Letter

4 messages

Tendai Matiyenga 7 May 2021 at 07:59

Hi sir.
Please find attached my research consent letter for your attention.
Regards,
TD Matiyenga
Cell: 0780836551

2 attachments

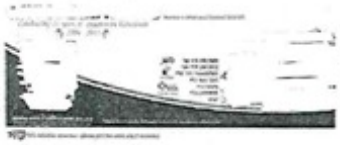


image002.png
55K

Headmaster consent letter.docx
14K

To: Tendai Matiyenga 7 May 2021 at 10:38

Dear Tendayi

I am happy for you to use to assist you with your research. I attach a copy of our policy regarding such research. Please indicate by email that you are happy to comply with our conditions.

And good luck with the research

REMEMBER TO BE SAFE ☺: SANITISE ÷ SOCIALLY DISTANCE ÷ MASK UP ÷ VENTILATE!

Kind regards

Principal

[redacted] School NPC [redacted]

POLICY: Educational surveys and research

1. Only written requests to the principal to conduct an educational survey or research at the school will be considered.

2. Such written requests will include the following information:
   2.1. The full details of the student / person wishing to conduct the survey;
   2.2. The university and course for which the researcher is registered;
   2.3. Details of whether the proposed research has the approval of the university;
   2.4. The name and contact details of the university supervisor/ promoter overseeing the research;
   2.5. The purpose of the research and / or why the survey is being done;
   2.6. An undertaking from the researcher that the information gleaned from the survey would remain strictly confidential and would only be seen by those people involved in the assessment of the researcher.

3. If such a request is approved, the principal will formally write to the researcher informing him / her that the request had been approved and the conditions under which the research will be conducted.

4. A digital copy of the completed dissertation / thesis / essay that the research results in must be provided (at the expense of the researcher) to the principal as soon as possible after the study has been completed, if the principal makes such a request.

5. The principal, in consultation with the person / student undertaking the research, will decide on the dissemination of the results of the research both internally and externally.

[redacted]
Principal
30 April 2021

Appendix 4: Parent/ Guardian consent letter

11 Feathers Hill

Vharanani Street
Bendor
Polokwane
0699

11 February 2021

Dear Parent/ Guardian

RE: **Requesting permission to conduct research**

I am an MEd (Mathematics Education) student at University of Limpopo. As part of my study, I am carrying out research on **Using GeoGebra to enhance Grade 11 learners' conception of transformations of functions**.

I kindly request that you give consent for your child to participate in the research project. The research will involve teaching grade 11 learners Transformations of Functions using GeoGebra, administering class activities and conducting task based interviews. I assure that the research process will not tamper with the normal teaching and learning activities of the school. In addition, anonymity of the school and the participants will be maintained throughout the research process. Please indicate your consent by signing the letter and return it back to me.

If you have any comments or questions about this research, please contact my supervisor Dr P Mutodi at paul.mutodi2@ul.ac.za.

Yours Faithfully
Tendai David Matiyenga
(Masters Student)

Name of Parent:……
Signature: ……
Date: …….

Appendix 5: Assent Form

Name of Study: **Using GeoGebra to enhance Grade 11 learners' conception of transformations of functions.**

I understand that I have been asked to participate in the study and I will be asked to take part in task based interviews.
I understand that I do not have to participate. If I do participate, I can quit at any time. I also understand that I do not have to answer any questions I do not want to answer or do anything I do not want to do.
No one else will know what I have said or done in the study except for the researcher.
This study is being done by Mr TD Matiyenga. His phone number is 0780836551 and his email addresses are t.matiyenga@mitchellhouse.co.za and daviematiyenga@gmail.com.
If I have any questions or concerns about the study, I can call and ask him about them.
When I sign my name, this means that I agree to participate in the study and that all of my questions have been answered. I have also been given a copy of this form.

Name: .....

Signature: …….

Date: …….

Appendix 6: TREC Approval

University of Limpopo
Department of Research Administration and Development
Private Bag X1106, Sovenga, 0727, South Africa
Tel: (015) 268 3935, Fax: (015) 268 2306, Email: anastasia.ngobe@ul.ac.za

TURFLOOP RESEARCH ETHICS COMMITTEE

ETHICS CLEARANCE CERTIFICATE

MEETING: 14 October 2021

PROJECT NUMBER: TREC/219/2021: PG

PROJECT:

Title:	Using Geogebra to Enhance Grade 11 Learners' Conception of Transformations of Functions
Researcher:	TD Matiyenga
Supervisor:	Dr P Mutodi
Co-Supervisor/s:	Dr BK Maphutha
School:	Education
Degree:	Master of Education in Mathematics Education

PROF P MASOKO
CHAIRPERSON: TURFLOOP RESEARCH ETHICS COMMITTEE

The Turfloop Research Ethics Committee (TREC) is registered with the National Health Research Ethics Council, Registration Number: **REC-0310111-031**

Note:

i) This Ethics Clearance Certificate will be valid for one (1) year, as from the abovementioned date. Application for annual renewal (or annual review) need to be received by TREC one month before lapse of this period.

ii) Should any departure be contemplated from the research procedure as approved, the researcher(s) must re-submit the protocol to the committee, together with the Application for Amendment form.

iii) PLEASE QUOTE THE PROTOCOL NUMBER IN ALL ENQUIRIES.

Appendix 7: Lessons Outline

	Topic/ Content	Teacher/ Pupil Activities
Lesson 1	Translations	Using standard forms for hyperbola, parabola and exponential functions, the teacher asked the learners to use sliders on GeoGebra to investigate the effects of changing any of the parameters a, p and q. They were to write their observations in their exercise followed by a class discussion led by the teacher after about thirty minutes. A short exercise for further practice was given to learners at the end of the lesson.
Lesson 2	Deductions from sketch graphs	Discussion on the previous day's homework. Teacher gave learners a new worksheet that required learners to make certain deductions from sketch graphs when the equation of the graph is given. Learners were expected to determine coordinates, length of lines, points of intersections of graphs and transformations of functions. Learners were encouraged to work in pairs. Teacher was moving around the class, checking for progress and assisting needy learners in the process. Learners were also free to confirm their answers on GeoGebra.
Lesson 3	Summary of topic	A class discussion on the previous day's worksheet. A homework for data collection for this study was also handed out for learners to go and do the work individually at home. They were to hand in the work the following day.

Appendix 8: Lesson Observation Sheet

Date:
Lesson:
Teacher:

Purpose: The purpose of this observation sheet is to observe the instructional methods and strategies used in the classroom as well as learners' participation and work.

This sheet allows the observer to answer the following questions:

What type of transformations of functions are being used or emphasized?

Are the APOS and ACE constructs evident?

Are learners just copying and taking notes?

Are they asking questions?

Are there any class discussions being conducted?

Does the teacher provide learners with activities and opportunities to discuss, think, reflect, analyze or explore certain problems?

Is GeoGebra incorporated in some activities?

Is it used by learners or by teacher or both?

Is GeoGebra providing learners with opportunities to access and deal with the different opportunities that it offers?

Appendix 9: Translation of graphs worksheet

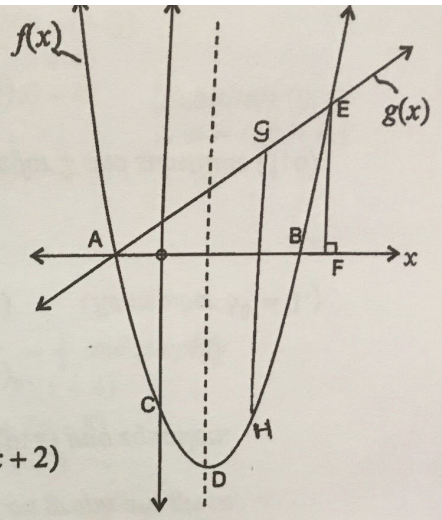
1. Draw the graph of $y = ax + q$ on GeoGebra. What are the effects of moving sliders for a and q?
2. Draw the graph of $y = a(x + p)^2 + q$ on GeoGebra. What are the effects of moving sliders for a, p and q?
3. Draw the graph of $y = \frac{a}{x+p} + q$ on GeoGebra. What are the effects of moving sliders for a, p and q?
4. Draw the graph of $y = a \cdot b^{x+p} + q$ on GeoGebra. What are the effects of moving sliders for a, p and q?
5. Does moving sliders for a, p and q have the same effect on the different graphs?
6. What will the equation $y = 3^{x-1} - 1$ be if the graph is translated 2 units to the left and 2 units up?
7. What will the equation of $y = -2(x - 1)^2 + 3$ be if it is translated 3 units to the left and 2 units up?
8. Draw the graph of $y = ax^2 + bx + c$ on GeoGebra. What do you notice? What are the effects of moving sliders for a, b and c?
9. What will the equation $y = x^2 - 2x - 3$ be when moved 2 units to the right and translated 3 units down?

Appendix 10: Deductions from sketch graphs worksheet (Adapted from: The Solution for Revision and Examination Preparation- Grade 11 Mathematics)

Example 19.

In the diagram $f(x) = x^2 - 4x - 5$ and $g(x) = x + 1$.

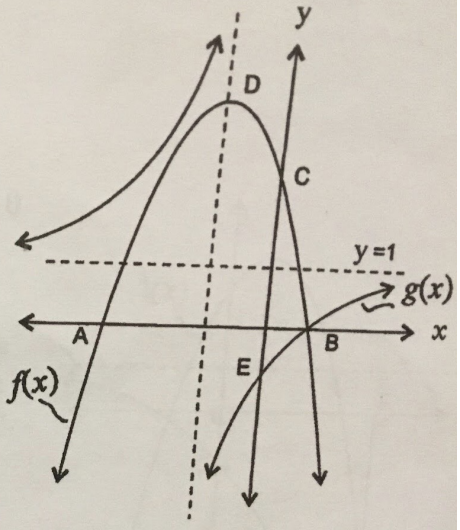
19.1 Calculate the lengths of OA, OB and OC.
19.2 Calculate the coordinates of D, the turning point of $f(x)$.
19.3 Calculate the length of EF.
19.4 Calculate the maximum length of GH.
19.5 What is the range of $f(x)$?
19.6 Give the coordinates of the turning point of $f(x+2)$ and describe the translation that took place.



Example 20.

The diagram represents the graphs of $f(x) = -x^2 - 2x + 3$ and $g(x) = \frac{-2}{x+p} + q$.
D is the turning point of the parabola.

20.1 Determine the average gradient of $f(x)$ between the points $x = 2$ en $x = 3$.
20.2 What are the values of p and q ?
20.3 Hence, calculate the coordinates of E.
20.4 Give the domain and the range of $g(x)$.
20.5 Calculte $g(-5)$



Excample 21.

The figure shows the graphs of the functions $f(x) = x^2 - x - 2$ and $g(x) = 2^{x+1} - 1$.

21.1 Determine the coordinates of C.
21.2 What is the equation of the asymptote of the function $y = 2^{x+1} - 1$?
21.3 Determine the length of EF if OE = 3 units.
21.4 If $h(x) = g(x) + 2$, give the equation of the asymptote of $h(x)$ as well as the coordinates of the *y-intercept* of $h(x)$.